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Constant-factor approximations for asymmetric TSP on nearly-embeddable graphs

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Abstract

In the Asymmetric Traveling Salesperson Problem (ATSP) the goal is to find a closed walk of minimum cost in a directed graph visiting every vertex. We consider the approximability of ATSP on topologically restricted graphs. It has been shown by Oveis Gharan and Saberi [14] that there exists polynomial-time constant-factor approximations on planar graphs and more generally graphs of constant orientable genus. This result was extended to non-orientable genus by Erickson and Sidiropoulos [9].

We show that for any class of *nearly-embeddable* graphs, ATSP admits a polynomial-time constant-factor approximation. More precisely, we show that for any fixed $k \geq 0$, there exist $\alpha, \beta > 0$, such that ATSP on n-vertex k-nearly-embeddable graphs admits a α-approximation in time $O(n^\beta)$. The class of k-nearly-embeddable graphs contains graphs with at most k apices, k vortices of width at most k, and an underlying surface of either orientable or non-orientable genus at most k. Prior to our work, even the case of graphs with a single apex was open. Our algorithm combines tools from rounding the Held-Karp LP via thin trees with dynamic programming.

We complement our upper bounds by showing that solving ATSP exactly on graphs of pathwidth k (and hence on k-nearly embeddable graphs) requires time $n^{\Omega(k)}$, assuming the Exponential-Time Hypothesis (ETH). This is surprising in light of the fact that both TSP on undirected graphs and Minimum Cost Hamiltonian Cycle on directed graphs are FPT parameterized by treewidth.

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1 Introduction

An instance of the Asymmetric Traveling Salesman Problem (ATSP) consists of a directed graph $\vec{G}$ and a (not necessarily symmetric) cost function $c : E(\vec{G}) \to \mathbb{R}^+$.The goal is to find a spanning closed walk of $\vec{G}$ with minimum total cost. This is one of the most well-studied NP-hard problems.

Asadpour *et al.* [2] obtained a polynomial-time $O(\log n/\log\log n)$-approximation algorithm for ATSP, which was the first asymptotic improvement in almost 30 years [13, 4, 10, 19]. Building on their techniques, Oveis Gharan and Saberi [14] described a polynomial-time $O(\sqrt{g}\log g)$-approximation algorithm when the input includes an embedding of the input graph into an orientable surface of genus g. Erickson and Sidiropoulos [9] improved the dependence on the genus by obtaining a $O(\log g/\log\log g)$-approximation.

Anari and Oveis Gharan [1] have recently shown that the integrality gap of the natural linear programming relaxation of ATSP proposed by Held and Karp [17] is $\log\log^{O(1)} n$. This implies a polynomial-time $\log\log^{O(1)} n$-approximation algorithm for the *value* of ATSP. We remark that the best known lower bound on the integrality gap of the Held-Karp LP is 2 [5]. Obtaining a polynomial-time constant-factor approximation algorithm for ATSP is a central open problem in optimization.

1.1 Our contribution

We study the approximability of ATSP on topologically restricted graphs. Prior to our work, a constant-factor approximation algorithm was known only for graphs of bounded genus. We significantly extend this result by showing that there exist a polynomial-time constant-factor approximation algorithm for ATSP on nearly embeddable graphs. These graphs include graphs with bounded genus, with a bounded number of apices and a bounded number of vortices of bounded pathwidth. For any $a, g, k, p \geq 0$, we say that a graph is (a, g, k, p)-nearly embeddable if it is obtained from a graph of Euler genus g by adding a apices and k vortices of pathwidth p (see [21, 20, 8] for more precise definitions). The following summarizes our result.

Theorem 1.1. *Let $a, g, k \geq 0$, $p \geq 1$. There exists a $O(a(g+k+1)+p^2)$-approximation algorithm for ATSP on (a, g, k, p)-nearly embeddable digraphs, with running time $n^{O((a+p)(g+k+1)^4)}$.*

The above algorithm is obtained via a new technique that combines the Held-Karp LP with a dynamic program that solves the problem on vortices. We remark that it is not known whether the integrality gap of the LP is constant for graphs of constant pathwidth.

We complement this result by showing that solving ATSP exactly on graphs of pathwidth p (and hence on p-nearly embeddable graphs) requires time $n^{\Omega(p)}$, assuming the Exponential-Time Hypothesis (ETH). This is surprising in light of the fact that both TSP on undirected graphs and Minimum Cost Hamiltonian Cycle on directed graphs are FPT parameterized by treewidth. The following summarizes our lower bound.

Theorem 1.2. *Assuming ETH, there is no $f(p)n^{o(p)}$ time algorithm for ATSP on graphs of pathwidth at most p for any computable function f.*

1.2 Overview of the algorithm

We now give a high level overview of the main steps of the algorithm and highlight some of the main challenges.

Step 1: Reducing the number of vortices. We first reduce the problem to the case of nearly embeddable graphs with a single vortex. This is done by iteratively merging pairs of vortices. We can merge two vortices by adding a new handle on the underlying surface-embedded graph. For the remainder we will focus on the case of graphs with a single vortex.

Step 2: Traversing a vortex. We obtain an exact polynomial-time algorithm for computing a closed walk that visits all the vertices in the vortex. We remark that this subsumes as a special case the problem of visiting all the vertices in a single face of a planar graph, which was open prior to our work.

Let us first consider the case of a vortex in a planar graph. Let $\vec{W}$ be an optimal walk that visits all the vertices in the vortex. Let F be the face on which the vortex is attached. We give a dynamic program that maintains a set of partial solutions for each subpath of F. We prove correctness of the algorithm by establishing structural properties of $\vec{W}$. The main technical difficulty is that $\vec{W}$ might be self-crossing. We first decompose $\vec{W}$ into a collection $\mathcal{W}$ of non-crossing walks. We form a conflict graph $\mathcal{I}$ of $\mathcal{W}$ and consider a spanning forest $\mathcal{F}$ of $\mathcal{I}$. This allows us to prove correctness via induction on the trees of $\mathcal{F}$.

The above algorithm can be extended to graphs of bounded genus. The main difference is that the dynamic program now computes a set of partial solutions for each bounded collection of subpaths of F.

Finally, the algorithm is extended to the case of nearly-embeddable graphs by adding the apices to the vortex without changing the cost of the optimum walk.

Step 3: Finding a thin forest in the absence of vortices. The constant-factor approximation for graphs of bounded genus was obtained by showing by constructing thin forests with a bounded number of components in these graphs [14, 9]. We extend this result by constructing thin forests with a bounded number of components in graphs of bounded genus and with a bounded number of additional apices. Prior to our work even the case of planar graphs with a single apex was open; in fact, no constant-factor approximation algorithm was known for these graphs.

Step 4: Combining the Held-Karp LP with the dynamic program. We next combine the dynamic program with the thin forest construction. We first compute an optimal walk $\vec{W}$ visiting all the vertices in the vortex, and we contract the vortex into a single vertex. A natural approach would be to compute a thin forest in the contracted graph. Unfortunately this fails because such a forest might not be thin in the original graph. In order to overcome this obstacle we change the feasible solution of the Held-Karp LP by taking into account $\vec{W}$, and we modify the forest construction so that it outputs a subgraph that is thin with respect to this new feasible solution.

Step 5: Rounding the forest into a walk. Once we have a thin spanning subgraph of G we can compute a solution to ATSP via circulations, as in previous work.

1.3 Organization

The rest of the paper is organized as follows. Section 2 introduces some basic notation. Section 3 defines the Held-Karp LP for ATSP. Section 4 presents the main algorithm, using the dynamic

program and the thin forest construction as a black box. Section 5 presents the technique for combining the dynamic program with the Held-Karp LP. Section 6 gives the algorithm for computing a thin tree in a 1-apex graph. This algorithm is generalized to graphs with a bounded number of apices in Section 7, and to graphs of bounded genus and with a bounded number of apices in Section 8. In Section 9 we show how to modify the thin forest construction so that we can compute a spanning thin subgraph in a nearly-embeddable graph, using the solution of the dynamic program.

The dynamic program is given in Sections 10, 11, 12, 13, and 14. More precisely, Section 10 introduces a certain preprocessing step. Section 11 establishes a structural property of the optimal solution. Section 12 presents the dynamic program for a vortex in a planar graph. Sections 13 and 14 generalize this dynamic program to graphs of bounded genus and with a bounded number of apices respectively.

Finally, Section 15 presents the lower bound.

2 Notation

In this section we introduce some basic notation that will be used throughout the paper.

Graphs. Unless otherwise specified, we will assume that for every pair of vertices in a graph there exists a unique shortest path; this property can always achieved by breaking ties between different shortest paths in a consistent manner (e.g. lexicographically). Moreover for every edge of a graph (either directed or undirected) we will assume that its length is equal to the shortest path distance between its endpoints. Let $\vec{G}$ be some digraph. Let G be the undirected graph obtained from $\vec{G}$ by ignoring the directions of the edges, that is $V(G) = V(\vec{G})$ and $E(G) = \{\{u,v\} : (u,v) \in E(\vec{G}) \text{ or } (v,u) \in E(\vec{G})\}$. We say that G is the *symmetrization* of $\vec{G}$. For some $x : E(\vec{G}) \to \mathbb{R}$ we define $\mathsf{cost}_{\vec{G}}(x) = \sum_{(u,v)\in E(\vec{G})} x((u,v)) \cdot d_{\vec{G}}(u,v)$. For a subgraph $S \subseteq G$ we define $\mathsf{cost}_G(S) = \sum_{e\in E(S)} c(e)$. Let z be a weight function on the edges of G. For any $A, B \subseteq V(G)$ we define $z(A,B) = \sum_{a\in A, b\in B} z(\{a,b\})$.

Asymmetric TSP. Let $\vec{G}$ be a directed graph with non-negative arc costs. For each arc $(u,v) \in E(\vec{G})$ we denote the cost of (u,v) by $c(u,v)$. A *tour* in $\vec{G}$ is a closed walk in $\vec{G}$. The *cost* of a tour $\tau = v_1, v_2, \ldots, v_k, v_1$ is defined to be $\mathsf{cost}_{\vec{G}}(\tau) = d_{\vec{G}}(v_k, v_1) + \sum_{i=1}^{k-1} d_{\vec{G}}(v_i, v_{i+1})$. Similarly the cost of an open walk $W = v_1, \ldots, v_k$ is defined to be $\mathsf{cost}_{\vec{G}}(W) = \sum_{i=1}^{k-1} d_{\vec{G}}(v_i, v_{i+1})$. The cost of a collection $\mathcal{W}$ of walks is defined to be $\mathsf{cost}_{\vec{G}}(\mathcal{W}) = \sum_{W\in\mathcal{W}} \mathsf{cost}_{\vec{G}}(W)$. We denote by $\mathsf{OPT}_{\vec{G}}$ the minimum cost of a tour traversing all vertices in $\vec{G}$. For some $U \subseteq V(\vec{G})$ we denote by $\mathsf{OPT}_{\vec{G}}(U)$ the minimum cost of a tour in $\vec{G}$ that visits all vertices in U.

3 The Held-Karp LP

We recall the Held-Karp LP for ATSP [16]. Fix a directed graph $\vec{G}$ and a cost function $c : E(\vec{G}) \to \mathbb{R}^+$. For any subset $U \subseteq V$, we define

$$\begin{aligned} \delta^+_{\vec{G}}(U) &:= \{(u,v) \in E(\vec{G}) : u \in U \text{ and } v \notin U\} \\ \text{and} \quad \delta^-_{\vec{G}}(U) &:= \delta^+_{\vec{G}}(V \setminus U). \end{aligned}$$

We omit the subscript $\vec{G}$ when the underlying graph is clear from context. We also write $\delta^+(v) = \delta^+(\{v\})$ and $\delta^-(v) = \delta^-(\{v\})$ for any single vertex v.

Let G be the symmetrization of $\vec{G}$. For any $U \subseteq V(G)$, we define

$$\delta_G(U) := \{\{u, v\} \in E(G) : u \in U \text{ and } v \notin U\}.$$

Again, we omit the subscript G when the underlying graph is clear from context. We also extend the cost function c to undirected edges by defining

$$c(\{u, v\}) := \min\{c((u, v)), c((v, u))\}.$$

For any function $x\colon E(\vec{G}) \to \mathbb{R}$ and any subset $W \subseteq E(\vec{G})$, we write $x(W) = \sum_{a \in W} x(a)$. With this notation, the Held-Karp LP relaxation is defined as follows.

$$\begin{array}{lrl}
\text{minimize} & \sum_{a \in A} c(a) \cdot x(a) & \\
\text{subject to} & x(\delta^+(U)) \geq 1 & \text{for all nonempty } U \subsetneq V(\vec{G}) \\
& x(\delta^+(v)) = x(\delta^-(v)) & \text{for all } v \in V(\vec{G}) \\
& x(a) \geq 0 & \text{for all } a \in E(\vec{G})
\end{array}$$

We define the *symmetrization* of x as the function $z\colon E(G) \to \mathbb{R}$ where

$$z(\{u, v\}) := x((u, v)) + x((v, u))$$

for every edge $\{u, v\} \in E(G)$. For any subset $W \subseteq E(G)$ of edges, we write $z(W) := \sum_{e \in W} z(e)$. Let $\vec{W} \subseteq \vec{G}$. Let $\alpha, s > 0$. We say that $\vec{W}$ is *α-thin* (w.r.t. z) if for all $U \subseteq V$ we have

$$|E(\vec{W}) \cap \delta(U)| \leq \alpha \cdot z(\delta(U)).$$

We also say that $\vec{W}$ is *(α, s)-thin* (w.r.t. x) if $\vec{W}$ is α-thin (w.r.t. z) and

$$c(E(\vec{W})) \leq s \cdot \sum_{e \in E(\vec{G})} c(e) \cdot x(e).$$

We say that z is *$\vec{W}$-dense* if for all $(u, v) \in E(\vec{W})$ we have $z(\{u, v\}) \geq 1$. We say that z is *ε-thick* if for all $U \subsetneq V(G)$ with $U \neq \emptyset$ we have $z(\delta(U)) \geq \varepsilon$.

4 An approximation algorithm for nearly-embeddable graphs

The following Lemma is implicit in the work of Erickson and Sidiropoulos [9] (see also [3]).

Lemma 4.1. *Let $\vec{G}$ be a digraph and let x be a feasible solution for the Held-Karp LP for $\vec{G}$. Let $\alpha, s > 0$, and let S be a (α, s)-thin spanning subgraph of G (w.r.t. x), with at most k connected components. Then, there exists a polynomial-time algorithm which computes a collection of closed walks $C_1, \ldots, C_{k'}$, for some $k' \leq k$, such that their union visits all the vertices in $V(\vec{G})$, and such that $\sum_{i=1}^{k} \mathsf{cost}_{\vec{G}}(C_i) \leq (2\alpha + s) \sum_{e \in E(\vec{G})} c(e) \cdot x(e)$.*

The following is the main technical Lemma that combines a solution to the Held-Karp LP with a walk traversing the vortex that is computed via the dynamic program. The proof of Lemma 4.2 is deferred to Section 5.

Lemma 4.2. *Let $a, g, p > 0$, let $\vec{G}$ be a $(a, g, 1, p)$-nearly embeddable graph, and let G be its symmetrization. There exists an algorithm with running time $n^{O((a+p)g^4)}$ which computes a feasible solution x for the Held-Karp LP for $\vec{G}$ with cost $O(\mathsf{OPT}_{\vec{G}})$ and a spanning subgraph S of G with at most $O(a+g)$ connected components, such that S is $(O(a \cdot g + p^2), O(1))$-thin w.r.t. x.*

Using Lemma 4.2 we are now ready to obtain an approximation algorithm for nearly-embeddable graphs with a single vortex.

Theorem 4.3. *Let $a, g \geq 0$, $p \geq 1$. There exists a $O(a \cdot g + p^2)$-approximation algorithm for ATSP on $(a, g, 1, p)$-nearly embeddable digraphs, with running time $n^{O((a+p)(g+1)^4)}$.*

Proof. We follow a similar approach to [9]. The only difference is that in [9] the algorithm uses an optimal solution to the Held-Karp LP. In contrast, here we use a feasible solution that is obtained by Lemma 4.2, together with an appropriate thin subgraph.

Let $\vec{G}$ be $(a, g, 1, p)$-nearly embeddable digraph. By using Lemma 4.2, we find in time $n^{O((a+p)g^4)}$ a feasible solution x for the Held-Karp LP for $\vec{G}$ with cost $O(\mathsf{OPT}_{\vec{G}})$ and a spanning subgraph S of G with at most $O(a+g)$ connected components, such that S is $(O(a \cdot g + p^2), O(1))$-thin w.r.t. x. Now we compute in polynomial time a collection of closed walks $C_1, \ldots, C_{k'}$, for some $k' \in O(a+g)$, that visit all the vertices in $V(\vec{G})$, and such that the total cost of all walks is at most $O((a \cdot g + p^2) \cdot OPT_{\vec{G}})$, using Lemma 4.1. For every $i \in \{1, \ldots, k'\}$, let $v_i \in V(\vec{G})$ be an arbitrary vertex visited by C_i. We construct a new instance $(\vec{G}', c')$ of ATSP as follows. Let $V(\vec{G}') = \{v_1, \ldots, v_{k'}\}$. For any $u, v \in V(\vec{G}')$, we have an edge (u, v) in $E(\vec{G}')$, with $c'(u, v)$ being the shortest-path distance between u and v in G with edge weights given by c. By construction we have $\mathsf{OPT}_{\vec{G}'} \leq \mathsf{OPT}_{\vec{G}}$. We find a closed tour C in $\vec{G}'$ with $\mathsf{cost}_{\vec{G}'}(C) = \mathsf{OPT}_{\vec{G}'}$ in time $2^{O(|V(\vec{G}')|)} \cdot n^{O(1)} = 2^{O(a+g)} \cdot n^{O(1)}$. By composing C with the k' closed walks $C_1, \ldots, C_{k'}$, and shortcutting as in [12], we obtain a solution for the original instance, of total cost $O(a \cdot g + p^2) \cdot \mathsf{OPT}_{\vec{G}}$. □

We are now ready to prove the main algorithmic result of this paper.

Proof of Theorem 1.1. We may assume $k \geq 2$ since otherwise the assertion follows by Theorem 4.3. We may also assume w.l.o.g. that $p \geq 2$. Let $\vec{G}$ be a (a, g, k, p)-nearly embeddable digraph. It suffices to show that there exists a polynomial time computable $(a, g+k-1, 1, 2p)$-nearly embeddable digraph $\vec{G}'$ with $V(\vec{G}') = V(\vec{G})$ such that for all $u, v \in V(\vec{G})$ we have $d_{\vec{G}}(u, v) = d_{\vec{G}'}(u, v)$. We compute $\vec{G}'$ as follows. Let $\vec{H}_1, \ldots, \vec{H}_k$ be the vortices of $\vec{G}$ and let $\vec{F}_1, \ldots, \vec{F}_k$ be the faces on which they are attached. For each $i \in \{1, \ldots, k\}$ pick distinct $e_i, f_i \in E(\vec{F}_i)$, with $e_i = \{w_i, w_i'\}$, $f_i = \{z_i, z_i'\}$. There exists a path decomposition $B_{i,1}, \ldots, B_{i,\ell_i}$ of $\vec{H}_i$, of width at most $2p$, and such that $B_{i,1} = e_i$, and $B_{i,\ell_i} = f_i$. For each $i \in \{1, \ldots, k-1\}$, we add edges (w_i, z_i), (z_i, w_i), (w_i', z_i'), and (z_i', w_i') to $\vec{G}'$, and we set their length to be equal to the shortest path distance between their endpoints in $\vec{G}$. We also add a handle connecting punctures in the disks bounded by $\vec{F}_i$ and $\vec{F}_{i+1}$ respectively, and we route the four new edges along this handle. Since we add $k-1$ handles in total the Euler genus of the underlying surface increases by at most $k-1$. We let $\vec{H}$ be the single vortex in $\vec{G}'$ with $V(\vec{H}) = \bigcup_{i=1}^{k} V(\vec{H}_i)$ and $E(\vec{H}) = \left(\bigcup_{i=1}^{k} E(\vec{H}_i)\right) \cup \left(\bigcup_{i=1}^{k-1} \{(w_i, z_i), (z_i, w_i), (w_i', z_i'), (z_i', w_i')\}\right)$. It is immediate that

$$B_{1,1}, \ldots, B_{1,\ell_1}, \{f_1, e_2\}, B_{2,1}, \ldots, B_{2,\ell_2}, \{f_2, e_3\}, \ldots, B_{k,1}, \ldots, B_{k,\ell_k}$$

is a path decomposition of $\vec{H}$ of width at most $2p$. Thus $\vec{G}'$ is $(a, g+k-1, 1, 2p)$-nearly embeddable, which concludes the proof. □

5 Combining the Held-Karp LP with the dynamic program

In this Section we show how to combine the dynamic program that finds an optimal closed walk traversing all the vertices in a vortex, with the Held-Karp LP. The following summarizes our exact algorithm for traversing the vortex in a nearly-embeddable graph. The proof of Theorem 5.1 is deferred to Section 14.

Theorem 5.1. *Let $\vec{G}$ be an n-vertex $(a, g, 1, p)$-nearly embeddable graph and let $\vec{H}$ be the single vortex of $\vec{G}$. Then there exists an algorithm which computes a walk $\vec{W}$ visiting all vertices in $V(\vec{H})$ of total length at most $\mathsf{OPT}_{\vec{G}}(V(\vec{H}))$ in time $n^{O((a+p)g^4)}$.*

Definition 5.2 ($\vec{W}$-augmentation)**.** *Let $\vec{G}$ be a directed graph. Let $x : E(\vec{G}) \to \mathbb{R}$ and let $\vec{W} \subseteq \vec{G}$. We define the $\vec{W}$-augmentation of x to be the function $x' : E(\vec{G}) \to \mathbb{R}$ such that for all $e \in E(\vec{G})$ we have*

$$x'(e) = \begin{cases} x(e) + 1 & \text{if } e \in E(\vec{W}) \\ x(e) & \text{otherwise} \end{cases}$$

The following summarizes the main technical result for computing a thin spanning subgraph in a nearly embeddable graph. The proof of Lemma 5.3 is deferred to Section 9.

Lemma 5.3. *Let $\vec{G}$ be a $(a, g, 1, p)$-nearly embeddable digraph, let $\vec{H}$ be its vortex, and let $\vec{W}$ be a walk in $\vec{G}$ visiting all vertices in $V(\vec{H})$. Let G, H, and W be the symmetrizations of $\vec{G}$, $\vec{H}$, and $\vec{W}$ respectively. Let $z : E(G) \to \mathbb{R}_{\geq 0}$ be α-thick for some $\alpha \geq 2$, and $\vec{W}$-dense. Then there exists a polynomial time algorithm which given $\vec{G}$, $\vec{H}$, A, $\vec{W}$, z, and an embedding of $\vec{G} \setminus (A \cup \vec{H})$ into a surface of genus g, outputs a subgraph $S \subseteq G \setminus H$, satisfying the following conditions:*

(1) $W \cup S$ is a spanning subgraph of G and has $O(a + g)$ connected components.

(2) $W \cup S$ is $O(a \cdot g + p^2)$-thin w.r.t. z.

We are now ready to prove the main result of this section.

Proof of Lemma 4.2. Let $\vec{H}$ be the single vortex of $\vec{G}$. We compute an optimal solution $y : E(\vec{G}) \to \mathbb{R}$ for the Held-Karp LP for $\vec{G}$. We find a tour $\vec{W}$ in $\vec{G}$ visiting all vertices in $V(\vec{H})$, with $\mathsf{cost}_{\vec{G}}(\vec{W}) = O(\mathsf{OPT}_{\vec{G}})$ using Theorem 5.1. Let $x : E(\vec{G}) \to \mathbb{R}$ be the $\vec{W}$-augmentation of y. Since for all $e \in E(\vec{G})$ we have $x(e) \geq y(e)$, it follows that x is a feasible solution for the Held-Karp LP. Moreover since $\mathsf{cost}_{\vec{G}}(\vec{W}) = O(\mathsf{OPT}_{\vec{G}})$, we obtain that $\mathsf{cost}_{\vec{G}}(x) = \mathsf{cost}_{\vec{G}}(y) + \mathsf{cost}_G(\vec{W}) = O(\mathsf{OPT}_{\vec{G}})$. Let z be the symmetrization of x.

Note that z is 2-thick and $\vec{W}$-dense. Therefore, by Lemma 5.3 we can find a subgraph $S \subseteq G \setminus H$ such that $T = W \cup S$ is a $O(a \cdot g + p^2)$-thin spanning subgraph of G (w.r.t. z), with at most $O(g + a)$ connected components. Therefore, there exists a constant α such that for every $U \subseteq V(G)$ we have $|T \cap \delta(U)| \leq \alpha \cdot (a \cdot g + p^2) \cdot z(\delta(U))$. We can assume that $\alpha \geq 1$. Now we follow a similar approach to [9].

Let $m = \lfloor n^2/\alpha \rfloor$. We define a sequence of functions $z_0, \ldots, z_m$, and a sequence of spanning forests $T_1, \ldots, T_m$ satisfying the following conditions.

(1) For any $i \in \{0, \ldots, m\}$, z_i is non-negative, 2-thick and $\vec{W}$-dense.

(2) For any $i \in \{1, \ldots, m\}$, T_i has at most $O(a + g)$ connected components.

(3) For every $U \subseteq V(G)$ we have $|T_{i+1} \cap \delta(U)| \leq \alpha \cdot (a \cdot g + p^2) \cdot z_i(\delta(U))$.

We set $z_0 = 3\lfloor zn^2 \rfloor / n^2$. Now suppose for $i \in \{0, \ldots, m-1\}$ we have defined z_i. We define z_{i+1} and T_{i+1} as follows. We apply Lemma 5.3 and we obtain a subgraph T_{i+1} of G with at most $O(a+g)$ connected components such that for every $U \subseteq V(G)$ we have $|T_{i+1} \cap \delta(U)| \leq \alpha \cdot (a \cdot g + p^2) \cdot z_i(\delta(U))$. Also, for every $e \in E(G)$ we set $z_{i+1}(e) = z_i(e)$ if $e \notin T_{i+1}$, and $z_{i+1}(e) = z_i(e) - 1/n^2$ if $e \in T_{i+1}$. Now by using the same argument as in [9], we obtain that z_{i+1} is non-negative and 2-thick. By the construction, we know that z is $\vec{W}$-dense and thus for all $(u, v) \in \vec{W}$ we have $z_0(\{u, v\}) \geq 3$. Note that for all $e \in E(G)$ we have $z_{i+1}(e) \geq z_i(e) - 1/n^2$. Thus for all $e \in E(G)$ and for all $i \in \{0, \ldots, m\}$ we have $z_i(e) \geq 2$. Therefore for all $i \in \{0, \ldots, m\}$ we have that z_i is $\vec{W}$-dense.

Now, similar to [9] we set the desired S to be the subgraph T_i that minimizes $\mathsf{cost}_G(T_i)$, which implies that S is a $(O(a \cdot g + p^2), O(1))$-thin spanning subgraph with at most $O(a + g)$ connected components. □

6 Thin trees in 1-apex graphs

The following is implicit in the work of Oveis Gharan and Saberi [14].

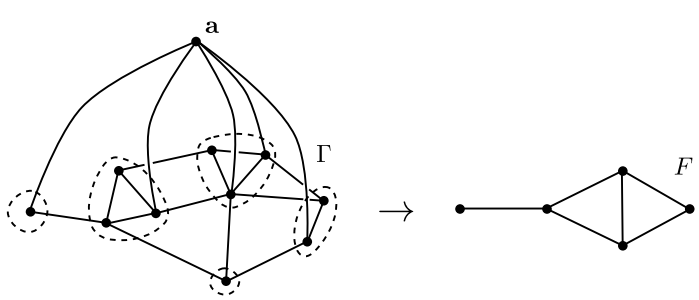
Theorem 6.1. *If G is a planar graph and z is an α-thick weight function on the edges of G for some $\alpha > 0$, then there exists a $10/\alpha$-thin spanning tree in G w.r.t. z.*

For the remainder of this section, let G be an 1-apex graph with planar part Γ and apex $\mathbf{a}$. Let z be a 2-thick weight function on the edges of G. We will find a $O(1)$-thin spanning tree in G (w.r.t. z). We describe an algorithm for finding such a tree in polynomial time. The algorithm proceeds in five phases.

Phase 1. We say that a cut U is *tiny* (w.r.t. z) if $z(\delta(U)) < 0.1$. We start with Γ and we proceed to partition it via tiny cuts. Each time we find a tiny cut U, we partition the remaining graph by deleting all edges crossing U. This process will stop in at most n steps. Let Γ' be the resulting subgraph of Γ where $V(\Gamma') = V(\Gamma)$ and $E(\Gamma') \subset E(\Gamma)$.

Phase 2. By the construction, we know that there is no tiny cuts in each connected component of Γ'. Therefore, following [14], in each connected component C of Γ', we can find a $O(1)$-thin spanning tree T_C (w.r.t. z). More specifically, we will find a 100-thin spanning tree in each of them.

Phase 3. We define a graph F with $V(F)$ being the set of connected components of Γ' and $\{C, C'\} \in E(F)$ iff there exists an edge between some vertex in C and some vertex in C' in Γ. We set the weight of $\{C, C'\}$ to be $z(C, C')$. We call F the *graph of components*.



We define a graph G' obtained from G by contracting every connected component of Γ' into a single vertex. We remark that we may get parallel edges in G'.

Phase 4. In this phase, we construct a tree T' in G'. We say that a vertex in F is *originally heavy*, if it has degree of at most 15 in F. Since F is planar, the minimum degree of F is at most 5. We contract all vertices in $V(G') \setminus \{\mathbf{a}\}$ into the apex sequentially. In each step, we find a vertex in F with degree at most 5, we contract it to the apex in G', and we delete it from F. Since the remaining graph F is always planar, there is always a vertex of degree at most 5 in it, and thus we can continue this process until all vertices of $V(G') \setminus \{\mathbf{a}\}$ are contracted to the apex.

Initially we consider all vertices of F having no parent. In each step when we contract a vertex C with degree at most 5 in F to the apex in G', for each neighbor C' of C in F, we make C the *parent* of C' if C' does not have any parents so far. Note that each vertex in F can be the parent of at most 5 other vertices, and can have at most one parent.

Every time we contract a vertex C to the apex, we add an edge e to T'. If C is originally heavy, we add an arbitrary edge e from C to the apex; we will show in Lemma 6.3 that $z(C, \{a\}) \geq 0.5$, which implies that such an edge always exists in G. Otherwise, we add an arbitrary edge from C to its parent (which is a neighbor vertex, therefore such an edge exists in G). We will show in the next section that each vertex in F is originally heavy or it has a parent (or both). Therefore, T' is a tree on G'.

Phase 5. In this last phase we compute a tree T in G. We set $E(T) = E(T') \cup \bigcup_{C \in V(F)} E(T_C)$. We prove in the next subsection that T is a $O(1)$-thin spanning tree in G.

6.1 Analysis

We next show that T is a $O(1)$-thin spanning tree in G.

Lemma 6.2. *The weight of every edge in F is less than* 0.1.

Proof. Let $\{C, C'\} \in E(F)$. By construction, each component of Γ' is formed by finding a tiny cut in some other component. Suppose C was formed either simultaneously with C' or later than C' by finding a tiny cut in some C''. If $C' \subseteq C''$ then $z(C, C') < z(\delta(C)) < 0.1$. Otherwise, the total weight of edges from C to C' is a part of a tiny cut which means that $z(C, C') < 0.1$. □

Lemma 6.3. *Let C be an originally heavy vertex in F. Then $z(C, \{\boldsymbol{a}\}) \geq 0.5$.*

Proof. For every neighbor C' of C in F, by Lemma 6.2 we have that the weight of $\{C, C'\}$ is less than 0.1. By the assumption on z, we have that $z(\delta(C)) \geq 2$. Now since C has degree of at most 15, we have that $z(C, \{\mathbf{a}\}) \geq 0.5$, as desired. □

Lemma 6.4. *Each vertex $C \in V(F)$ is originally heavy or it has a parent (both cases might happen for some vertices).*

Proof. Let $C \in V(F)$. If it is originally heavy, we are done. Otherwise, it has degree of at least 15. We know that all vertices in F are going to be contracted to $\mathbf{a}$ at some point, and we only contract vertices with degree at most 5 in each iteration. This means that at least 10 other neighbors of C were contracted to the apex before we decided to contract C to the apex. Therefore one of them is the parent of C. □

Lemma 6.5. *T is a spanning tree in G.*

Proof. Suppose G has n vertices and F has m vertices. After the second phase of our algorithm, we obtain a spanning forest on Γ with m components and $n-m-1$ edges. Each time we contract a vertex of F to the apex, we add a single edge to T. Therefore, T has $n-1$ edges. It is now sufficient to show that T is connected.

We will show that for every vertex u in Γ, there is path between u and $\mathbf{a}$ in T. Let u be a vertex of Γ. Suppose u is in some component C_u which is a vertex of F. If C_u is originally heavy, then there is an edge e in T between a vertex $v \in C_u$ and the apex. Since we have a spanning tree in C_u, there is a path between u and v in T. Therefore, there is path between u and the apex a in T.

Otherwise, C_u must have some parent C_{u_1} and there is an edge between these two components. Therefore, there is a path between u and each vertex of these two components. Now, the same argument applies for C_{u_1}. Either it is originally heavy or it has a parent C_{u_2}. If it is originally heavy, we are done. Otherwise, we use the same argument for C_{u_2}. Note that by construction and the definition of a parent, we do not reach the same component in this sequence. Therefore, at some point, we reach a component C_{u_k} which is originally heavy and we are done. □

Now we are ready to show that T is a $O(1)$-thin tree in G (w.r.t. z). We have to show that there exists some constant α such that for every cut U, $|E(T) \cap \delta(U)| \leq \alpha \cdot z(\delta(U))$. Let U be a cut in G. We can assume w.l.o.g. that $\mathbf{a} \notin U$, since otherwise we can consider the cut $V(G) \setminus U$. We partition $E(T) \cap \delta(U)$ into three subsets:

(1) $T_1 = \{\{\mathbf{a}, v\} \in E(T) \cap \delta(U) : v \in V(\Gamma)\}$.

(2) $T_2 = \{\{u, v\} \in E(T) \cap \delta(U) : u$ and v are in the same component of $\Gamma'\}$.

(3) $T_3 = \{\{u, v\} \in E(T) \cap \delta(U) : u$ and v are in different components of $\Gamma'\}$.

Lemma 6.6. *There exists some constant α_1 such that $|T_1| \leq \alpha_1 \cdot z(\delta(U))$.*

Proof. Let $e = \{\mathbf{a}, v\} \in T_1$ where $v \in V(\Gamma)$. Let $C_v \in V(F)$ such that $v \in C_v$. By the construction of T, C_v is originally heavy. If $C_v \subseteq U$, we can charge e to $z(C_v, \{\mathbf{a}\})$, which we know is at least 0.5. Otherwise suppose C_v is not a subset of U. By the assumption we have $\mathbf{a} \notin U$ and thus $v \in U$ which implies that $U \cap C_v \neq \emptyset$. By the construction, we know that there is no tiny cuts in C_v. Therefore, $z(\delta(U) \cap E(G[C_v])) \geq 0.1$. Thus we can charge e to the total weight of the edges in $\delta(U) \cap E(G[C_v])$. Note that for each $C_v \in V(F)$, there is at most one edge in T_1 between $\mathbf{a}$ and C_v. Therefore we have that $|T_1| \leq 10 \cdot z(\delta(U))$. □

Lemma 6.7. *There exists some constant α_2 such that $|T_2| \leq \alpha_2 \cdot z(\delta(U))$.*

Proof. We have

$$\begin{aligned}
|T_2| &= \left| \bigcup_{C \in V(F)} (E(C) \cap T_2) \right| \\
&= \left| \bigcup_{C \in V(F)} (E(C) \cap E(T) \cap \delta(U)) \right| \\
&= \left| \bigcup_{C \in V(F)} (E(T_C) \cap \delta(U)) \right| \\
&\leq \sum_{C \in V(F)} 100 \cdot z(\delta(U) \cap E(C)) \\
&\leq 100 \cdot z(\delta(U)),
\end{aligned}$$

concluding the proof. □

Lemma 6.8. *There exists some constant α_3 such that $|T_3| \leq \alpha_3 \cdot z(\delta(U))$.*

Proof. We partition T_3 into three subsets:

(1) $T_{31} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_v \neq \emptyset\}$.

(2) $T_{32} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_u \neq \emptyset\}$.

(3) $T_{33} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_v = \emptyset, C_u \subseteq U\}$.

First for each $e = \{u, v\} \in T_{31}$ where $v \in C_v$ for some $C_v \in V(F)$, we have that $U_v = U \cap C_v$ is a cut in C_v which is not tiny. By the construction, C_v can be the parent of at most five other vertices in F and it can have at most one parent. Therefore, there are at most six edges in T_3 with a vertex in C_v. So we can charge e and at most five other edges to $z(\delta(U_v))$. Since $z(\delta(U_v)) \geq 0.1$ we get $|T_{31}| \leq 60 \cdot z(\delta(U))$.

Second for each $e = \{u, v\} \in T_{32}$ where $u \in C_u$ for some $C_u \in V(F)$, we have that $U_u = U \cap C_u$ is a cut in C_u which is not tiny. Therefore, the same argument for C_v as in the first case, applies here for C_u and we get $|T_{32}| \leq 60 \cdot z(\delta(U))$.

Finally, for T_{33} we need to find a constant α_{33} such that $|T_{33}| \leq \alpha_{33} \cdot z(\delta(U))$. First, we define a new cut U_1 as follows. For every $C \in V(F)$ with $C \cap U \neq \emptyset$, if $C \cap U \neq C$, we add all the other vertices of C to U and we say that C is *important*. This process leads to a new cut U_1 such that for every $C \in F$, either $C \cap U_1 = \emptyset$ or $C \subseteq U_1$. Let $U_2 = \{C \in V(F) : C \subseteq U_1\}$. Let $X = \{C \in V(F) : C \notin U_2\}$ and $Y = X \cup \{\mathbf{a}\}$.

Let

$$T_{331} = \{\{u, v\} \in T_{33} : u \in U, u \in C \text{ for some } C \in V(F) \text{ with } \deg_{F[U_2]}(C) < 19\}.$$

Let also $T_{332} = T_{33} \setminus T_{331}$.

For each edge $e = \{u, v\} \in T_{331}$ where $u \in U$ and $u \in C_u$ for some $C_u \in V(F)$, we have $\deg_{F[U_2]}(C_u) < 19$. By Lemma 6.2, we know that for any $C, C' \in V(F)$, $z(C, C') \leq 0.1$. Therefore,

we get $z(C_u, Y) \geq 0.2$. Note that there are at most six edges in T_{331} with a vertex in C_u. So we can charge e and at most five other edges to $z(C_u, Y)$. Therefore, $|T_{331}| \leq 30 \cdot z(\delta(U))$.

Let $V_1 = \{C \in U_2 : \deg_{F[U_2]}(C) \geq 19\}$ and $V_2 = \{C \in U_2 : \deg_{F[U_2]}(C) \leq 5\}$. By Euler's formula, we know that the average degree of a planar graph is at most 6. Since $F[U_2]$ is planar, we get $|V_1| \leq |V_2|$. For any $C \in V_2$, if C is important, then $C \cap U$ is a cut for C and we have $z(C \cap U, Y) \geq 0.1$. If C is not important, then we have $z(C, Y) \geq 1.5$. Note that for any $C' \in V_1$, there are at most six edges in T_{332} with a vertex in C'. Therefore, we have $|T_{332}| \leq 60 \cdot z(\delta(U))$.

Now since $T_3 = T_{31} \cup T_{32} \cup T_{331} \cup T_{332}$, we have $|T_3| \leq 210 \cdot z(\delta(U))$ completing the proof. □

Lemma 6.9. *T is a $O(1)$-thin spanning tree in G.*

Proof. By Lemma 6.5 we know that T is a spanning tree. For any $U \subseteq V(G)$, by Lemmas 6.6, 6.7 and 6.8 we get $|T| \leq 320 \cdot z(\delta(U))$. This completes the proof. □

We are now ready to prove the main result of this Section.

Theorem 6.10. *Let G be a 1-apex graph and let $z : E(G) \to \mathbb{R}_{\geq 0}$ be β-thick for some $\beta > 0$. Then there exists a polynomial time algorithm which given G and z outputs a $O(1/\beta)$-thin spanning tree in G (w.r.t. z).*

Proof. For $\beta \geq 2$, by Lemma 6.9 we know that we can find a 320-thin spanning tree in G. For any β with $0 < \beta < 2$, the assertion follows by scaling z by a factor of $2/\beta$. □

7 Thin forests in graphs with many apices

Let $a \geq 1$. In this section, we describe an algorithm for finding thin-forests in an a-apex graph. The high level approach is analogous to the case of 1-apex graphs. We construct a similar graph F of components and contract each vertex of F to some apex.

Let $e_0 = \{u_0, v_0\} \in E(G)$. Let G' be obtained from G by contracting e_0. We define a new weight function z' on the edges of G' as follows. For any $\{u, v_0\} \in E(G)$, we set $z'(\{u, v_0\}) = z(\{u, v_0\}) + z(\{u, u_0\})$. For any other edge e we set $z'(e) = z(e)$. We say that z' is *induced* by z. Similarly, when G' is obtained by contracting a subset X of edges in G, we define z' by inductively contracting the edges in X in some arbitrary order.

For the remainder of this section let G be a a-apex graph with the set of apices $A = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_a\}$. Let Γ be the planar part of G. Let z be a 2-thick weight function on the edges of G. The algorithm proceeds in 5 phases.

Phase 1. We say that a cut U is *tiny* (w.r.t. z) if $z(\delta(U)) < 1/(100 \cdot a)$. Similar to the case of 1-apex graphs, we start with Γ and repeatedly partition it via tiny cuts until there are no more such cuts and we let Γ' be the resulting graph.

Phase 2. For each connected component C of Γ' we find a $O(a)$-thin tree T_C using Theorem 6.1.

Phase 3. We define F and G' exactly the same way as in the case of 1-apex graphs.

Lemma 7.1. *For every $\{C, C'\} \in E(F)$, we have $z(C, C') \leq 1/(100 \cdot a)$.*

Proof. The same argument as in Lemma 6.2 applies here. The only difference here is that a cut U is tiny if $z(\delta(U)) < 1/(100 \cdot a)$. □

Phase 4. We construct a forest T' on G'. Let $m = |V(F)|$. We define a sequence of planar graphs $F_0, F_1, \cdots, F_m$, a sequence of graphs $G'_0, G'_1, \cdots, G'_m$ and a sequence of weight functions $z_0, z_1, \cdots, z_m$ as follows. Let $F_0 = F$, $G'_0 = G'$ and $z_0 = z$. We also define a sequence of forests $P_0, \ldots, P_m$ where each P_j contains a tree rooted at each $\mathbf{a}_i \in A$. We set P_0 to be the forest that contains a tree for each $\mathbf{a}_i \in A$ and with no other vertices.

Let $C \in V(F_j)$ for some j. For any $\mathbf{a}_i \in A$, we say that C is *$\mathbf{a}_i$-heavy* in F_j if $z_j(C, \{\mathbf{a}_i\}) \geq 1/a$. Let $C' \in V(F)$. For any $\mathbf{a}_i \in A$, we say that C' is *originally $\mathbf{a}_i$-heavy* if C' is $\mathbf{a}_i$-heavy in F.

We maintain the following inductive invariant:

(I1) For any $j \in \{0, \ldots, m-1\}$, let $v \in V(F_j)$ be a vertex of minimum degree. Then either there exists some $\mathbf{a}_i \in A$ such that v is originally $\mathbf{a}_i$-heavy or $v \in V(P_j)$.

Consider some $i \in \{0, \ldots, m-1\}$. Let $v_i \in V(F_i)$ be a vertex with minimum degree. If v_i is originally $\mathbf{a}_j$-heavy for some $\mathbf{a}_j \in A$, then we contract v_i to $\mathbf{a}_j$. Otherwise, by the inductive invariant (I1), we have that $v_i \in V(P_i)$. Thus there exists a tree in P_i containing v_i that is rooted in some $\mathbf{a}_j \in A$; we contract v_i to $\mathbf{a}_j$. In either case, by contracting v_i to $\mathbf{a}_j$ we obtained G'_{i+1} from G'_i. We also delete v_i from F_i to obtain F_{i+1}. We let z_{i+1} be the weight function on G'_{i+1} induced by z_i.

Finally, we need to define P_{i+1}. If v_i was originally $\mathbf{a}_j$-heavy then we add v_i to P_i via an edge $\{v_i, \mathbf{a}_j\}$. For each $u \in V(F_i)$ that is a neighbor of v_i, and is not in $V(P_i)$, we add u to P_{i+1} by adding the edge $\{v_i, u\}$ iff the following conditions hold:

(i) For all $\mathbf{a}_r \in A$, we have that u is not $\mathbf{a}_r$-heavy in F_i.

(ii) u is $\mathbf{a}_j$-heavy in F_{i+1}.

In this case we say that v is the *parent* of u. This completes the description of the process that contracts each vertex in $V(G')$ into some apex.

Lemma 7.2. *Let $j \in \{0, 1, \ldots, m-1\}$. Let $C \in V(F_j)$ be a vertex with minimum degree. Then there exists $\mathbf{a}_i \in A$ such that C is $\mathbf{a}_i$-heavy in F_j.*

Proof. Since C has at most 5 neighbors in F_j, by Lemma 7.1 we have $z_j(C, A) \geq 2 - 5/(100 \cdot a) \geq 1$. Therefore by averaging, there exists an apex $\mathbf{a}_i$ such that $z_j(C, \{\mathbf{a}_i\}) \geq 1/a$. □

Lemma 7.3. *For any $j \in \{0, \ldots, m\}$ and for any $v \in V(\Gamma) \cap V(P_j)$, we have $\deg_{P_j}(v) \leq 6$.*

Proof. By the construction, in each step we pick a vertex of minimum degree and contract it into some apex. Since F_i is planar, its minimum degree is at most 5. This means that for any $v \in V(\Gamma) \cap V(P_j)$, v can be the parent of at most five other vertices and can have at most one parent. This completes the proof. □

Lemma 7.4. *The inductive invariant (I1) is maintained.*

Proof. For any $j \in \{0, \ldots, m-1\}$, let $v \in V(F_j)$ be a vertex of minimum degree. If there exists some $\mathbf{a}_i \in A$ such that v is originally $\mathbf{a}_i$-heavy, then we are done. Suppose for all $\mathbf{a}_i \in A$, v is not originally $\mathbf{a}_i$-heavy. By Lemma 7.2 we know that there exists some $\mathbf{a}_l \in A$ such that v is $\mathbf{a}_l$-heavy in F_j. Let $j^* \in \{1, \ldots, j\}$ be minimum such that v is not $\mathbf{a}_t$-heavy in F_{j^*-1} for all $\mathbf{a}_t \in A$, and v is $\mathbf{a}_{t'}$-heavy in F_{j^*} for some $\mathbf{a}_{t'} \in A$. Let $u \in V(F_{j^*-1})$ be vertex that is contracted to some apex in step j^*. It follows by construction that u is the parent of v in P_{j^*}. Since $j \geq j^*$ it follows that $v \in V(P_j)$, concluding the proof. □

Now we are ready to describe how to construct T' in G'. For any $l \in \{0, \ldots, m-1\}$, let $C \in V(F_l)$ be a vertex of minimum degree. If C is originally $\mathbf{a}_i$-heavy for some $\mathbf{a}_i \in A$ and we contract C to $\mathbf{a}_i$, we pick an arbitrary edge e between C and $\mathbf{a}_i$ and we add it to T'. Otherwise, by Lemma 7.4 we have $C \in V(P_l)$. This means that C has a parent C'. In this case, we pick an arbitrary edge e between C and C' and we add it to T'.

Phase 5. We construct a forest T in G the same way as in the 1-apex case. We set $E(T) = E(T') \cup \bigcup_{C \in V(F)} E(T_C)$.

This completes the description of the algorithm.

7.1 Analysis

By the construction, T has a connected components. We will show that T is a $O(a)$-thin spanning forest. Let U be a cut in G. Similar to the 1-apex case, we partition $E(T) \cap \delta(U)$ into three subsets:

(1) $T_1 = \{\{\mathbf{a}_i, v\} \in E(T) \cap \delta(U) : \mathbf{a}_i \in A, v \in V(\Gamma)\}$.

(2) $T_2 = \{\{u, v\} \in E(T) \cap \delta(U) : u$ and v are in the same component of $\Gamma'\}$.

(3) $T_3 = \{\{u, v\} \in E(T) \cap \delta(U) : u$ and v are in different components of $\Gamma'\}$.

Lemma 7.5. *There exists a constant α_1 such that $|T_1| \leq \alpha_1 \cdot a \cdot z(\delta(U))$.*

Proof. A similar argument as in the case of 1-apex graph applies here with two differences. First, a cut U is tiny if $z(\delta(U)) < 1/(100 \cdot a)$. Second, for any $\mathbf{a}_i \in A$ and $C \in V(F)$ where C is originally $\mathbf{a}_i$-heavy, we have that $z(C, \{\mathbf{a}_i\}) \geq 1/a$. Therefore, we get $|T_1| \leq 100 \cdot a \cdot z(\delta(U))$. □

Lemma 7.6. *There exists a constant α_2 such that $|T_2| \leq \alpha_2 \cdot a \cdot z(\delta(U))$.*

Proof. Again, a similar argument as in the case of 1-apex graphs applies here. The only difference here is the definition of tiny cut. Therefore, we get $|T_2| \leq 100 \cdot a \cdot z(\delta(U))$. □

Lemma 7.7. *There exists a constant α_3 such that $|T_3| \leq \alpha_3 \cdot a \cdot z(\delta(U))$.*

Proof. Similar to the 1-apex case, we partition T_3 into three subsets:

(1) $T_{31} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_v \neq \emptyset\}$.

(2) $T_{32} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_u \neq \emptyset\}$.

(3) $T_{33} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_v = \emptyset, C_u \subseteq U\}$.

The arguments for T_{31} and T_{32} are the same as in 1-apex graphs. The only difference here is that a cut U is tiny if $z(\delta(U)) < 1/(100 \cdot a)$. Therefore, we have $|T_{31}| \leq 600 \cdot a \cdot z(\delta(U))$ and $|T_{32}| \leq 600 \cdot a \cdot z(\delta(U))$.

Now for T_{33} we want to find a constant α_{33} such that $|T_{33}| \leq \alpha_{33} \cdot a \cdot z(\delta(U))$. We define two new cuts U_1 and U_2 as follows. For every $C \in V(F)$ with $C \cap U \neq \emptyset$, if $C \cap U \neq C$, we add all other vertices of C to U (delete all other vertices of C from U) to obtain U_1 (U_2) and we say that C is *U-important*. Let $U_1' = \{C \in V(F) : C \subseteq U_1\}$ and $U_2' = \{C \in V(F) : C \subseteq U_2\}$.

For any $e = \{u, v\} \in T_{33}$ where $u \in U$, $v \notin U$, $u \in C_u$ and $v \in C_v$ for some $C_u, C_v \in V(F)$, by the construction of T_3, we have that both C_u and C_v have been contracted to the same apex $\mathbf{a}_i$ for some $\mathbf{a}_i \in A$. Let $j \in \{0, \ldots, m\}$ be the step during which C_u is contracted to $\mathbf{a}_i$. Let $B = \{C \in V(F) : C \text{ is contracted to } \mathbf{a}_i\}$. Let D be the connected component of $F[B]$ containing C_u. Let $D^{\mathsf{in}}_{U'_1} = D[U'_1]$, $D^{\mathsf{out}}_{U'_1} = D[V(D) \setminus U'_1]$, $D^{\mathsf{in}}_{U'_2} = D[U'_2]$, $D^{\mathsf{out}}_{U'_2} = D[V(D) \setminus U'_2]$.

We consider the following two cases:

Case 1: $\mathbf{a}_i \notin U$. We know that $C_u \in U$. By the construction, we have that $z_j(C_u, \{\mathbf{a}_i\}) \geq 1/a$. If $z_0(C_u, \{\mathbf{a}_i\}) \geq 1/(100 \cdot a)$, we can charge e to $z_0(C_u, \{\mathbf{a}_i\})$ and we know that there are at most six edges in T_{33} with a vertex in C_u.

Otherwise, we have that $z_0(C_u, (V(D) \setminus C_u)) \geq 99/(100 \cdot a)$. If $z_0(C_u, (V(D^{\mathsf{out}}_{U'_1}) \setminus C_u)) \geq 1/(100 \cdot a)$, then by the construction of $D^{\mathsf{out}}_{U'_1}$ we get $z_0(C_u, (G \setminus U)) \geq 1/(100 \cdot a)$. Therefore we can charge e to $z_0(C_u, (G \setminus U))$.

Otherwise, we have that $z_0(C_u, (V(D^{\mathsf{in}}_{U'_1}) \setminus C_u)) \geq 98/(100 \cdot a)$. This implies that $\deg_{D^{\mathsf{in}}_{U'_1}}(C_u) \geq 98$. Now note that $D^{\mathsf{in}}_{U'_1}$ is a planar graph and its average degree is at most 6. Now a similar argument as in the 1-apex case applies here. Let $V_1 = \{C \in D^{\mathsf{in}}_{U'_1} : \deg_{D^{\mathsf{in}}_{U'_1}}(C) \geq 98\}$. Let $V_2 = \{C \in D^{\mathsf{in}}_{U'_1} : \deg_{D^{in}_{U'_1}}(C) \leq 5\}$. By planarity of $D^{\mathsf{in}}_{U'_1}$, we have that $|V_1| \leq |V_2|$. For any $C \in V_2$, if C is U-important, then $C \cap U$ is a cut for C which is not tiny. Therefore we have $z(C \cap U, G \setminus U) \geq 1/(100 \cdot a)$. If C is not U-important, we have $z(C, G \setminus U) \geq 95/(100 \cdot a)$. Now note that for any $C' \in V_1$ there are at most six edges in T_{33} with a vertex in C'. Therefore we have $|T_{33}| \leq 1000 \cdot a \cdot z(\delta(U))$.

Case 2: $\mathbf{a}_i \in U$. Let $X_1 = \{C \in D^{out}_{U'_2} : \deg_{D^{out}_{U'_2}}(C) \geq 98\}$. Let $X_2 = \{C \in D^{out}_{U'_2} : \deg_{D^{out}_{U'_2}}(C) \leq 5\}$. We know that $C_v \notin U$. We follow a similar approach as in the first case by considering U'_2, $D^{\mathsf{in}}_{U'_2}$ and $D^{\mathsf{out}}_{U'_2}$. The same argument applies here by replacing U'_1, $D^{\mathsf{in}}_{U'_1}$, $D^{\mathsf{out}}_{U'_1}$, X_1 and X_2 with U'_2, $D^{\mathsf{out}}_{U'_2}$, $D^{\mathsf{in}}_{U'_2}$, V_1 and V_2 respectively. Therefore, we get $|T_{33}| \leq 1000 \cdot a \cdot z(\delta(U))$.

Now from what we have discussed, we have $|T_{31}| \leq 600 \cdot a \cdot z(\delta(U))$, $|T_{32}| \leq 600 \cdot a \cdot z(\delta(U))$ and $|T_{33}| \leq 1000 \cdot a \cdot z(\delta(U))$. Therefore, we get $|T_3| \leq 2200 \cdot a \cdot z(\delta(U))$ completing the proof. □

Lemma 7.8. *T is a $O(a)$-thin spanning forest in G with at most a connected components.*

Proof. By the construction, T is a spanning forest and has at most a connected components. Let $\alpha = \alpha_1 + \alpha_2 + \alpha_3$, where α_1, α_2 and α_3 are obtained by Lemmas 7.5, 7.6, 7.7. Therefore for any $U \subset V(G)$, we have $|T| \leq \alpha \cdot a \cdot z(\delta(U)) = 2400 \cdot a \cdot z(\delta(U))$ completing the proof. □

We are now ready to prove the main result of this Section.

Theorem 7.9. *Let $a \geq 1$ and let G be a a-apex graph with set of apices $A = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_a\}$. Let $z : E(G) \to \mathbb{R}_{\geq 0}$ be β-thick for some $\beta > 0$. Then there exists a polynomial time algorithm which given G, A and z outputs a $O(a/\beta)$-thin spanning forest in G (w.r.t. z) with at most a connected components.*

Proof. By Lemma 7.8, for $\beta \geq 2$, we know that we can find a $(2400a)$-thin spanning forest in G (w.r.t. z) with at most a connected components. For any other $0 < \beta < 2$, the claim follows by scaling z by a factor of $2/\beta$. □

8 Thin forests in higher genus graphs with many apices

The following theorem is implicit in the work of Erickson and Sidiropoulos [9].

Theorem 8.1 (Erickson and Sidiropoulos [9])**.** *Let G be a graph with $\mathsf{eg}(G) = g$, and let z be a β-thick weight function on the edges of G for some $\beta \geq 0$. Then there exists a polynomial time algorithm which given G, z, and an embedding of G into a surface of Euler genus g, outputs a $O(1/\beta)$-thin spanning forest in G (w.r.t. z), with at most g connected components.*

In this section, we study the problem in higher genus graphs. First, the following two Lemmas can be obtained by Euler's formula.

Lemma 8.2. *Let G be a graph of genus $g \geq 1$ with $|V(G)| \geq 10g$. Then there exists $v_0 \in V(G)$ with $\deg_G(v_0) \leq 7$.*

Lemma 8.3. *Let G be an n vertex graph of genus $g \geq 1$. Then the average degree of vertices of G is at most $6 + 12(g-1)/n$.*

For the remainder of this section, let G be an a-apex graph with the set of apices $A = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_a\}$ on a surface of genus g. Let $\Gamma = G \setminus A$, where Γ is a graph of genus g. Let z be a 2-thick weight function on the edges of G. We will find a $O(a \cdot g)$-thin (w.r.t. z) spanning forest in G with at most $O(a+g)$ connected components. The high level approach is similar to the case where Γ was planar. The algorithm proceeds in 5 phases.

Phase 1. We say that a cut U is *tiny* (w.r.t. z) if $z(\delta(U)) < 1/(1000 \cdot a \cdot g)$. We construct Γ' the same way as in Section 7. The only difference here is the definition of tiny cut.

Phase 2. Similar to the planar case, for each connected component C of Γ' we find a $O(a \cdot g)$-thin forest T_C, with at most g connected components, using Theorem 8.1

Phase 3. We define F and G' the exact same way as in Section 7.

Lemma 8.4. *For every $\{C, C'\} \in E(F)$, we have $z(C, C') \leq 1/(1000 \cdot a \cdot g)$.*

Proof. The same argument as in Lemma 6.2 applies here. The only difference here is the definition of tiny cut. □

Phase 4. We construct a spanning forest T' on G', with at most $a+10g$ connected components. We follow a similar approach as in the planar case. Let $m = |V(F)| - 10g$. If $m \leq 0$, we set $E(T') = \emptyset$ and we skip to the next phase. Otherwise, we define two sequences of graphs $F_0, F_1, \ldots, F_m$, $G'_0, G'_1, \ldots, G'_m$, a sequence of weight functions $z_0, z_1, \ldots, z_m$, a sequence of forests $P_0, P_1, \ldots, P_m$ satisfying the inductive invariant (I1) the exact same way as in Section 7. For any $j \in \{0, 1, \ldots, m-1\}$, $C \in V(F_j)$ and $\mathbf{a}_i \in A$, we also define the notion of $\boldsymbol{a}_i$*-heavy* and *originally* $\boldsymbol{a}_i$*-heavy* the same way as in Section 7. The only differences here is that $m = |V(F)| - 10g$ instead of $|V(F)|$.

Lemma 8.5. *Let $j \in \{0, 1, \ldots, m\}$. Let $C \in V(F_j)$ be a vertex of minimum degree. Then there exists $\boldsymbol{a}_i \in A$ such that C is $\boldsymbol{a}_i$-heavy in F_j.*

Proof. By the construction, $|V(F_j)| \geq 10g$. Therefore by Lemma 8.2, we have $\deg_{F_j}(C) \leq 7$. Therefore by Lemma 8.4, we get $z_j(C, A) \geq 2 - 7/(1000 \cdot a \cdot g) \geq 1$. This implies that there exists $\mathbf{a}_i \in A$ such that $z_j(C, \{\mathbf{a}_i\}) \geq 1/a$. □

Lemma 8.6. *For any $j \in \{0, \ldots, m\}$ and for any $v \in \Gamma \cap P_j$, we have $\deg_{P_j}(v) \leq 8$.*

Proof. A similar argument as in the planar case applies here. The only difference here is that the minimum degree is at most 7. Therefore, every vertex can be the parent of at most 7 other vertices and can have at most one parent. □

Lemma 8.7. *The inductive invariant (I1) is maintained.*

Proof. The exact same argument as in Section 7 applies here. □

Now we construct a forest T' on G' the same way as in Section 7.

Lemma 8.8. *T' has at most $a + 10g$ connected components.*

Proof. If $|V(F)| \leq 10g$, then we are done. Otherwise, we have $|F_m| = 10g$. Now since $|A| = a$, by the construction, we have that the number of connected components of T' is at most $a + 10g$. □

Phase 5. We construct a forest T in G the exact same way as in Section 7, by setting $E(T) = E(T') \cup \bigcup_{C \in V(F)} E(T_C)$.

This completes the description of the algorithm.

8.1 Analysis

Lemma 8.9. *T is a spanning forest in G, with at most $O(a + g)$ connected components.*

Proof. For any $C \in V(F)$, let g_C be the genus of $\Gamma[C]$. By Theorem 8.1 we know that the number of connected components in T_C is at most g_C. Therefore, by Lemma 8.8 we have that the number of connected components in T is at most $a + 10g + \sum_{C \in V(F)} g_C \leq a + 11g$. □

For the thinness of T, we follow a similar approach as in the planar case. There are two main differences here: First for any $j \in \{0, 1, \ldots, m\}$, by Lemma 8.3 we have that the average degree of F_j is at most $20g$. Second a cut U is tiny if $z(\delta(U)) < 1/(1000 \cdot a \cdot g)$.

Let U be a cut in G. Similar to the planar case, we partition $E(T) \cap \delta(U)$ into three subsets:

(1) $T_1 = \{\{\mathbf{a}_i, v\} \in E(T) \cap \delta(U) : \mathbf{a}_i \in A, v \in V(\Gamma)\}$.

(2) $T_2 = \{\{u, v\} \in E(T) \cap \delta(U) : u$ and v are in the same component of $\Gamma'\}$.

(3) $T_3 = \{\{u, v\} \in E(T) \cap \delta(U) : u$ and v are in different components of $\Gamma'\}$.

Also similar to the planar case, we partition T_3 into three subsets:

(1) $T_{31} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_v \neq \emptyset\}$.

(2) $T_{32} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_u \neq \emptyset\}$.

(3) $T_{33} = \{\{u, v\} \in T_3 : u \in U, v \notin U, \exists C_u, C_v \in V(F)$ such that $u \in C_u, v \in C_v, U \cap C_v = \emptyset, C_u \subseteq U\}$.

Lemma 8.10. *For any index $i \in \{1, 2, 31, 32\}$, there exists a constant α_i such that $|T_i| \leq \alpha_i \cdot a \cdot g \cdot z(\delta(U))$.*

Proof. A similar argument as in Lemmas 7.5, 7.6 and 7.7 applies here. There are two differences here. First the definition of a tiny cut is different. Second, for each $C \in V(F)$, C can be the parent of at most seven vertices and can have at most one parent. Therefore we get $|T_1| \leq 1000 \cdot a \cdot g \cdot z(\delta(U))$, $|T_2| \leq 1000 \cdot a \cdot g \cdot z(\delta(U))$, $|T_{31}| \leq 8000 \cdot a \cdot g \cdot z(\delta(U))$ and $|T_{32}| \leq 8000 \cdot a \cdot g \cdot z(\delta(U))$. □

Lemma 8.11. *There exists a constant α_{33} such that $|T_{33}| \leq \alpha_{33} \cdot a \cdot g \cdot z(\delta(U))$.*

Proof. Let $e = \{u, v\} \in T_{33}$. We follow a similar approach as in the planar case. We define U_1, U_2, U_1', U_2', B, $D^{\mathsf{in}}_{U_1'}$, $D^{\mathsf{out}}_{U_1'}$, $D^{\mathsf{in}}_{U_2'}$ and $D^{\mathsf{out}}_{U_2'}$ the exact same way as in Lemma 7.7. Let $V_1 = \{C \in D^{\mathsf{in}}_{U_1'} : \deg_{D^{\mathsf{in}}_{U_1'}}(C) \geq 98g\}$, $V_2 = \{C \in D^{\mathsf{in}}_{U_1'} : \deg_{D^{\mathsf{in}}_{U_1'}}(C) \leq 20g\}$, $X_1 = \{C \in D^{out}_{U_2'} : \deg_{D^{out}_{U_2'}}(C) \geq 98g\}$, and $X_2 = \{C \in D^{out}_{U_2'} : \deg_{D^{out}_{U_2'}}(C) \leq 5g\}$. By Lemma 8.3 we have that $|V_1| \leq |V_2|$ and $|X_1| \leq |X_2|$. With these definitions, the rest of the proof is the same as in Lemma 7.7, and thus we get $|T_{33}| \leq 10000 \cdot a \cdot g \cdot z(\delta(U))$. □

Lemma 8.12. *T is a $O(a \cdot g)$-thin spanning forest in G, with at most $O(a+g)$ connected components.*

Proof. By combining Lemmas 8.9, 8.10 and 8.11 and we get $|T| \leq 24000 \cdot a \cdot g \cdot z(\delta(U))$, which proves the assertion. □

We are now ready to prove the main result of this Section.

Theorem 8.13. *Let $a, g \geq 1$. Let G be a graph and $A \subseteq V(G)$, with $|A| = a$, such that $H = G \setminus A$ is a graph of genus g. Let $z : E(G) \to \mathbb{R}_{\geq 0}$ be β-thick for some $\beta > 0$. Then there exists a polynomial time algorithm which given G, A, an embedding of H on a surface of genus g, and z outputs a $O((a \cdot g)/\beta)$-thin spanning forest in G (w.r.t. z) with at most $O(a + g)$ connected components.*

Proof. For $\beta \geq 2$, by Lemma 8.12, we can find a $(24000 \cdot a \cdot g)$-thin spanning forest with at most $a + 11g$ connected components. For $0 < \beta < 2$, the claim follows by scaling z by a factor of $2/\beta$. □

9 Thin subgraphs in nearly-embeddable graphs

9.1 $(0, g, 1, p)$-nearly embeddable graphs

For the remainder of this subsection, let $\vec{G}$ be a $(0, g, 1, p)$-nearly embeddable digraph and let G be its symmetrization. Let $\vec{H}$ be the single vortex of $\vec{G}$ of width p, attached to some face $\vec{F}$ of $\vec{G}$. Let H and F be the symmetrizations of $\vec{H}$ and $\vec{F}$ respectively. Let $\{B_v\}_{v \in V(F)}$ be a path-decomposition of H of width p. Let $\vec{W}$ be a closed walk in $\vec{G}$ visiting all vertices in $V(\vec{H})$ and let W be its symmetrization. Let $z : E(G) \to \mathbb{R}_{\geq 0}$ be α-thick, for some $\alpha \geq 2$, and $\vec{W}$-dense. Let G' be the graph obtained by contracting F to a single vertex v^* in $G \setminus H$.

Following [9] we introduce the following notation. For any $u, v \in V(G)$, a *ribbon* R between u and v is the set of all parallel edges $e = \{u, v\}$ such that for every $e, e' \in R$, there exists a homeomorphism between e and e' on the surface. Let R' be a set of parallel edges in G. We say that an edge $e \in R'$ is central if the total weight of edges on each side of e in R' (containing e), is at least $z(R')/2$.

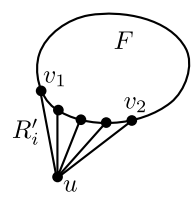
We will find a $O(1)$-thin spanning forest S in G' (w.r.t. z), with at most g connected components, such that S is $O(1)$-thin in G (w.r.t. z). We follow a similar approach to [9] to construct S. We apply some modifications that assure S is $O(1)$-thin in G (w.r.t. z).

9.1.1 The modified ribbon-contraction argument

If $|V(G')| \leq g$, then we set $E(S) = \emptyset$ and we are done. Otherwise, let $l = |V(G')| - g$. We define two sequences of graphs $G_0, \ldots, G_l$ and $G'_0, \ldots, G'_l$, with $G_0 = G$ and $G'_0 = G'$. For each $j \in \{0, \ldots, l\}$, G_j is obtained by uncontracting $v^* \in V(G'_j)$. Let $i \geq 0$ and suppose we have defined G'_i. Let R_i be the heaviest ribbon in G'_i (w.r.t. z). Let $R'_i \subseteq E(G_i)$ be the corresponding set of edges in G_i. We contract all the edges in R_i and we let G'_{i+1} be the graph obtained after contracting R_i. We also perform the contraction in a way such that for all $i \in \{0, \ldots, l-1\}$ we have $v^* \in G'_i$.

Let $i \in \{0, \ldots, l-1\}$. If $R_i = \{u, v\}$ where $u, v \neq v^*$, similar to [9], we let e_i be a central edge in R_i and we add e_i to S. Otherwise, suppose that $R_i = \{u, v^*\}$ for some $u \in V(G'_i)$. If there exists an edge $e \in R_i$ with $e \in W$ or $z(e) \geq 0.1$, we let $e_i = e$ and we add it to S. Otherwise, we can assume that there is no edge $e \in R$ with $e \in W$ or $z(e) \geq 0.1$.

Let Q_i be the set of vertices $v \in V(F)$ with an endpoint in R'_i. By the construction, Q_i is a subpath of F. Let $v_1, v_2 \in V(F)$ be the endpoints of Q_i. Let W'_i be the restriction of W on $\bigcup_{v \in Q_i} B_v$. Let W''_i be the subgraph of W'_i obtained by deleting all edges e with both endpoints in B_{v_1} or B_{v_2}.



For any subgraph C of W, we define the *i-load* of C as follows. The i-load of C is the total weight of all edges in R'_i with an endpoint in C. Let C_i be the connected component of W''_i with the maximum i-load. Let $Y_i = \{e \in R''_i :$ e has an endpoint in $C_i\}$. We let e_i be a central edge in Y_i and we add e_i to S.

We set $T = S \cup W$. We will show that T is a $O(p^2)$-thin spanning subgraph of G (w.r.t. z), with at most g connected components.

Lemma 9.1. *T is a spanning subgraph of G with at most g connected components.*

Proof. By the construction, S has at most g connected components in G'. Now note that W is a closed walk visiting H in G. Therefore, all vertices of F are in the same connected component in T. This means that T has at most g connected components. □

Lemma 9.2. *For any $i \in \{0, \ldots, l-1\}$, W'_i has at most $2p$ connected components.*

Proof. This follows immediately from the fact that $\{B_v\}_{v \in V(F)}$ is a path-decomposition of width p and there is no edge in $R_i \cap W$. □

Lemma 9.3. *For any $i \in \{0, \ldots, l-1\}$, W''_i has at most $p(p+1)$ connected components.*

Proof. By Lemma 9.2 we know that W'_i has at most $2p$ connected components. W''_i is obtained by deleting at most $p(p-1)$ edges of W'_i. Therefore, W''_i has at most $p(p+1) = 2p + p(p-1)$ connected components. □

Lemma 9.4. *For any $i \in \{0, \ldots, l-1\}$, the i-load of W'_i is at least* 0.4.

Proof. The i-load of W_i' is $z(R_i')$. Following [9] we know that $z(R_i) \geq 2/5$ and thus $z(R_i') \geq 2/5$. □

Lemma 9.5. *For any $i \in \{0, \ldots, l-1\}$, the i-load of W_i'' is at least* 0.2.

Proof. By Lemma 9.4 the i-load of W' is at least 0.4. By the construction, we have $z(\{u, v_1\}) \leq 0.1$ and $z(\{u, v_2\}) \leq 0.1$. By deleting edges with both endpoints in B_{v_1} or B_{v_2}, we decrease the i-load by at most 0.2. Therefore, the i-load of W_i'' is at least 0.2. □

Lemma 9.6. *For any $i \in \{0, \ldots, l-1\}$, the i-load of C_i is at least $1/5p(p+1)$.*

Proof. By Lemma 9.5 we know that the i-load of W_i'' is at least 0.2. By Lemma 9.3 there are at most $p(p+1)$ connected components in W_i''. Therefore the i-load of C_i is at least $1/5p(p+1)$. □

Lemma 9.7. *There exists a constant β such that for any $U \subseteq V(G)$, we have $|S \cap \delta(U)| \leq \beta \cdot p^2 \cdot z(\delta(U))$.*

Proof. First we partition $S \cap \delta(U)$ into two subsets:

(1) $S_1 = \{\{u, v\} \in S \cap \delta(U) : u, v \notin V(F)\}$.

(2) $S_2 = \{\{u, v\} \in S \cap \delta(U) : v \in V(F)\}$.

By the construction, following [9] we have $|S_1| \leq 20 \cdot z(\delta(U))$. Let $e = \{u, v\} \in S_2$. Let $i \in \{0, \ldots, l-1\}$ be the step that we add e to S. If $e \in W$, we can charge it to $z(e) \geq 1/2$ and we are done. Suppose $e \notin W$. If there exists an edge $e' \in E(C_i) \cap \delta(U)$, we know that by the construction, e' does not have both endpoints in B_{v_1} or B_{v_2}. Therefore, for all $j \neq i$ we have $e' \notin E(C_j)$. Thus we can charge e to $z(e') \geq 1/2$ and we are done. Otherwise, suppose there is no edge in $E(C_i) \cap \delta(U)$. In this case, by the construction, for all $e'' \in R_i$ with an endpoint in C_i, we have $e'' \in \delta(U)$. Now we know that e is the central edge in Y_i. By Lemma 9.6 we know that the i-load of C_i is at least $1/5p(p+1)$. Therefore, we can charge e to the i-load of C_i and we get $|S_2| \leq 10 \cdot p^2 z(\delta(U))$. Therefore, we have $|S \cap \delta(U)| \leq 20 \cdot p^2 \cdot z(\delta(U))$. □

Lemma 9.8. *Let $\vec{G}$ be a $(0, g, 1, p)$-nearly embeddable digraph, let $\vec{H}$ be its vortex, and let $\vec{W}$ be a walk in $\vec{G}$ visiting all vertices in $V(\vec{H})$. Let G, H, and W be the symmetrizations of $\vec{G}$, $\vec{H}$, and $\vec{W}$ respectively. Let $z : E(G) \to \mathbb{R}_{\geq 0}$ be α-thick for some $\alpha \geq 2$, and $\vec{W}$-dense. Then there exists a polynomial time algorithm which given $\vec{G}$, $\vec{H}$, $\vec{W}$, z, and an embedding of $\vec{G} \setminus \vec{H}$ into a surface of genus g, outputs a subgraph $S \subseteq G \setminus H$, satisfying the following conditions:*

(1) $W \cup S$ is a spanning subgraph of G and has $O(g)$ connected components.

(2) $W \cup S$ is $O(p^2)$-thin w.r.t. z.

Proof. The assertion follows immediately by Lemmas 9.1 and 9.7. □

9.2 $(a, g, 1, p)$-nearly embeddable graphs

For the remainder of this subsection, let $a, g, k, p \geq 0$ and $\vec{G}$ be an n-vertex $(a, g, 1, p)$-nearly embeddable digraph and let G be its symmetrization. Let $\vec{H}$ be the single vortex of width p, attached to a face $\vec{F}$ of $\vec{G}$. Let H and F be the symmetrization of $\vec{H}$ and $\vec{F}$ respectively. Let $A \subseteq V(G)$ with $|A| = a$ be the set of apices of G, where $\Gamma = G \setminus (A \cup H)$ is a graph of genus g. Let $\vec{W}$ be a walk in $\vec{G}$ visiting all vertices in $V(\vec{H})$ and let W be its symmetrization. Let $z : E(G) \to \mathbb{R}_{\geq 0}$ be α-thick, for some $\alpha \geq 2$, and $\vec{W}$-dense. Let G' be the graph obtained by contracting F to a single vertex v^* in $G \setminus H$.

We follow a similar algorithm as in Section 8 to find a $O(a \cdot g + p^2)$-thin spanning forest S in G', with at most $O(a + g)$ connected components. We modify the algorithm such that S is a $O(a \cdot g + p^2)$-thin subgraph of G.

We first start with Γ and construct Γ' the same way as in Section 8. For each connected component C of Γ', we want to find a $O(p^2)$-thin spanning forest T_C, with at most g connected components. Let C_{v^*} be the connected component of Γ' with $v^* \in C_{v^*}$. C_{v^*} is a graph of genus at most g. For this component, we apply the modified ribbon-contraction argument on Subsection 9.1 to find $T_{C_{v^*}}$. Therefore, $T_{C_{v^*}}$ is a $O(p^2)$-thin spanning forest in C_{v^*} with at most g connected components. The rest of the algorithm is the same as in Section 8 and we find a $O(a \cdot g + p^2)$-thin spanning forest S in G', with at most $O(a + g)$ connected components. Let $T = S \cup W$.

Lemma 9.9. *T is a spanning subgraph of G with at most $O(a + g)$ connected components.*

Proof. The same proof as in Lemma 9.1 applies here. The only difference here is that S has at most $O(a + g)$ connected components in G'. Therefore, T has at most $O(a + g)$ connected components in G. □

Lemma 9.10. *S is a $O(a \cdot g + p^2)$-thin subgraph of G (w.r.t. z).*

Proof. Let $U \subseteq V(G)$ be a cut. Similar to Subsection 9.1, we partition $S \cap \delta(U)$ into two subsets.

(1) $S_1 = \{\{u, v\} \in S \cap \delta(U) : u, v \notin V(F)\}$.

(2) $S_2 = \{\{u, v\} \in S \cap \delta(U) : v \in V(F)\}$.

First, by the construction and Lemma 8.12, we have $|S_1| \leq 24000 \cdot a \cdot g \cdot z(\delta(U))$. Now we partition S_2 into three subsets.

(1) $S_{21} = \{\{a_j, v\} \in S_2 : a_j \in A, v \in V(F)\}$.

(2) $S_{22} = \{\{u, v\} \in S_2 : v \in V(F), u$ and v are in different components of $\Gamma'\}$.

(3) $S_{23} = \{\{u, v\} \in S_2 : v \in V(F), u, v \in C_{v^*}\}$.

By the construction, we know that $|S_{21}| \leq 1$ and $|S_{22}| \leq 7$. Also, by Lemma 9.7 we have $|S_{23}| \leq 20 \cdot p^2 \cdot z'(\delta(U))$. Therefore, we have $|S \cap \delta(U)| \leq 8(24000 \cdot a \cdot g + 20p^2)z(\delta(U))$. □

Lemma 9.11. *T is a $O(a \cdot g + p^2)$-thin subgraph of G (w.r.t. z).*

Proof. By Lemma 9.10 we know that S is a $O(a \cdot g + p^2)$-thin subgraph of G (w.r.t. z). Now note that z is $\vec{W}$-dense. Therefore, $T = S \cup W$ is a $O(a \cdot g + p^2)$-thin subgraph of G (w.r.t. z). □

We are now ready to prove the main result of this Section.

Proof of Lemma 5.3. It follows by Lemmas 9.9 and 9.11. □

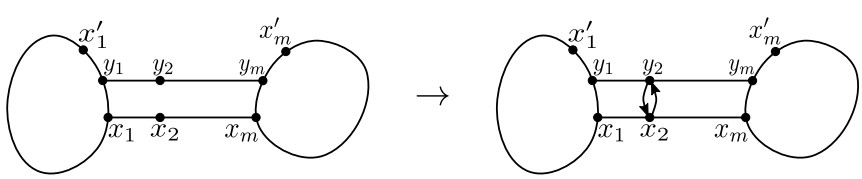
10 A preprocessing step for the dynamic program

Definition 10.1 (Facial normalization)**.** *Let $g \geq 0$, $p \geq 1$ and let $\vec{G}$ be a $(0, g, 1, p)$-nearly embeddable graph. Let $\vec{F}$ be the face on which the vortex is attached. We say that $\vec{G}$ is* facially normalized *if the symmetrization of $\vec{F}$ is a simple cycle and every $v \in V(\vec{F})$ has at most one incident edge that is not in $E(\vec{F})$.*

Lemma 10.2. *Let $g \geq 0$, $p \geq 1$ and let $\vec{G}$ be a $(0, g, 1, p)$-nearly embeddable graph. There exists a polynomial-time computable $(0, g, 1, p)$-nearly embeddable facially normalized graph $\vec{G}'$ such that the following holds. Let $\vec{H}$ be the vortex in $\vec{G}$ and let $\vec{H}'$ be the vortex in $\vec{G}'$. Then $\mathsf{OPT}_{\vec{G}}(V(\vec{H})) = \mathsf{OPT}_{\vec{G}'}(V(\vec{H}'))$. Moreover there exists a polynomial-time algorithm which given any closed walk $\vec{W}'$ in $\vec{G}'$ that visits all vertices in $V(\vec{H}')$, outputs some closed walk $\vec{W}$ in $\vec{G}$ that visits all vertices in $V(\vec{H})$ with $\mathsf{cost}_{\vec{G}}(\vec{W}) = \mathsf{cost}_{\vec{G}'}(\vec{W}')$.*

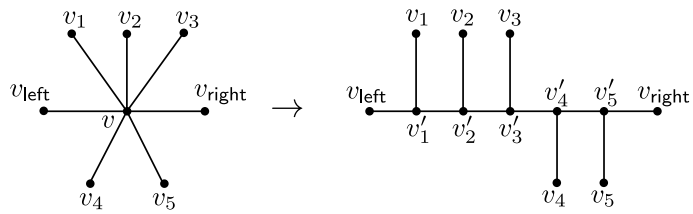
Proof. Let $\vec{F}$ be the face in $\vec{G}$ that $\vec{H}$ is attached to. Let F be the symmetrization of $\vec{F}$. We first construct a $(0, g, 1, p)$-nearly embeddable graph $\vec{G}''$, with a vortex $\vec{H}''$ attached to a face $\vec{F}''$ such that the symmetrization of $\vec{F}''$ is a simple cycle. Initially we set $\vec{G}'' = \vec{G}$. If $\vec{F}$ is a simple cycle then there is nothing to be done. Otherwise, suppose that $\vec{F}$ is not a simple cycle. Therefore, ∂F contains a family of simple cycles $\mathcal{C} = \{C_1, \ldots, C_k\}$ for some k, and a family of simple paths $\mathcal{P} = \{P_1, \ldots, P_l\}$ for some l, such that every $P_i \in \mathcal{P}$ is a path $x_1, x_2, \ldots, x_m$ where $x_1 \in C_\alpha$ and $x_m \in C_\beta$ for some $C_\alpha, C_\beta \in \mathcal{C}$. We allow $\mathcal{P}$ to contain paths of length 0.

For every $P = x_1, x_2, \ldots, x_m \in \mathcal{P}$, where $x_1 \in C$ and $x_m \in C'$ for some $C, C' \in \mathcal{C}$, we update $\vec{G}''$ as follows. Let $x'_1 \in V(C)$ and $x'_m \in V(C')$ be neighbors of x_1 and x_m. We first duplicate P to get a new path $P' = y_1, y_2, \ldots, y_m$. For every edge $e \in E(P)$, we set the cost of the corresponding edge $e' \in P'$ equal to the cost of e. Also, for every $j \in \{1, \ldots, m\}$, we add two edges (x_i, y_i) and (y_i, x_i) to $\vec{G}''$ with $\mathsf{cost}_{\vec{G}''}(x_i, y_i) = \mathsf{cost}_{\vec{G}''}(y_i, x_i) = 0$. Also, we delete edges (x_1, x'_1), (x'_1, x_1), (x_m, x'_m), and (x'_m, x_m) and we add edges (y_1, x'_1), (x'_1, y_1), (y_m, x'_m) and (x'_m, y_m) with the same cost respectively.



By the construction, $\vec{G}''$ is a $(0, g, 1, p)$-nearly embeddable graph, such that $\vec{F}''$ is a simple cycle. Also suppose that $\vec{W}''$ is a closed walk in $\vec{G}''$ that visits all vertices in $V(\vec{H}'')$. Then we can find a closed walk $\vec{W}$ in $\vec{G}$ that visits all vertices in $V(\vec{H})$ with $\mathsf{cost}_{\vec{G}}(\vec{W}) = \mathsf{cost}_{\vec{G}''}(\vec{W}'')$.

Now we construct a facially normalized graph $\vec{G}'$. Initially we set $\vec{G}' = \vec{G}''$. For every $v \in V(\vec{F}'')$ that has more than one incident edge in $E(\vec{G}'') \setminus E(\vec{F}'')$ we update $\vec{G}'$ as follows. Let $v_{\mathsf{left}}, v_{\mathsf{right}} \in V(\vec{F}'')$ be the left and right neighbors of v on $\vec{F}''$. Let $\mathcal{V} = \{v_1, \ldots, v_m\}$ be the set of all neighbors of v in $V(\vec{G}'') \setminus V(\vec{F}'')$. Let $\mathcal{V}' = \{v'_1, \ldots, v'_m\}$. First we delete v from $\vec{G}'$ and we add $\mathcal{V}'$ to $V(\vec{G}')$. For every $(v, v_i) \in E(\vec{G}'')$ we add (v'_i, v_i) to $E(\vec{G}')$ with $\mathsf{cost}_{\vec{G}'}(v'_i, v_i) = \mathsf{cost}_{\vec{G}''}(v, v_i)$. Also, for every $j \in \{1, \ldots, m-1\}$, we add (v'_j, v'_{j+1}) and (v'_{j+1}, v'_j) to $E(\vec{G}')$ with $\mathsf{cost}_{\vec{G}'}(v'_j, v_{j+1}) = 0$. Finally we add $(v'_1, v_{\mathsf{left}})$, $(v_{\mathsf{left}}, v'_1)$, $(v'_m, v_{\mathsf{right}})$ and $(v_{\mathsf{right}}, v'_m)$ to $\vec{G}'$ with the same costs as in $\vec{G}''$.



It is immediate that $\vec{G}'$ is the desired graph. □

Lemma 10.3. *Let $g \geq 0$, $p \geq 1$ and let $\vec{G}$ be a $(0, g, 1, p)$-nearly embeddable graph. There exists a polynomial-time computable $(0, g, 1, p)$-nearly embeddable facially normalized graph $\vec{G}''$ such that the following conditions hold:*

(1) Let $\vec{H}$ be the vortex in $\vec{G}$ and let $\vec{H}''$ be the vortex in $\vec{G}''$. Then $\mathsf{OPT}_{\vec{G}}(V(\vec{H})) = \mathsf{OPT}_{\vec{G}'}(V(\vec{H}''))$.

(2) There exists a polynomial-time algorithm which given any closed walk $\vec{W}''$ in $\vec{G}''$ that visits all vertices in $V(\vec{H}'')$, outputs some closed walk $\vec{W}$ in $\vec{G}$ that visits all vertices in $V(\vec{H})$ with $\mathsf{cost}_{\vec{G}}(\vec{W}) = \mathsf{cost}_{\vec{G}''}(\vec{W}'')$.

(3) Let $\vec{\Gamma}$ be the genus-g piece of $\vec{G}''$. Let $\vec{F}''$ be the face of $\vec{\Gamma}$ on which the vortex $\vec{H}''$ is attached. Then any $v \in V(\vec{\Gamma}) \setminus V(\vec{F}'')$ has degree at most 4.

(4) There exists some closed walk $\vec{W}^*$ in $\vec{G}''$ that visits all vertices in $V(\vec{H}'')$, with $\mathsf{cost}_{\vec{G}''}(\vec{W}^*) = \mathsf{OPT}_{\vec{G}''}(V(\vec{H}''))$, and such that every edge in $\vec{\Gamma}$ is traversed at most once by $\vec{W}^*$.

We say that a graph $\vec{G}''$ satisfying the above conditions is cross normalized.

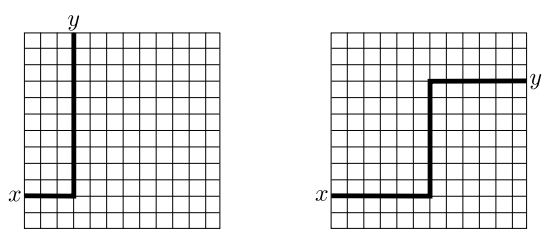
Proof. We begin with computing the facially normalized graph $\vec{G}'$ given by Lemma 10.2. Clearly $\vec{G}'$ satisfies conditions (1) and (2).

We next modify $\vec{G}'$ so that it also satisfies (3). This can be done as follows. Let $\vec{\Gamma}'$ be the genus-g piece of $\vec{G}'$ and let $\vec{F}'$ be the face on which the vortex is attached. We replace each $v \in V(\vec{\Gamma}') \setminus V(\vec{F}')$ of degree $d > 4$ by a tree T_v with d leaves and with maximum degree 4; we replace each edge incident to v an edge incident to a unique leaf, and we set the length of every edge in $E(T_v)$ to 0.

It remains to modify $\vec{G}'$ so that it also satisfies (4). Let $\vec{H}'$ be the vortex in $\vec{G}'$. Let $\vec{W}'$ be a walk in $\vec{G}'$ that visits all vertices in $\vec{H}'$ with $\mathsf{cost}_{\vec{G}'}(\vec{W}') = \mathsf{OPT}_{\vec{G}'}(V(\vec{H}'))$. We may assume w.l.o.g. that $\vec{W}'$ contains at most n^2 edges. Thus, every vertex in $v \in V(\vec{\Gamma}') \setminus V(\vec{F}')$ is visited at most n^2 times by $\vec{W}'$. We replace each $v \in V(\vec{\Gamma}') \setminus V(\vec{F}')$ by a grid A_v of size $3n^2 \times 3n^2$, with each edge having length 0. Each edge incident to v in $\vec{G}'$, corresponds to a unique sides of A_v so that the ordering of the sides agrees with the ordering of the edges around v (in ψ). We replace each $(u, v) \in E(\vec{\Gamma}')$, by a matching of size $3n^2$ between the corresponding sides of A_u and A_v, where each edge in the matching has length equal to the length of (u, v). Let $\vec{G}''$ be the resulting graph.

We obtain the desired walk $\vec{W}^*$ in $\vec{G}''$ as follows. Let $x_1, \ldots, x_{4\ell}$ be the vertices in the boundary of A_v, with $\ell = 3n^2 - 1$, appearing in this order along a clockwise traversal of A_v, and such that x_1 is the lower left corner. Then for any $i \in \{1, \ldots, 4\}$, the vertices $x_{(i-1)\ell+1}, \ldots, x_{(i-1)\ell+n^2}$ correspond to the copies of v on the i-th side of A_v. We traverse $\vec{G}'$ starting at some arbitrary vertex in $V(\vec{H}')$, and we inductively construct the walk $\vec{W}^*$. We consider each edge (u, v) in the order that it is traversed by $\vec{W}'$. Suppose that (u, v) is the t-th edge traversed by $\vec{W}'$, for some $t \in \{1, \ldots, n^2\}$. For

each $i \in \{1, \ldots, 4\}$, we let the t-th copy of v on the i-th side of A_v to be $x_{(i-1)\ell+t}$. We distinguish between the following cases: (i) If $u, v \in V(\vec{H}')$, then $(u, v) \in E(\vec{G}'')$ and we simply traverse (u, v) in $\vec{G}''$. (ii) If $u \in V(\vec{H}')$ and $v \notin V(\vec{H}')$ then we traverse the edge in $\vec{G}''$ that connects u to the t-th copy of v in the appropriate side of A_v. (iii) If $u \notin V(\vec{H}')$ and $v \in V(\vec{H}')$ then we traverse the edge in $\vec{G}''$ that connects the t-th copy of u in the appropriate side of A_u to v. (iv) If $u, v \notin V(\vec{H}')$ then we traverse the edge in $\vec{G}''$ that connects the t-th copy of u to the t-th copy of v in the appropriate sides of A_u and A_v respectively. Finally, for any pair of consecutive edges (u, v), (v, w) traversed by $\vec{W}'$, with $v \notin V(\vec{H}')$, we need to add a path P in A_v connecting two copies of v in the corresponding sides of A_v. Since A_v is a grid of size $3n^2 \times 3n^2$ this can be done so that all these paths are edge-disjoint. More precisely, this can be done as follows. Suppose that (u, v) is the t-th edge traversed by $\vec{W}'$, for some $t \in \{1, \ldots, n^2\}$. If P connects vertices x and y in consecutive sides of A_v, then we proceed as follows. We may assume w.l.o.g that $x = x_t$ and $y = x_{\ell+t+1}$, since other cases can be handled in a similar way. We set P to be the unique path starting at x_t, following $t + 1$ horizontal edges in A_v, and finally following $\ell - t$ vertical edges to $x_{\ell+t+1}$. Otherwise, if P connects vertices x and y in opposite sides of A_v, then we proceed as follows. We may assume w.l.o.g that $x = x_t$ and $y = x_{2\ell+t+1}$, since other cases can be handled in a similar way. We set P to be the unique path starting at x_t, following $t + n^2$ horizontal edges in A_v, and then following $\ell - 2t$ vertical edges, and finally following $\ell - t - n^2 - 1$ horizontal edges to $x_{2\ell+t+1}$.



By the construction, it is immediate that all the paths P constructed above are pairwise edge-disjoint, which implies that every edge in $E(\vec{G}'')$ is visited by $\vec{W}^*$ at most once, concluding the proof. □

11 Uncrossing an optimal walk traversing a vortex

Let $g \geq 0$, $p \geq 1$ and let $\vec{G}$ be a $(0, g, 1, p)$-nearly embeddable graph. By Lemma 10.3 we may assume w.l.o.g. that $\vec{G}$ is facially normalized and cross normalized. Let $\vec{G}' \subseteq \vec{G}$ be the piece of genus g and fix a drawing ψ of $\vec{G}'$ into a surface of genus g. Let $\vec{H}$ be the single vortex in $\vec{G}$ and suppose that $\vec{H}$ is attached to some face $\vec{F}$ of $\vec{G}'$. Fix an optimal solution $\vec{W}_{\mathsf{OPT}}$, that is a closed walk in $\vec{G}$ that visits all vertices in $\vec{H}$ minimizing $\mathsf{cost}_{\vec{G}}(\vec{W})$; if there are multiple such walks pick consistently one with a minimum number of edges. Since $\vec{G}$ is cross normalized we may assume w.l.o.g. that $\vec{W}_{\mathsf{OPT}}$ traverses every edge in $E(\vec{G}') \setminus E(\vec{F})$ at most once.

11.1 The structure of an optimal solution

Definition 11.1 (Shadow)**.** *Let $\mathcal{W}$ be a collection of walks in $\vec{G}$. We define* shadow *of $\mathcal{W}$ (w.r.t. $\vec{G}'$) to be the collection of open and closed walks obtained by restricting every walk in $\mathcal{W}$ on $\vec{G}'$ (note that a walk in $\mathcal{W}$ can give rise to multiple walks in $\mathcal{W}'$, and every open walk in $\mathcal{W}'$ must have both endpoints in $\vec{F}$).*

We say that two edje-disjoint paths P, P' in $\vec{G}'$ *cross* (w.r.to ψ) if there exists $v \in V(P) \cap V(P')$ such that v has degree 4 (recall that $\vec{G}$ is cross normalized), with neighbors $u_1, \ldots, u_4$, such that the edges $\{v, u_1\}, \ldots, \{v, u_4\}$ appear in this order around v in the embedding ψ, P contains the subpath u_1, v, u_3, and P' contains the subpath u_2, v, u_4. We say that two walks in $\vec{G}'$ cross (at v, w.r.to ψ) if they contain crossing subpaths. Finally, a walk is *self-crossing* if it contains two disjoint crossing subpaths.

Lemma 11.2 (Uncrossing an optimal walk of a vortex)**.** *There exists a collection $\mathcal{W} = \{\vec{W}_1, \ldots, \vec{W}_\ell\}$ of closed walks in $\vec{G}$ satisfying the following conditions:*

(1) Every edge in $E(\vec{G}') \setminus E(\vec{F})$ is traversed in total at most once by all the walks in $\mathcal{W}$.

(2) $V(\vec{W}_1) \cup \ldots \cup V(\vec{W}_\ell)$ is a strongly-connected subgraph of $\vec{G}$.

(3) $V(\vec{H}) \subseteq V(\vec{W}_1) \cup \ldots \cup V(\vec{W}_\ell)$.

(4) $\sum_{i=1}^{\ell} \mathsf{cost}_{\vec{G}}(\vec{W}_i) \leq \mathsf{OPT}_{\vec{G}}(V(\vec{H}))$.

(5) Let $\mathcal{W}'$ be the shadow of $\mathcal{W}$. Then the walks in $\mathcal{W}'$ are non-self-crossing and pairwise non-crossing.

Proof. Initially, we set $\mathcal{W} = \{\vec{W}_{\mathsf{OPT}}\}$. Recall that since $\vec{G}$ is cross normalized, every edge in $E(\vec{G}') \setminus E(\vec{F})$ is traversed at most once by $\vec{W}_{\mathsf{OPT}}$. Clearly, this choice of $\mathcal{W}$ satisfies conditions (1)–(4). We proceed to iteratively modify $\mathcal{W}$ until condition (4) is also satisfied, while inductively maintaining (1)–(4).

Suppose that the current choice for $\mathcal{W}$ does not satisfy (5). This means that either there exist two distinct crossing walks in $\mathcal{W}$, or there exists some self-crossing walk in $\mathcal{W}$. In either case, it follows that there exist subpaths P, P' of the walks in $\mathcal{W}$ that are crossing (w.r.to ϕ). This means that there exists $v \in V(P) \cap V(P')$ and $e_1, e_2 \in E(P)$, $e_3, e_4 \in E(P')$ such that $\psi(e_1)$, $\psi(e_4)$, $\psi(e_2)$, $\psi(e_3)$ appear in this order around $\psi(v)$. We modify P and P' by swapping e_1 and e_3. It is immediate that the above operation preserves conditions (1)–(4). Moreover, after performing the operation, the total number of crossings and self-crossings (counted with multiplicities) between the walks in $\mathcal{W}$ decreases by at least one. Since the original number of crossings is finite, it follows that the process terminates after a finite number of iterations. By the inductive condition, it is immediate that when the process terminates the collection $\mathcal{W}$ satisfies condition (5), concluding the proof. □

For the remainder of this section let $\mathcal{W}$ and $\mathcal{W}'$ be as in Lemma 11.2. Let $\mathcal{I}$ be a graph with $V(\mathcal{I}) = \mathcal{W}'$ and with

$$E(\mathcal{I}) = \left\{ \{W, W'\} \in \binom{\mathcal{W}'}{2} : V(W) \cap V(W') \neq \emptyset \right\}.$$

Let $\vec{W}, \vec{Z}$ be distinct closed walks in some digraph, and let $v \in V(\vec{W}) \cap V(\vec{Z})$. Suppose that $\vec{W} = x_1, \ldots, x_k, v, x_{k+1}, \ldots, x_{k'}, x_1$ and $\vec{Z} = y_1, \ldots, y_r, v, y_{r+1}, \ldots, y_{r'}, y_1$. Let $\vec{S}$ be the closed walk $x_1, \ldots, x_k, v, y_{r+1}, \ldots, y_{r'}, y_1, \ldots, y_r, v, x_{k+1}, \ldots, x_{k'}, x_1$. We say that $\vec{S}$ is obtained by *shortcutting* $\vec{W}$ and $\vec{Z}$ (at v).

Let $\mathcal{J}$ be a subgraph of $\mathcal{I}$. Let $\mathcal{W}^{\mathcal{J}}$ be a collection of walks in $\vec{G}$ constructed inductively as follows. Initially we set $\mathcal{W}^{\mathcal{J}} = \mathcal{W}$. We consider all $\{W, W'\} \in E(\mathcal{J})$ in an arbitrary order. Note

that since $\{W, W'\} \in E(\mathcal{J})$, it follows that W and W' cross at some $v \in V(\vec{G}')$. Let R and R' be the walks in $\mathcal{W}^{\mathcal{J}}$ such that W and W' are sub-walks of R and R' respectively. If $R \neq R'$ then we replace R and R' in $\mathcal{W}^{\mathcal{J}}$ by the walk obtained by shortcutting R and R' at v. This completes the construction of $\mathcal{W}^{\mathcal{J}}$. We say $\mathcal{W}^{\mathcal{J}}$ is obtained by *shortcutting* $\mathcal{W}$ *at* $\mathcal{J}$.

Lemma 11.3. *There exists some forest $\mathcal{F}$ in $\mathcal{I}$ such that the collection of walks obtained by shortcutting $\mathcal{W}$ at $\mathcal{F}$ contains a single walk.*

Proof. Let $\mathcal{F}$ be a forest obtained by taking a spanning subtree in each connected component of $\mathcal{I}$. Let $W, W' \in \mathcal{W}$. Let $\mathcal{W}^{\mathcal{F}}$ be obtained by shortcutting $\mathcal{W}$ at $\mathcal{F}$. It suffices to show that W and W' become parts of the same walk in $\mathcal{W}^{\mathcal{F}}$. By condition (2) of Lemma 11.2 we have that there exists a sequence of walks $W_1, \ldots, W_t \in \mathcal{W}$, with $W_1 = W$, $W_t = W'$, and such that for any $i \in \{1, \ldots, t-1\}$, there exists some walk $A_i \in W_i \cap \vec{G}'$, and some walk $B_{i+1} \in W_{i+1} \cap \vec{G}'$ such that $\{A_i, B_{i+1}\} \in E(\mathcal{I})$. Therefore A_i and B_{i+1} are in the same connected component of $\mathcal{I}$. Thus there exits some tree $\mathcal{T}_i$ in $\mathcal{F}$ such that $A_i, B_{i+1} \in V(\mathcal{T}_i)$. It follows that after shortcutting $\mathcal{W}$ at $\mathcal{F}$, the walks A_i and B_{i+1} become parts of the same walk. By induction on $i \in \{1, \ldots, t-1\}$, it follows that A_1 and B_t become parts of the same walk in $\mathcal{W}^{\mathcal{F}}$, and thus so do W and W', concluding the proof. □

For the remainder let $\mathcal{F}$ be the forest given by Lemma 11.3.

Lemma 11.4. *All leaves of $\mathcal{F}$ intersect F.*

Proof. Suppose that there exists some leaf $\vec{W}$ of $\mathcal{F}$ with $V(\vec{W}) \cap V(F) = \emptyset$. Then simply removing $\vec{W}$ from $\mathcal{W}$ leaves a new collection of walks that visits all vertices in $V(\vec{H})$ and such that the union of all walks is a strongly-connected subgraph of $\vec{G}$. Thus after shortcutting all these walks we may obtain a new single walk $\vec{R}$ that visits all vertices in $V(\vec{H})$ with $\mathsf{cost}_{\vec{G}}(\vec{R}) \leq \mathsf{cost}_{\vec{G}}(\vec{W}_{\mathsf{OPT}})$ and with fewer edges than $\vec{W}_{\mathsf{OPT}}$, contradicting the choice of $\vec{W}_{\mathsf{OPT}}$. □

12 The dynamic program for traversing a vortex in a planar graph

For the remainder of this section let $\vec{G}$ be a n-vertex $(0, 0, 1, p)$-nearly embeddable graph (that is, planar with a single vortex). Let $\vec{H}$ be the vortex in $\vec{G}$ and suppose it is attached on some face $\vec{F}$. Fix an optimal solution $\vec{W}_{\mathsf{OPT}}$, that is a closed walk in $\vec{G}$ that visits all vertices in $\vec{H}$ minimizing $\mathsf{cost}_{\vec{G}}(\vec{W})$; if there are multiple such walks pick consistently one with a minimum number of edges. Let F be the symmetrization of $\vec{F}$. We present an algorithm for computing a walk traversing all vertices in $V(\vec{H})$ based on dynamic programming. By Lemma 10.3 we may assume w.l.o.g. that $\vec{G}$ is facially normalized and cross normalized.

Fix a path-decomposition $\{B_v\}_{v \in V(F)}$ of $\vec{H}$ of width p.

Let $\mathcal{S}$ be a collection of walks in $\vec{G}$. For any $v \in V(\vec{G})$ we denote by $\mathsf{in\text{-}degree}_{\mathcal{S}}(v)$ the number of times that the walks in $\mathcal{S}$ enter v; similarly, we denote by $\mathsf{out\text{-}degree}_{\mathcal{S}}(v)$ the number of times that the walks in $\mathcal{S}$ exit v. We define $\vec{G}[\mathcal{S}]$ to be the graph with $V(\vec{G}[\mathcal{S}]) = \bigcup_{W \in \mathcal{S}} V(W)$ and $E(\vec{G}[\mathcal{S}]) = \bigcup_{W \in \mathcal{S}} E(W)$.

12.1 The dynamic program

Let $\mathcal{P}$ be the set of all subpaths of F, where we allow allow for simplicity in notation that a path be closed. Let u, v be the endpoints of P. Let

$$\vec{H}_P = \vec{H}\left[\bigcup_{x \in V(P)} B_x\right].$$

Let $\mathcal{C}_P$ be the set of all possible partitions of $B_u \cup B_v$. Let $\mathcal{D}_P^{\mathsf{in}} = \{0, \ldots, n\}^{B_u \cup B_v}$, $\mathcal{D}_P^{\mathsf{out}} = \{0, \ldots, n\}^{B_u \cup B_v}$, that is, every element of $\mathcal{D}_P^{\mathsf{in}} \cup \mathcal{D}_P^{\mathsf{out}}$ is a function $f : B_u \cup B_v \to \{0, \ldots, n\}$. Let $\mathcal{A} = V(F)^2$.

12.1.1 The dynamic programming table

The dynamic programming table is indexed by all pairs (P, ϕ) where $P \in \mathcal{P}$ and

$$\phi = (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p) \in \mathcal{I}_P$$

where

$$\mathcal{I}_P = \mathcal{C}_P \times \mathcal{D}_P^{\mathsf{in}} \times \mathcal{D}_P^{\mathsf{out}} \times (\mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}) \times (V(\vec{G}) \cup \mathsf{nil}) \times (V(\vec{G}) \cup \mathsf{nil}) \times (V(\vec{G}) \cup \mathsf{nil}).$$

A *partial solution* is a collection of walks in $\vec{G}$.

We say that a partial solution $\mathcal{S}$ is *compatible* with (P, ϕ) if the following conditions are satisfied:

(T1) For every $x \in V(\vec{H}_P)$ there exists some walk in $\mathcal{S}$ that visits x. That is $V(\vec{H}_P) \subseteq \bigcup_{Q \in \mathcal{S}} V(Q)$.

(T2) If $a \neq \mathsf{nil}$ and $a \in \mathcal{A}$, let $a = (u', v')$. Let Q_1 be the shortest path from u' to l in $\vec{G}$. Let Q_2 be the shortest path from l to r in $\vec{G}$. Let Q_3 be the shortest path from r to v' in $\vec{G}$. Let Q_1^* be the walk from u' to v' obtained by the concatenation of Q_1, Q_2 and Q_3. Then Q_1^* is a sub-walk of some walk in $\mathcal{S}$. We refer to Q^* as the *grip* of $\mathcal{S}$, and in this case we say that it is an *unbroken grip*. If $a \in V(P)^2$ then we say that the unbroken grip is *closed* and otherwise we say that it is *open*. Otherwise, if $a \in (\mathcal{A} \times \mathcal{A})$, let $a = ((u_1', v_1'), (u_2', v_2'))$. Let Q_1' be the shortest path from u_1' to l in $\vec{G}$. Let Q_2' be the shortest path from l to v_1' in $\vec{G}$. Let Q_2^* be the path from u_1' to v_1' obtained by the concatenation of Q_1' and Q_2'. Let Q_1'' be the shortest path from u_2' to r in $\vec{G}$. Let Q_2'' be the shortest path from r to v_2' in $\vec{G}$. Let Q_3^* be the path from u_2' to v_2' obtained by the concatenation of Q_1'' and Q_2''. Then Q_2^* and Q_3^* are sub-walks of some walks in $\mathcal{S}$. We refer to (Q_2^*, Q_3^*) as the *broken grip* of $\mathcal{S}$.

(T3) If $a = \mathsf{nil}$ or $a \in \mathcal{A}$, then every open walk in $\mathcal{S}$ has both endpoints in $B_u \cup B_v$, except possibly for one walk $W \in \mathcal{S}$ that contains the grip as a sub-walk. If $a = (Q_1, Q_2) \in (\mathcal{A} \times \mathcal{A})$, then every open walk in $\mathcal{S}$ has both endpoints in $B_u \cup B_v$, except possibly for at most two walks $W_1, W_2 \in \mathcal{S}$ that contain Q_1 and Q_2 as sub-walks (note that Q_1 and Q_2 might be sub-walks of the same walk in $\mathcal{S}$).

(T4) For all $x \in B_u \cup B_v$ we have

$$f^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{S}}(x) \qquad \text{and} \qquad f^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{S}}(x).$$

(T5) For any $x, y \in B_u \cup B_v$ we have that if x and y are in the same set of the partition C then they are in the same weakly-connected component of $\bigcup_{W \in \mathcal{S}} W$. Moreover for any $z \in V(\vec{H}_P)$ there exists $z' \in B_u \cap B_v$ such that z and z' are in the same weakly-connected component of $\bigcup_{W \in \mathcal{S}} W$.

12.1.2 Merging partial solutions

We compute the values of the dynamic programming table inductively as follows. Let $P, P_1, P_2 \in \mathcal{P}$ such that $E(P_1) \neq \emptyset$, $E(P_2) \neq \emptyset$, $E(P_1) \cap E(P_2) = \emptyset$, and $P = P_1 \cup P_2$. Let $u \in V(P_1)$, $w \in V(P_1) \cap V(P_2)$, $v \in V(P_2)$ such that u, w are the endpoints of P_1 and w, v are the endpoints of P_2. Let

$$\phi = (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p) \in \mathcal{I}_P,$$
$$\phi_1 = (C_1, f_1^{\mathsf{in}}, f_1^{\mathsf{out}}, a_1, l_1, r_1, p_1) \in \mathcal{I}_{P_1},$$
$$\phi_2 = (C_2, f_2^{\mathsf{in}}, f_2^{\mathsf{out}}, a_2, l_2, r_2, p_2) \in \mathcal{I}_{P_2}.$$

For any $i \in \{1, 2\}$ let $\mathcal{S}_i$ be a partial solution that is compatible with (P_i, ϕ_i). We proceed to compute a collection of walks $\mathcal{S}$ that is compatible with (P, ϕ). This is done in phases, as follows:

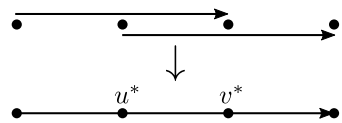
Merging phase 1: Joining the walks. We check that for all $x \in B_w$ we have $f_1^{\mathsf{in}}(x) = f_2^{\mathsf{out}}(x)$ and $f_2^{\mathsf{in}}(x) = f_1^{\mathsf{out}}(x)$. If not then the merging procedure return nil. For any $x \in B_w$ and for any $i \in \{1, 2\}$ let $E_i(x)^{\mathsf{in}}$ (resp. $E_i(x)^{\mathsf{out}}$) be the multiset of all edges in all walks in $\mathcal{S}_i$ that are incoming to (resp. outgoing from) x counted with multiplicities. Since $f_1^{\mathsf{out}}(x) = f_2^{\mathsf{in}}(x)$ and $f_2^{\mathsf{out}}(x) = f_1^{\mathsf{in}}(x)$ it follows that $|E_1^{\mathsf{in}}(x) \cup E_2^{\mathsf{in}}(x)| = |E_1^{\mathsf{out}}(x) \cup E_2^{\mathsf{out}}(x)|$. Pick an arbitrary bijection $\sigma_x : E_1^{\mathsf{in}}(x) \cup E_2^{\mathsf{in}} \to E_1^{\mathsf{out}}(x) \cup E_2^{\mathsf{out}}(x)$. We initially set $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$. For each $x \in B_w$ we proceed as follows. For each $e \in E_1^{\mathsf{in}(x)} \cup E_2^{\mathsf{in}}(x)$ we modify the walk traversing e so that immediately after traversing e it continues with the walk traversing $\sigma_x(e) \in E_1^{\mathsf{out}} \cup E_2^{\mathsf{out}}$.

Merging phase 2: Updating the grip. We check that at least one of the following conditions is satisfied:

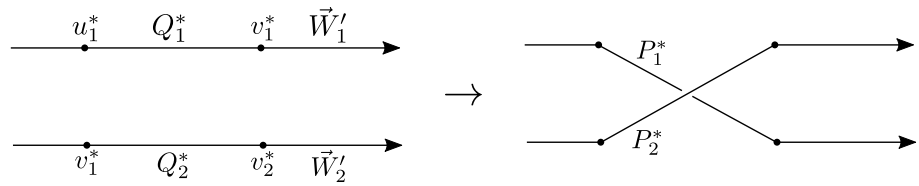
(1) Suppose that $a = l = r = p = a_1 = l_1 = r_1 = p_1 = a_2 = l_2 = r_2 = p_2 = \mathsf{nil}$. Then there is nothing to do.

(2) Suppose that $a_1 = l_1 = r_1 = p_1 = \mathsf{nil}$, $l = l_2$, $r = r_2$, $p = p_2$ and $a = a_2 = (u_2^*, v_2^*) \in \mathcal{A}$ with $\{u_2^*, v_2^*\} \cap V(P_1) \subseteq \{u, w\}$, or $a_2 = l_2 = r_2 = p_2 = \mathsf{nil}$, $l = l_1$, $r = r_1$, $p = p_1$ and $a = a_1 = (u_1^*, v_1^*) \in \mathcal{A}$ with $\{u_1^*, v_1^*\} \cap V(P_2) \subseteq \{w, v\}$. Then there is nothing to do.

(3) Suppose that $a_1 \neq \mathsf{nil}$, $a_2 \neq \mathsf{nil}$, and $a \neq \mathsf{nil}$. Suppose $a_1 = a_2 = a = (u^*, v^*) \in \mathcal{A}$, with $a \in V(P_1) \times V(P_2)$ or $a \in V(P_2) \times V(P_1)$. Suppose $l_1 = l_2 = l$, $r_1 = r_2 = r$ and $p_1 = p_2 = p$. Then we proceed as follows to ensure that (T2) holds. We may assume w.l.o.g. that $a \in V(P_1) \times V(P_2)$ since the remaining case can be handled in a similar way. Let Q^* be the grip between u^* and v^* in $\vec{G}$. It follows by (T2) that for any $i \in \{1, 2\}$ there exists a walk $\vec{W}_i \in \mathcal{S}_i$ that contains Q^* as a sub-walk. It follows by the definition of the merging phase 1 that for any $i \in \{1, 2\}$ there exists $\vec{W}_i' \in \mathcal{S}$ that contains Q^* as a sub-walk. We will modify $\mathcal{S}$ in order to ensure that (T2) holds. We remove Q^* from $\vec{W}_2'$ and we merge $\vec{W}_1'$ with $\vec{W}_2' \setminus Q^*$ (via concatenation).

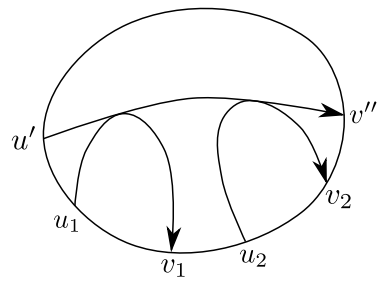


(4) Suppose that $a_1, a_2, a \in \mathcal{A}$. Suppose that $a_1 = (u_1^*, v_1^*)$, $a_2 = (u_2^*, v_2^*)$, $a = (u^*, v^*)$, with $u^* \in \{u_1^*, u_2^*\}$, and $v^* \in \{v_1^*, v_2^*\}$. Suppose that $l_1 = l$, $l_2 = r_2 = r$ and $p_1 = p_2 = p$, or $l_2 = l$, $l_1 = r_1 = r$ and $p_1 = p_2 = p$, or $l = r = p_1 = p_2$ and $l_2 = r_2$, or $l = r = p_1 = p_2$ and $l_1 = r_1$. Then we proceed as follows to ensure that (T2) holds. We may assume w.l.o.g. that $a = (u_2^*, v_1^*)$, $l_1 = l$, $l_2 = r_2 = r$ and $p_1 = p_2 = p$ since the other cases can be handled in a similar way. For any $i \in \{1, 2\}$ let Q_i^* be the grip of $\mathcal{S}_i$. It follows by (T2) that for any $i \in \{1, 2\}$ there exists a walk $\vec{W}_i \in \mathcal{S}_i$ that contains Q_i^* as a sub-walk. It follows that for any $i \in \{1, 2\}$ there exists $\vec{W}_i' \in \mathcal{S}$ that contains Q_i^* as a sub-walk. We will modify $\mathcal{S}$ in order to ensure that (T2) holds. If $a = a_1$ or $a = a_2$ then there is nothing left to do. Otherwise, let R_1' be the shortest path in $\vec{G}$ from u_1^* to r_1. Let R_1'' be the shortest path in $\vec{G}$ from r_1 to l_2. Let R_1''' be the shortest path in $\vec{G}$ from l_2 to v_2^*. Let R_1^* be the path in $\vec{G}$ from u_1^* to v_2^* obtained by concatenation of R_1', R_1'' and R_1'''. Let R_2' be the shortest path in $\vec{G}$ from u_2^* to p. Let R_2'' be the shortest path in $\vec{G}$ from p to l_1. Let R_2''' be the shortest path in $\vec{G}$ from l_1 to v_1^*. Let R_2^* be the path in $\vec{G}$ from u_2^* to v_1^* obtained by concatenation of R_2', R_2'' and R_2'''. We remove Q_1^* and Q_2^* from $\vec{W}_1'$ and $\vec{W}_2'$ and replace them by R_1^* and R_2^*.



(5) Suppose that $a_1 = \mathsf{nil}$ and $a = a_2 \in (\mathcal{A} \times \mathcal{A})$, or $a_2 = \mathsf{nil}$ and $a = a_1 \in (\mathcal{A} \times \mathcal{A})$. Then there is nothing to do.

(6) Suppose that $a_1 \in \mathcal{A}$, $a_2 \in \mathcal{A}$ and $a \in (\mathcal{A} \times \mathcal{A})$. Suppose $a_1 = (u_1, v_1)$, $a_2 = (u_2, v_2)$ and $a = ((u', v'), (u'', v''))$, with $v' \in \{v_1, v_2\}$ and $u'' \in \{u_1, u_2\}$. We may assume w.l.o.g that $a = ((u', v_1), (u_2, v''))$. Suppose that $l = l_1 = r_1$, $r = l_2 = r_2$ and $p = p_1 = p_2$. For any $i \in \{1, 2\}$, let Q_i be the grip of $\mathcal{S}_i$. It follows by (T2) that for any $i \in \{1, 2\}$ there exists a walk $\vec{W}_i \in \mathcal{S}_i$ that contains Q_i as a sub-walk. It follows that for any $i \in \{1, 2\}$ there exists $\vec{W}_i' \in \mathcal{S}$ that contains Q_i as a sub-walk. Let Q_1' be the shortest path in $\vec{G}$ from u_1 to l_1. Let Q_2' be the shortest path in $\vec{G}$ from l_1 to l_2. Let Q_3' be the shortest path in $\vec{G}$ from l_2 to v_2. Let Q_1^* be the path in $\vec{G}$ from u_1 to v_2 obtained by concatenation of Q_1', Q_2' and Q_3'. Let Q_1'' be the shortest path in $\vec{G}$ from u' to l_1. Let Q_2'' be the shortest path in $\vec{G}$ from l_1 to v_1. Let Q_2^* be the path in $\vec{G}$ from u' to v_1 obtained by concatenation of Q_1'' and Q_2''. Let Q_1''' be the shortest path in $\vec{G}$ from u_2 to l_2. Let Q_2''' be the shortest path in $\vec{G}$ from l_2 to v''. Let Q_3^* be the path in $\vec{G}$ from u_2 to v'' obtained by concatenation of Q_1''' and Q_2'''. Then we remove Q_1 and Q_2 from $\vec{W}_1'$ and $\vec{W}_2'$, and replace them by Q_1^*, Q_2^* and Q_3^*.



(7) Suppose that $a_1 \in (\mathcal{A} \times \mathcal{A})$, $a_2 \in \mathcal{A}$ and $a \in (\mathcal{A} \times \mathcal{A})$, or $a_1 \in \mathcal{A}$, $a_2 \in (\mathcal{A} \times \mathcal{A})$ and $a \in (\mathcal{A} \times \mathcal{A})$. We may assume w.l.o.g that $a_1 \in (\mathcal{A} \times \mathcal{A})$, $a_2 \in \mathcal{A}$ and $a \in (\mathcal{A} \times \mathcal{A})$. Suppose that $a_1 = ((u_1, v_1), (u_1', v_1'))$, $a_2 = (u_2, v_2)$ and $a = ((u, v), (u', v'))$, with $(u, v) = (u_1, v_1)$, $u' \in \{u_1, u_2\}$, and $v' \in \{v_1, v_2\}$, or $(u, v) = (u_2, v_2)$, $u' \in \{u_1, u_2\}$, and $v' \in \{v_1, v_2\}$. We may assume w.l.o.g that $(u, v) = (u_1, v_1)$ and $(u', v') = (u_2, v_1')$. Suppose that $l = l_1$, $r = l_2 = r_2$ and $p = p_1 = p_2$. Let (Q_1, Q_1') be the grip of $\mathcal{S}_1$ and let Q_2 be the grip of $\mathcal{S}_2$. Let R_1 be the shortest path in $\vec{G}$ from u_1' to r_1. Let R_2 be the shortest path in $\vec{G}$ from r_1 to r_2. Let R_3 be the shortest path in $\vec{G}$ from r_2 to v_2. Let R' be the path in $\vec{G}$ from u_1' to v_2 obtained by concatenation of R_1, R_2 and R_3. Let R_1' be the shortest path in $\vec{G}$ from u_2 to r_2. Let R_2' be the shortest path in $\vec{G}$ from r_2 to v_1'. Let R'' be the path in $\vec{G}$ from u_2 to v_1' obtained by concatenation of R_1' and R_2'. Then we remove Q_1' and Q_2, and replace them by R' and R''.

If none of the above holds then the merging procedure returns nil.

Merging phase 3: Checking connectivity. We check that condition (T5) holds for $\mathcal{S}$ and we return nil if it does not.

Lemma 12.1. *If the merging procedure outputs some partial solution $\mathcal{S}$ then $\mathcal{S}$ is compatible with ϕ.*

Proof. It follows immediately by the definition of compatibility. □

12.1.3 Initializing the dynamic programming table

For all $P \in \mathcal{P}$ containing at most one edge and for all $\phi \in \mathcal{I}_P$ with $\phi = (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p)$ we proceed as follows. We enumerate all partial solutions $\mathcal{S}$ that are compatible with (P, ϕ) and have minimum cost. Any walk in any such partial solution can intersect $\vec{G} \setminus \vec{H}$ only on the at most two oppositely-directed edges in $E(P)$. Moreover there are at most $O(n^7)$ possibilities for a, l, r and p. Thus the enumeration can clearly be done in time $n^{O(1)}$ by ensuring that for all walks $W \in \mathcal{S}$, their sub-walks that do not intersect $E(P)$ are shortest paths between vertices in $B_u \cup B_v$. The total running time of this initialization step is therefore $n^{O(p)}$.

12.1.4 Updating the dynamic programming table

For all $P \in \mathcal{P}$ containing $m > 1$ edges, and for all $P_1, P_2 \in \mathcal{P}$ with $E(P_1) \neq \emptyset$, $E(P_2) \neq \emptyset$, $E(P_1) \cap E(P_2) = \emptyset$ and $P_1 \cup P_2 = P$, and for all $\phi_1 \in \mathcal{I}_{P_1}$ and $\phi_2 \in \mathcal{I}_{P_2}$ we proceed as follows. Suppose that for all paths P' containing $m' < m$ edges and all $\phi' \in \mathcal{I}_{P'}$ we have computed the partial solutions in the dynamic programming table at (P', ϕ'). If there exist partial solutions

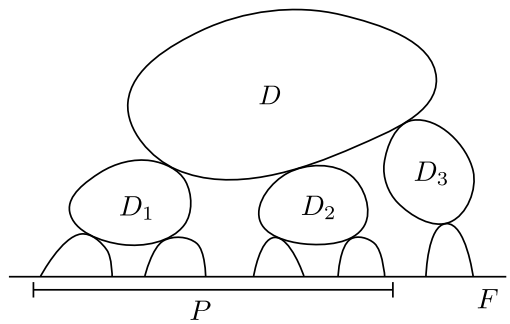


Figure 1: Example of a basic path.

$\mathcal{S}_1$ and $\mathcal{S}_2$ at (P_1, ϕ_1) and (P_2, ϕ_2) respectively, we call the merging process to merge $\mathcal{S}_1$ and $\mathcal{S}_2$. Suppose that the merging process returns a partial solution $\mathcal{S}$ at (P, ϕ) for some $\phi \in \mathcal{I}_P$. If there is no partial solution stored currently at (P, ϕ) then we store $\mathcal{S}$ at that location. Otherwise if there there exists a partial solution $\mathcal{S}'$ stored at (P, ϕ) and the cost of $\mathcal{S}$ is smaller than the cost of $\mathcal{S}'$ then we replace $\mathcal{S}'$ with $\mathcal{S}$.

12.2 Analysis

Let $\mathcal{W}$ be the collection of walks given by Lemma 11.2. Let $\mathcal{W}'$ be the shadow of $\mathcal{W}$. Let $\mathcal{F}$ be the forest obtained by Lemma 11.3. For every connected component $\mathcal{T}$ of $\mathcal{F}$ pick some $v_{\mathcal{T}} \in V(\mathcal{T})$ and consider $\mathcal{T}$ to be rooted at $v_{\mathcal{T}}$.

Let $\vec{G}'$ be the planar piece of $\vec{G}$, that is $\vec{G}' = \vec{G} \setminus (V(\vec{H}) \setminus (\vec{F}))$. Fix some planar drawing ψ of G'. Let $\mathcal{D}$ be the disk with $\partial\mathcal{D} = \psi(F)$ with $\psi(\vec{G}) \subset \mathcal{D}$.

Let $P \in \mathcal{P}$ with endpoints u, v, and let $\mathcal{T}$ be a subtree of some tree in $\mathcal{F}$. We say that P *covers* $\mathcal{T}$ if for all $D \in V(\mathcal{T})$ we have $V(D) \cap V(F) \subseteq V(P)$. We say that P *avoids* $\mathcal{T}$ if for all $D \in V(\mathcal{T})$ we have $V(D) \cap V(P) \subseteq \{u, v\}$.

Definition 12.2 (Basic path)**.** *Let $P \in \mathcal{P}$. Let $u, v \in V(P)$ be the endpoints of P. We say that P is* basic *(w.r.t. $\mathcal{W}$) if either $P \setminus \{u, v\}$ does not intersect any of the walks in $\mathcal{W}$ (in this case we call P* empty basic*) or the following holds. There exists some tree $\mathcal{T}$ in $\mathcal{F}$ and some $D \in V(\mathcal{T})$, with children $D_1, \ldots, D_k$, intersecting D in this order along a traversal of D, such that the following conditions are satisfied (see Figure 1):*

(1) For any $i \in \{1, \ldots, k\}$, let $\mathcal{T}_{D_i}$ be the subtree of $\mathcal{T}$ rooted at D_i and let $\mathcal{T}_D$ be the subtree of $\mathcal{T}$ rooted at D. Then at least one of the following two conditions is satisfied:

(1-1) P covers $\mathcal{T}_D$ and avoids $\mathcal{T} \setminus \mathcal{T}_D$.

(1-2) There exists $j \in \{1, \ldots, k\}$ such that for all $l \leq j$, P covers $\mathcal{T}_{D_l}$ and P avoids $\mathcal{T}_D \setminus \bigcup_{m=1}^{j} \mathcal{T}_{D_m}$.

(2) Let $\mathcal{T}'$ be a tree in $\mathcal{F}$ with $\mathcal{T}' \neq \mathcal{T}$. Then either P covers $\mathcal{T}'$ or P avoids $\mathcal{T}'$.

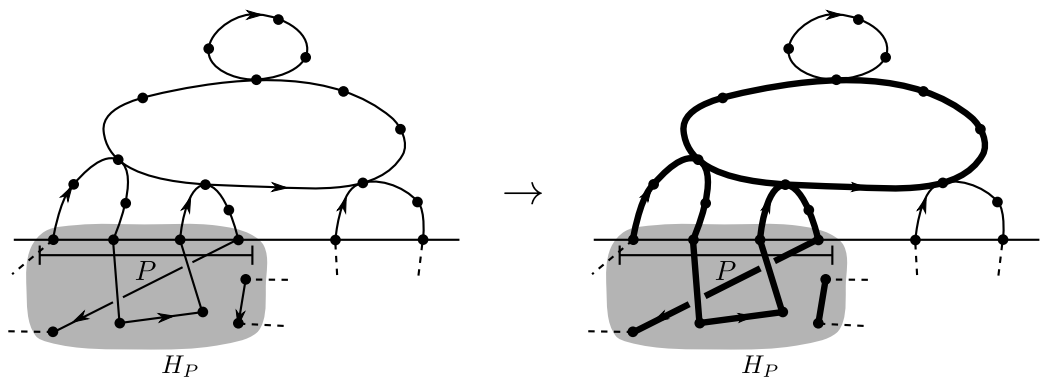


Figure 2: Part of the collection of walks in $\mathcal{W}$ (left) and the corresponding P-facial restriction of $\mathcal{W}$ depicted in bold (right).

Definition 12.3 (Facial restriction)**.** *Let $P \in \mathcal{P}$ be basic. Let $\mathcal{W}'$ be the collection of walks obtained by restricting every $W \in \mathcal{W}$ on H_P. If P is empty, we say that $\mathcal{W}'$ is the P-*facial restriction *of $\mathcal{W}$. Suppose that P is not empty. Let $\mathcal{T}$, D, $\mathcal{T}_D$, k and j be as in Definition 12.2. Let $\mathcal{F}' = \{\mathcal{T}' \in \mathcal{F} : P \text{ covers } \mathcal{T}'\}$ and let $\mathcal{R} = \bigcup_{\mathcal{T}' \in \mathcal{F}'} V(\mathcal{T}')$. If P covers $\mathcal{T}_D$ and avoids $\mathcal{T} \setminus \mathcal{T}_D$, we say that $\mathcal{R} \cup \mathcal{W}' \cup V(\mathcal{T}_D)$ is the P-*facial restriction *of $\mathcal{W}$. Otherwise, we say that $\mathcal{R} \cup \mathcal{W}' \cup \{D\} \cup \bigcup_{i=1}^{j} V(\mathcal{T}_{D_i})$ is the P-*facial restriction *of $\mathcal{W}$. Figure* **??** *depicts an example of a P-facial restriction.*

Definition 12.4 (Important walk)**.** *Let $P \in \mathcal{P}$ be a basic path. Let $\mathcal{W}'$ be the P-facial restriction of $\mathcal{W}$. Let $\mathcal{W}''$ be the shadow of $\mathcal{W}'$. Let Q be a walk in $G[\mathcal{W}'']$. We say that Q is P-*important *(w.r.to. $\mathcal{W}$) if the following conditions hold:*

(1) Both endpoints of Q are in $V(P)$.

(2) Q is the concatenation of walks $Q_1, \ldots, Q_\ell$ such that for each $i \in \{1, \ldots, \ell\}$ there exists some $W_i \in \mathcal{W}$ such that Q_i is a sub-walk of W_i, and for each $j \in \{1, \ldots, \ell - 1\}$ we have $\{W_j, W_{j+1}\} \in E(\mathcal{F})$.

Proposition 12.5. *For any $u, v \in V(P)$, there exists at most one P-important walk from u to v.*

Proof. It follows immediately by the fact that $\mathcal{F}$ is a forest. □

Lemma 12.6. *Let $P \in \mathcal{P}$ be basic w.r.t. $\mathcal{W}$ with endpoints $u, v \in V(F)$, where $u = v$ if P is closed. Let $\mathcal{W}_P$ be the P-facial restriction of $\mathcal{W}$. Let $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$. Let C be the partition of $B_u \cup B_v$ that corresponds to the weakly-connected components of Γ. For any $x \in B_u \cup B_v$ let $f^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_P}(x)$ and $f^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_P}(x)$. Then there exists some $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l, r, p \in (V(\vec{G}) \cup \mathsf{nil})$ such that the dynamic programming table contains some partial solution $\mathcal{S}$ at location $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$, such that $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$.*

Proof. Let us first assume that $\mathcal{F}$ contains only one tree. We will deal with the more general case later on. First suppose that P is an empty basic path. Let $P = x_1, x_2, \ldots, x_m$, where $x_1 = u$ and $x_m = v$. We will prove the assertion by induction on m. For the base case, suppose that $m = 2$, and

thus P contains only one edge. Let $a = l = r = p = \mathsf{nil}$. In this case, a partial solution $\mathcal{S}$ at location $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$ is computed in the initialization step of the dynamic programming table and clearly we have $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$ and we are done. Now suppose that $m > 2$ and we have proved the assertion for all $m' < m$. We first decompose P into two edge-disjoint paths P_1 and P_2, such that $V(P_1) \cap V(P_2) = w$ for some $1 < j < m$ and $w = x_j$. For $i \in \{1, 2\}$ let $\mathcal{W}_{P_i}$ be the P_i-facial restriction of $\vec{W}_{\mathsf{OPT}}$ and let $\Gamma_i = \bigcup_{\vec{W} \in \mathcal{W}_{P_i}} \vec{W}$. Let C_1 be the partition of $B_u \cup B_w$ that corresponds to the weakly-connected components of Γ_1 and let C_2 be the partition of $B_w \cup B_v$ that corresponds to the weakly-connected components of Γ_2. For any $x \in B_u \cup B_w$ let $f_1^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_{P_1}}(x)$ and $f_1^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_{P_1}}(x)$. Also for any $x \in B_w \cup B_v$ let $f_2^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_{P_2}}(x)$ and $f_2^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_{P_2}}(x)$. By the induction hypothesis, there exists partial solutions $\mathcal{S}_1$ and $\mathcal{S}_2$ that are compatible with $(P_1, (C_1, f_1^{\mathsf{in}}, f_1^{\mathsf{out}}, \mathsf{nil}, \mathsf{nil}, \mathsf{nil}, \mathsf{nil}))$ and $(P_2, (C_2, f_2^{\mathsf{in}}, f^{\mathsf{out}}, \mathsf{nil}, \mathsf{nil}, \mathsf{nil}, \mathsf{nil}))$ respectively, and we have $\mathsf{cost}_{\vec{G}}(\mathcal{S}_1) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_1})$, $\mathsf{cost}_{\vec{G}}(\mathcal{S}_2) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_2})$. The dynamic program will merge $\mathcal{S}_1$ and $\mathcal{S}_2$ to get the desired $\mathcal{S}$. Note that by the construction, for every $x \in B_w$ we have $f_1^{\mathsf{in}}(x) = f_2^{\mathsf{out}}(x)$ and $f_2^{\mathsf{in}}(x) = f_1^{\mathsf{out}}(x)$. Therefore, by the first merging phase, we can merge walks in $\mathcal{S}_1$ and $\mathcal{S}_2$. Also, we let $a = l = r = p = \mathsf{nil}$ and we proceed the second phase of merging. Finally, by the construction and definition of P-facial restriction, (T5) holds and the merging process returns a partial solution $\mathcal{S}$ compatible with $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$ with $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$, as desired.

Now suppose that P is not empty basic. Let $\mathcal{T}$, D, $\mathcal{T}_D$, k and j be as in Definition 12.2. We will prove the assertion by induction on $\mathcal{T}$. For the base case, suppose that D is a leaf of $\mathcal{T}$. Suppose that $P = x_1, x_2, \ldots, x_m$ for some $m > 0$, where $x_1 = u$ and $x_m = v$, and D is a (possibly closed) walk from $x_i \in V(P)$ to $x_j \in V(P)$. We may assume w.l.o.g. that $i \leq j$. There are some possible cases here based on i and j. First suppose that $i > 1$, $j < m$ and $j - i \geq 3$. In this case, let $P_1 = x_1, \ldots, x_i$, $P_2 = x_i, x_{i+1}$, $P_3 = x_{i+1}, \ldots, x_{j-1}$, $P_4 = x_{j-1}, x_j$ and $P_5 = x_j, \ldots, x_m$. Let $P_6 = P_1 \cup P_2$, $P_7 = P_6 \cup P_3$ and $P_8 = P_7 \cup P_4$. For $i \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ let $\mathcal{W}_{P_i}$ be the P_i-facial restriction of $\vec{W}_{\mathsf{OPT}}$ and let $\Gamma_i = \bigcup_{\vec{W} \in \mathcal{W}_{P_i}} \vec{W}$. We also define C_i, f_i^{in} and f_i^{out} as in the previous case. Note that P_1, P_3 and P_5 are empty basic paths. We let $a_1 = a_3 = a_5 = \mathsf{nil}$, $l_1 = l_3 = l_5 = \mathsf{nil}$, $r_1 = r_3 = r_5 = \mathsf{nil}$, $p_1 = p_3 = p_5 = \mathsf{nil}$ and thus we can find partial solutions $\mathcal{S}_1$, $\mathcal{S}_3$ and $\mathcal{S}_5$ compatible with $(P_1, (C_1, f_1^{\mathsf{in}}, f_1^{\mathsf{out}}, a_1, l_1, r_1, p_1))$, $(P_3, (C_3, f_3^{\mathsf{in}}, f_3^{\mathsf{out}}, a_3, l_3, r_3, p_3))$ and $(P_5, (C_5, f_5^{\mathsf{in}}, f_5^{\mathsf{out}}, a_5, l_5, r_5, p_5))$ respectively. We also have $\mathsf{cost}_{\vec{G}}(\mathcal{S}_1) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_1})$, $\mathsf{cost}_{\vec{G}}(\mathcal{S}_3) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_3})$ and $\mathsf{cost}_{\vec{G}}(\mathcal{S}_5) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_5})$. Let $a_2 = a_4 = a_6 = a_7 = a_8 = (x_i, x_j)$. If D does not have a parent in $\mathcal{T}$, then we let $l_2 = l_4 = l_6 = l_7 = l_8 = r_2 = r_4 = r_6 = r_7 = r_8 = p_2 = p_4 = p_6 = p_7 = p_8 \in V(D)$ to be an arbitrary vertex of D. Otherwise, suppose that D has a parent D' in $\mathcal{T}$. If D' does not have a parent in $\mathcal{T}$, then we let $l_2 = l_4 = l_6 = l_7 = l_8 = r_2 = r_4 = r_6 = r_7 = r_8 = p_2 = p_4 = p_6 = p_7 = p_8 \in V(D) \cap V(D')$. Otherwise, suppose that D' has a parent D'' in $\mathcal{T}$. In this case, we let $l_2 = l_4 = l_6 = l_7 = l_8 = r_2 = r_4 = r_6 = r_7 = r_8 \in V(D) \cap V(D')$ and $p_2 = p_4 = p_6 = p_7 = p_8 \in V(D') \cap V(D'')$. Therefore, by computing the initialization step, we can find a partial solution $\mathcal{S}_2$ compatible with $(P_2, (C_2, f_2^{\mathsf{in}}, f_2^{\mathsf{out}}, a_2, l_2, r_2, p_2))$ and a partial solution $\mathcal{S}_4$ compatible with $(P_4, (C_4, f_4^{\mathsf{in}}, f_4^{\mathsf{out}}, a_4, l_4, r_4, p_4))$, and we have $\mathsf{cost}_{\vec{G}}(\mathcal{S}_2) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_2})$ and $\mathsf{cost}_{\vec{G}}(\mathcal{S}_4) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_4})$. Now by merging $\mathcal{S}_1$ and $\mathcal{S}_2$, we get a partial solution $\mathcal{S}_6$ compatible with $(P_6, (C_6, f_6^{\mathsf{in}}, f_6^{\mathsf{out}}, a_6, l_6, r_6, p_6))$. By merging $\mathcal{S}_6$ and $\mathcal{S}_3$, we get a partial solution $\mathcal{S}_7$ compatible with $(P_7, (C_7, f_7^{\mathsf{in}}, f_7^{\mathsf{out}}, a_7, l_7, r_7, p_7))$. By merging $\mathcal{S}_7$ and $\mathcal{S}_4$, we get a partial solution $\mathcal{S}_8$ compatible with $(P_8, (C_8, f_8^{\mathsf{in}}, f_8^{\mathsf{out}}, a_8, l_8, r_8, p_8))$, and finally by merging $\mathcal{S}_8$ and $\mathcal{S}_5$, we get the desired partial solution $\mathcal{S}$ compatible with $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$ with $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$. If $i = 1$ or

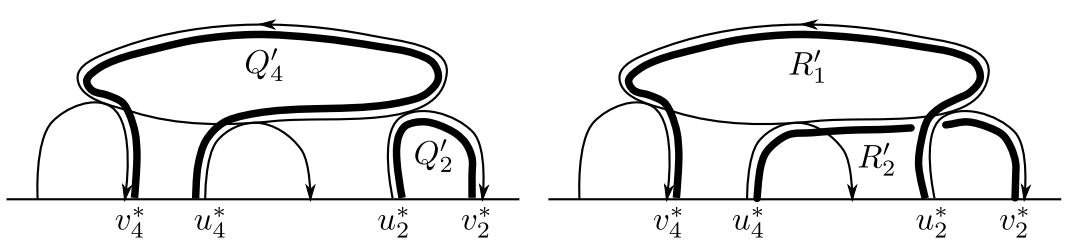
$j = m$, we will follow a similar approach. The only different is that instead of dividing P into five paths, we divide it into four paths. Finally, the last case is when $j - i < 3$. In this case, if $i \neq j$, we define the same subpaths P_1, P_2, P_4 and P_5, and we follow a similar approach. Otherwise, suppose that $i = j$. In this case, we let $P_1 = x_1, \ldots, x_i$ and $P_2 = x_i, \ldots, x_m$ and by following the same approach by mering two partial solutions, we get the desired $\mathcal{S}$.

Now suppose that $D \in V(\mathcal{T})$ is non-leaf. In this case, we prove the assertion by induction on j, where j comes from Definition 12.2. Note that we perform a second induction inside the first induction. For the base case, suppose that $j = 1$. In this case, D_1 is a child of D and P covers $\mathcal{T}_{D_1}$ and avoids $\mathcal{T}_D \setminus \mathcal{T}_{D_1}$. Therefore, by using the first induction hypothesis on D_1, there exists $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l, r, p \in V(\vec{G}) \cup \mathsf{nil}$ such that the dynamic programming table contains some partial solution $\mathcal{S}$ at location $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$ with $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$. Now for the same a, l, r, p and $\mathcal{S}$, we have that $\mathcal{S}$ is compatible with $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$, as desired. Now suppose that we have proved the assertion for all $1 \leq j' < j$. By Definition 12.2, there exists a basic path $P_1 \subseteq P$, where u is the first vertex of P_1, such that for all $l \leq j - 1$, P_1 covers $\mathcal{T}_{D_l}$ and avoids $\mathcal{T}_D \setminus (\bigcup_{m=1}^{j-1} \mathcal{T}_{D_m})$. Also there exists a basic path $P_2 \subseteq P$, where v is the last vertex of P_2, such that P_2 covers $\mathcal{T}_{D_j}$ and avoids $\mathcal{T} \setminus \mathcal{T}_{D_j}$. Let $u' \in V(F)$ and $v' \in V(F)$ be the other endpoints of P_1 and P_2 respectively. Let $P_3 \in \mathcal{P}$ be the path between u' and v' that does not contain v and let $P_4 = P_1 \cup P_3$. By the construction, P_3 and P_4 are basic. For $i \in \{1, 2, 3, 4\}$, let $\mathcal{W}_{P_i}$ be the P_i-facial restriction of $\vec{W}_{OPT}$ and let $\Gamma_i = \bigcup_{\vec{W} \in \mathcal{W}_{P_i}} \vec{W}$. Let C_1, C_2, C_3 and C_4 be the partitions of $B_u \cup B_{u'}$, $B_{v'} \cup B_v$, $B_{u'} \cup B_{v'}$ and $B_u \cup B_{v'}$ that corresponds to the weakly connected components of Γ_1, Γ_2, Γ_3 and Γ_4 respectively. For any $x \in B_u \cup B_{u'}$ let $f_1^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_{P_1}}(x)$ and $f_1^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_{P_1}}(x)$, for any $x \in B_{v'} \cup B_v$ let $f_2^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_{P_2}}(x)$ and $f_2^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_{P_2}}(x)$, for any $x \in B_{u'} \cup B_{v'}$, let $f_3^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_{P_3}}(x)$ and $f_3^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_{P_3}}(x)$, and for any $x \in B_u \cup B_{v'}$ let $f_4^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_{P_4}}(x)$ and $f_4^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_{P_4}}(x)$. By the second induction hypothesis, there exists some $a_1 \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l_1, r_1, p_1 \in V(\vec{G}) \cup \mathsf{nil}$, such that the dynamic programming table contains some partial solution $\mathcal{S}_1$ at location $(P_1, (C_1, f_1^{\mathsf{in}}, f^{\mathsf{out}}, a_1, l_1, r_1, p_1))$, with $\mathsf{cost}_{\vec{G}}(\mathcal{S}_1) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_1})$. Also by the first induction hypothesis, there exists some $a_2 \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l_2, r_2, p_2 \in V(\vec{G}) \cup \mathsf{nil}$, such that the dynamic programming table contains some partial solution $\mathcal{S}_2$ at location $(P_2, (C_2, f_2^{\mathsf{in}}, f^{\mathsf{out}}, a_2, l_2, r_2, p_2))$, with $\mathsf{cost}_{\vec{G}}(\mathcal{S}_2) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_2})$. Let $a_3 = l_3 = r_3 = p_3 = \mathsf{nil}$. Let $a_4 = a_1$, $l_4 = l_1$, $r_4 = r_1$ and $p_4 = p_1$. Since P_3 is basic, there exists a partial solution $\mathcal{S}_3$ compatible with $(P_3, (C_3, f_3^{\mathsf{in}}, f_3^{\mathsf{out}}, a_3, l_3, r_3, p_3))$. Now we merge $\mathcal{S}_1$ and $\mathcal{S}_3$ to get a partial solution $\mathcal{S}_4$ compatible with $(P_4, (C_4, f_4^{\mathsf{in}}, f_4^{\mathsf{out}}, a_4, l_4, r_4, p_4))$. Note that for every $x \in B_{u'}$, we have $f_3^{\mathsf{in}}(x) = f_1^{\mathsf{out}}(x)$ and $f_3^{\mathsf{out}}(x) = f_1^{\mathsf{in}}(x)$. Therefore, we can apply the first merging phase. Also we have $a_4 = a_1$, $l_4 = l_1$, $r_4 = r_1$ and $p_4 = p_1$, and thus we can apply the second merging phase. Finally, by the construction (T5) holds and we get a partial solution $\mathcal{S}_4$ compatible with $(P_4, (C_4, f_4^{\mathsf{in}}, f_4^{\mathsf{out}}, a_4, l_4, r_4, p_4))$. Now, we merge two partial solutions $\mathcal{S}_4$ and $\mathcal{S}_2$ to get the desired $\mathcal{S}$. Clearly, for every $x \in B_{v'}$ we have $f_4^{\mathsf{in}}(x) = f_2^{\mathsf{out}}(x)$ and $f_4^{\mathsf{out}}(x) = f_2^{\mathsf{in}}(x)$. Therefore, we can apply the first phase of merging. If $a_2 = l_2 = r_2 = p_2 = a_4 = l_4 = r_4 = p_4 = \mathsf{nil}$, then we let $a = l = r = p = \mathsf{nil}$. Otherwise, if $a_4 = l_4 = r_4 = p_4 = \mathsf{nil}$ and $a_2 = (u_2^*, v_2^*) \in \mathcal{A}$ with $\{u_2^*, v_2^*\} \cap V(P_4) \subseteq \{u, v'\}$, then we let $a = a_2$, $l = l_2$, $r = r_2$ and $p = p_2$. If $a_2 = l_2 = r_2 = p_2 = \mathsf{nil}$ and $a_4 = (u_4^*, v_4^*)$ with $\{u_4^*, v_4^*\} \cap V(P_2) \subseteq \{v', v\}$, then we let $a = a_4$, $l = l_4$, $r = r_4$ and $p = p_4$. Otherwise, if $a_2 \neq \mathsf{nil}$, $a_4 \neq \mathsf{nil}$, $a_4 = a_2$, $l_4 = l_2$, $r_4 = r_2$ and $p_4 = p_2$, then we let $a = a_4$, $l = l_4$, $r = r_4$ and $p = p_4$. Otherwise, if $a_2 = (u_2^*, v_2^*) \in \mathcal{A}$, $a_4 = (u_4^*, v_4^*) \in \mathcal{A}$, $l_4 = r_4$ and $p_1 = p_2$, then we let $a = (u^*, v^*)$, where $u^* \in \{u_2^*, u_4^*\}$ and $v^* \in \{v_2^*, v_4^*\}$, $l = l_2$, $r = r_4$ and $p = p_2$. Otherwise, if

$a_2 = \mathsf{nil}$ and $a_4 \in (\mathcal{A} \times \mathcal{A})$, then we let $a = a_4$, $l = l_4$, $r = r_4$ and $p = p_4$. Otherwise, if $a_4 = \mathsf{nil}$ and $a_2 \in (\mathcal{A} \times \mathcal{A})$, then we let $a = a_2$, $l = l_2$, $r = r_2$ and $p = p_2$. Otherwise, if $a_2 = (u_2^*, v_2^*) \in \mathcal{A}$, $a_4 = (u_4^*, v_4^*) \in \mathcal{A}$, $l_2 = r_2$, $l_4 = r_4$ and $p_2 = p_4$, then we let $a = ((u', v'), (u'', v''))$ where $v' \in \{v_2, v_4\}$ and $u'' \in \{u_2, u_4\}$, $l = l_2$, $r = r_2$ and $p = p_2$. Otherwise, if $a_4 = ((u_4, v_4), (u_4', v_4')) \in (\mathcal{A} \times \mathcal{A})$, $a_2 = (u_2, v_2) \in \mathcal{A}$, $l_2 = r_2$ and $p_2 = p_4$, then we let $a = ((u_4, v_4), (u_2, v_4'))$, $l = l_4$, $r = r_2$ and $p = p_2$. Therefore, after applying the second merging phase, we get a partial solution $\mathcal{S}$ compatible with $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$.

Now we have to show that $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$. Let us first suppose that D is a closed walk. We will deal with the case where D is an open walk later on. If $a_2 = l_2 = r_2 = p_2 = \mathsf{nil}$ or $a_4 = l_4 = r_4 = p_4 = \mathsf{nil}$, then this is immediate. If $a_2 = a_4$, $l_2 = l_4$, $r_2 = r_4$ and $p_2 = p_4$, then we have $a = a_2$, $l = l_2$, $r = r_2$ and $p = p_2$, and thus this case is also immediate. Suppose that $a_2 = (u_2^*, v_2^*) \neq \mathsf{nil}$, $a_4 = (u_4^*, v_4^*) \neq \mathsf{nil}$, $l_4 = r_4$ and $p_1 = p_2$, and $a = (u^*, v^*)$, where $u^* \in \{u_2^*, u_4^*\}$ and $v^* \in \{v_2^*, v_4^*\}$. We may assume w.l.o.g that $a = (u_2^*, v_4^*)$. We have that $l = l_2$, $r = r_4$ and $p = p_2$. For $i \in \{2, 4\}$ let Q_i^* be the grip of $\mathcal{S}_i$, and let $\vec{W}_i \in \mathcal{S}_i$ that contains Q_i^* as a sub-walk. Let Y_1 be the shortest-path in $\vec{G}$ from u_2^* to l_2. Let Y_2 be the shortest-path in $\vec{G}$ from l_2 to l_4. Let Y_3 be the shortest-path in $\vec{G}$ from l_4 to v_4^*. Let R_1^* be the path in $\vec{G}$ from u_2^* to v_4^* obtained by concatenation of Y_1, Y_2 and Y_3. Let Y_1' be the shortest-path in $\vec{G}$ from u_4^* to r_4. Let Y_2' be the shortest-path in $\vec{G}$ from r_4 to l_2. Let Y_3' be the shortest-path in $\vec{G}$ from l_2 to v_2^*. Let R_2^* be the path in $\vec{G}$ from u_4^* to v_2^* obtained by concatenation of Y_1', Y_2' and Y_3'. Now by the construction, we have that $\mathsf{cost}_{\vec{G}}(\mathcal{S}) = \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(R_1^*) + \mathsf{cost}_{\vec{G}}(R_2^*) - \mathsf{cost}_{\vec{G}}(Q_2^*) - \mathsf{cost}_{\vec{G}}(Q_4^*)$. Note that by the construction, there exists a P-important walk from u_2^* to v_4^* (w.r.t. $\mathcal{W}$) and a P-important walk from u_4^* to v_2^* (w.r.t. $\mathcal{W}$). By Proposition 12.5 these walks are unique. Let R_1' be the P-important walk from u_2^* to v_4^* (w.r.t. $\mathcal{W}$) and let R_2' be the P-important walk from u_4^* to v_2^* (w.r.t. $\mathcal{W}$). Also, there exists a P_2-important walk from u_2^* to v_2^* (w.r.t. $\mathcal{W}$) and a P_4-important walk from u_4^* to v_4^* (w.r.t. $\mathcal{W}$), and thus by Proposition 12.5 these walks are unique. Let Q_2' be the P_2-important walk from u_2^* to v_2^* (w.r.t. $\mathcal{W}$) and let Q_4' be the P_4-important walk from u_4^* to v_4^* (w.r.t. $\mathcal{W}$). By the definition of important walks, we have that $\mathsf{cost}_{\vec{G}}(R_1^*) \leq \mathsf{cost}_{\vec{G}}(R_1')$, $\mathsf{cost}_{\vec{G}}(R_2^*) \leq \mathsf{cost}_{\vec{G}}(R_2')$, $\mathsf{cost}_{\vec{G}}(Q_2^*) \leq \mathsf{cost}_{\vec{G}}(Q_2')$ and $\mathsf{cost}_{\vec{G}}(Q_4^*) \leq \mathsf{cost}_{\vec{G}}(Q_4')$. Also by the construction, we have that $\mathsf{cost}_{\vec{G}}(R_1') + \mathsf{cost}_{\vec{G}}(R_2') = \mathsf{cost}_{\vec{G}}(Q_2') + \mathsf{cost}_{\vec{G}}(Q_4')$. Therefore, we have

$$\begin{aligned}
\mathsf{cost}_{\vec{G}}(\mathcal{S}) &= \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(R_1^*) + \mathsf{cost}_{\vec{G}}(R_2^*) - \mathsf{cost}_{\vec{G}}(Q_2^*) - \mathsf{cost}_{\vec{G}}(Q_4^*) \\
&\leq \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(R_1') + \mathsf{cost}_{\vec{G}}(R_2') - \mathsf{cost}_{\vec{G}}(Q_2^*) - \mathsf{cost}_{\vec{G}}(Q_4^*) \\
&= (\mathsf{cost}_{\vec{G}}(\mathcal{S}_2) - \mathsf{cost}_{\vec{G}}(Q_4^*)) + (\mathsf{cost}_{\vec{G}}(\mathcal{S}_4) - \mathsf{cost}_{\vec{G}}(Q_2^*)) + \mathsf{cost}_{\vec{G}}(R_1') + \mathsf{cost}_{\vec{G}}(R_2') \\
&\leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_1}) + \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_2}) \\
&= \mathsf{cost}_{\vec{G}}(\mathcal{W}_P).
\end{aligned}$$



Now suppose that D is an open walk. If $a_2 = \mathsf{nil}$ and $a_4 \in (\mathcal{A} \times \mathcal{A})$, or $a_4 = \mathsf{nil}$ and $a_2 \in (\mathcal{A} \times \mathcal{A})$,

then this is immediate. If $a_2 = (u_2^*, v_2^*) \in \mathcal{A}$, $a_4 = (u_4^*, v_4^*) \in \mathcal{A}$, $l_2 = r_2$, $l_4 = r_4$ and $p_2 = p_4$, then we have $a = ((u', v'), (u'', v''))$ where $v' \in \{v_2^*, v_4^*\}$ and $u'' \in \{u_2^*, u_4^*\}$. We may assume w.l.o.g that $v' = v_2^*$ and $u'' = u_4^*$. Then we have $l = l_2$, $r = r_2$ and $p = p_2$. Let R_1 be the shortest-path in $\vec{G}$ from u' to l_2. Let R_2 be the shortest-path in $\vec{G}$ from l_1 to v'. Let R' be the path from u' to v' obtained by the concatenation of R_1 and R_2. Let Y_1 be the shortest path in $\vec{G}$ from u'' to l_4. Let Y_2 be the shortest path in $\vec{G}$ from l_4 to v''. Let Y' be the path from u'' to v'' obtained by the concatenation of Y_1 and Y_2. Let Z_1 be the shortest path in $\vec{G}$ from u_2^* to l_2. Let Z_2 be the shortest path in $\vec{G}$ from l_2 to l_4. Let Z_3 be the shortest path in $\vec{G}$ from l_4 to v_4^*. Let Z' be the path from u_2^* to v_4^* obtained by the concatenation of Z_1, Z_2 and Z_3. Let Q_2^* be the grip of $\mathcal{S}_2$ and let Q_4^* be the grip of $\mathcal{S}_4$. By the construction, we have that $\mathsf{cost}_{\vec{G}}(\mathcal{S}) = \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(R') + \mathsf{cost}_{\vec{G}}(Y') + \mathsf{cost}_{\vec{G}}(Z') - \mathsf{cost}_{\vec{G}}(Q_2^*) - \mathsf{cost}_{\vec{G}}(Q_4^*)$. By the construction, there exists a P-important walk from u_2^* to v_4^*, a P-important walk from u' to v_2^*, and a P-important walk from u_4^* to v''. Therefore by Proposition 12.5, these important walks are unique. Let R'', Y'' and Z'' be the important walks from u' to v', u'' to v'', and u_2^* to v_4^* respectively. Therefore, we have that

$$\begin{aligned}
\mathsf{cost}_{\vec{G}}(\mathcal{S}) &= \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(R') + \mathsf{cost}_{\vec{G}}(Y') + \mathsf{cost}_{\vec{G}}(Z') - \mathsf{cost}_{\vec{G}}(Q_2^*) - \mathsf{cost}_{\vec{G}}(Q_4^*) \\
&\leq \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(R'') + \mathsf{cost}_{\vec{G}}(Y'') + \mathsf{cost}_{\vec{G}}(Z'') - \mathsf{cost}_{\vec{G}}(Q_2^*) - \mathsf{cost}_{\vec{G}}(Q_4^*) \\
&\leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_1}) + \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_2}) \\
&= \mathsf{cost}_{\vec{G}}(\mathcal{W}_P).
\end{aligned}$$

If $a_4 = ((u_4, v_4), (u_4', v_4')) \in (\mathcal{A} \times \mathcal{A})$, $a_2 = (u_2, v_2) \in \mathcal{A}$, $l_2 = r_2$ and $p_2 = p_4$, then we have $a = ((u_4, v_4), (u_2, v_4'))$, $l = l_4$, $r = r_2$ and $p = p_2$. Let (Q_4^*, Q_4^{**}) be the grip of $\mathcal{S}_4$, and let Q_2^* be the grip of $\mathcal{S}_2$. We may assume w.l.o.g that Q_4^* is a path from u_4 to v_4, and Q_4^{**} is a path from u_4' to v_4'. Let Y_1 be the shortest path in $\vec{G}$ from u_2 to l_2. Let Y_2 be the shortest path in $\vec{G}$ from l_2 to v_4'. Let Y' be the path from u_2 to v_4' obtained by the concatenation of Y_1 and Y_2. Let Z_1 be the shortest path in $\vec{G}$ from u_4' to r_4. Let Z_2 be the shortest path in $\vec{G}$ from r_4 to l_2. Let Z_3 be the shortest path in $\vec{G}$ from l_2 to v_2. Let Z' be the path from u_4' to v_2 obtained by the concatenation of Z_1, Z_2 and Z_3. By the construction, we have that $\mathsf{cost}_{\vec{G}}(\mathcal{S}) = \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(Y') + \mathsf{cost}_{\vec{G}}(Z') - \mathsf{cost}_{\vec{G}}(Q_2^*) - cost_{\vec{G}}(Q_4^{**})$. By the construction, there exists a P-important walk from u_4' to v_2, and a P-important walk from u_2 to v_4'. Therefore by Proposition 12.5, these important walks are unique. Let Y'' and Z'' be the important walks from u_2 to v_4', and u_4' to v_2 respectively. Therefore we have

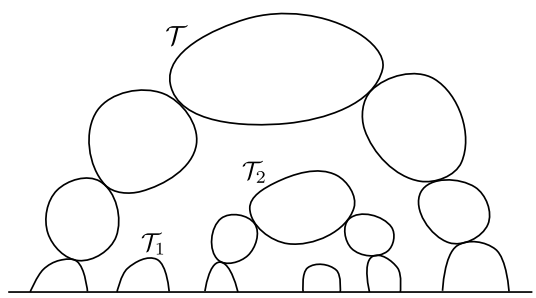
$$\begin{aligned}
\mathsf{cost}_{\vec{G}}(\mathcal{S}) &= \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(Y') + \mathsf{cost}_{\vec{G}}(Z') - \mathsf{cost}_{\vec{G}}(Q_2^*) - cost_{\vec{G}}(Q_4^{**}) \\
&\leq \mathsf{cost}_{\vec{G}}(\mathcal{S}) = \mathsf{cost}_{\vec{G}}(\mathcal{S}_2) + \mathsf{cost}_{\vec{G}}(\mathcal{S}_4) + \mathsf{cost}_{\vec{G}}(Y'') + \mathsf{cost}_{\vec{G}}(Z'') - \mathsf{cost}_{\vec{G}}(Q_2^*) - cost_{\vec{G}}(Q_4^{**}) \\
&\leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_1}) + \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_2}) \\
&= \mathsf{cost}_{\vec{G}}(\mathcal{W}_P).
\end{aligned}$$

Now suppose that $\mathcal{F}$ contains more than one tree. Let $A = \{\mathcal{T}_1, \ldots, \mathcal{T}_m\}$ be the set of all trees in $\mathcal{F}$. For every $\mathcal{T} \in A$ we define the *level* of $\mathcal{T}$, $L(\mathcal{T})$, as follows. Let D be the root of $\mathcal{T}$. We set $L(\mathcal{T}) = 0$, if there exists a basic path P' that covers $\mathcal{T}$ and avoids all $\mathcal{T}' \in \mathcal{F} \setminus \{\mathcal{T}\}$. We call a minimal such path, a *corresponding* basic path for $\mathcal{T}$ and we denote it by $P_{\mathcal{T}}$; it is immediate

that there is a unique such minimal path. Let $\mathcal{F}_0 = \{\mathcal{T} \in \mathcal{F} : L(\mathcal{T}) = 0\}$. Now for $i \geq 0$, suppose that we have defined trees of level i and $\mathcal{F}_i$. Suppose that $L(\mathcal{T}) \notin \{0, \ldots, i\}$. We set $L(\mathcal{T}) = i + 1$ if there exists a basic path P' that covers $\mathcal{T}$ such that for all $\mathcal{T}' \in \bigcup_{j=0}^{i} \mathcal{F}_j$, P' either avoids or covers $\mathcal{T}'$, and for all $\mathcal{T}'' \in (\mathcal{F} \setminus \{\mathcal{T}\}) \setminus \bigcup_{j=0}^{i} \mathcal{F}_j$, P' avoids $\mathcal{T}''$. We also call a minimal such path corresponding basic for $\mathcal{T}$ and we denote it by $P_{\mathcal{T}}$. Let $\mathcal{F}_{i+1} = \{\mathcal{T} \in \mathcal{F} : L(\mathcal{T}) = i + 1\}$.

We say that some $\mathcal{T} \in \mathcal{F}$ is *outer-most* if there is no $\mathcal{T}' \in \mathcal{F}$ such that $L(\mathcal{T}') > L(\mathcal{T})$ and $P_{\mathcal{T}} \subset P_{\mathcal{T}'}$.

Let us first suppose that there exists only one outer-most tree $\mathcal{T} \in \mathcal{F}$, such that $P_{\mathcal{T}} \subseteq P$. We will deal with the more general case later. Also, suppose that $\mathcal{T} \in \mathcal{F}_m$ for some $m \geq 0$. We will prove the assertion by induction on m. We also prove that for this case, we have $a = l = r = p = \mathsf{nil}$. For the base case, if $m = 0$, then by the construction, $\mathcal{T}$ is the only tree with a corresponding basic path $P_{\mathcal{T}} \subseteq P$. For this case, we have already established the assertion and we are done. Now suppose that we have proved the assertion for all $m' < m$. Let $\mathcal{F}' = \mathcal{F} \setminus \{\mathcal{T}\}$. Let $\mathcal{T}_1, \ldots, \mathcal{T}_t \in \mathcal{F}'$ be all outer-most trees in $\mathcal{F}'$, such that for $j \in \{1, \ldots, t\}$ we have $P_{\mathcal{T}_j} \subseteq P$, and they intersect F in this order. By the construction, for every $j \in \{1, \ldots, t\}$ we have $L(\mathcal{T}_j) < m$. For every $j \in \{1, \ldots, t\}$ let $P_{\mathcal{T}_j}$ be a corresponding basic path for $\mathcal{T}_j$. By the induction hypothesis, for every $j \in \{1, \ldots, t\}$ there exists a partial solution $\mathcal{S}_j$ for $P_{\mathcal{T}_j}$. Now, we can apply the same argument when we had only one tree $\mathcal{T} \in \mathcal{F}$ for $\mathcal{T}$. Note that for each $j \in \{1, \ldots, t\}$, we have $a_j = l_j = r_j = p_j = \mathsf{nil}$. The only difference is that the intermediate basic paths here are not necessarily empty basic paths, and each $\mathcal{T}_j$ appears as an intermediate basic path, with a partial solution $\mathcal{S}_j$. Therefore, by following a similar approach to the previous cases and merging appropriately we get a partial solution $\mathcal{S}$, as desired.



Now, suppose that there exist more than one outer-most trees $\mathcal{T} \in \mathcal{F}$, where $P_{\mathcal{T}} \subseteq P$. Let $B = \{\mathcal{T}_1, \ldots, \mathcal{T}_t\}$ be the set of all such trees, where they intersect consecutive subpaths of P in this order. In this case, we prove the assertion by induction on t. For the base case where $t = 1$, we have already proved the assertion. Now suppose that we have proved the assertion for all $t' < t$. Let $B' = \{\mathcal{T}_1, \ldots, \mathcal{T}_{t-1}\}$. Let $P' \subseteq P$ be the subpath of P such that B' is the set of all outer-most trees, with $P_{\mathcal{T}_j} \subseteq P'$ for all $1 \leq j \leq t - 1$. By the induction hypothesis, there exists a partial solution $\mathcal{S}'$ for P'. Let $P'' = P \subseteq P'$. By the construction, P'' is a corresponding basic path for $\mathcal{T}_t$, and thus by the induction hypothesis, there exists a partial solution S'' for P''. Therefore, by merging $\mathcal{S}'$ and $\mathcal{S}''$ we get a partial solution $\mathcal{S}$ for P, as desired. □

Theorem 12.7. *Let $\vec{G}$ be an n-vertex $(0, 0, 1, p)$-nearly embeddable graph (that is, planar with a single vortex) and let $\vec{H}$ be the vortex of $\vec{G}$. Then there exists an algorithm which computes a walk $\vec{W}$ visiting all vertices in $V(\vec{H})$ of total length $\mathsf{OPT}_{\vec{G}}(V(\vec{H}))$ in time $n^{O(p)}$.*

Proof. We have $F \in \mathcal{P}$ and it is immediate to check that F is basic. Since F is a cycle, both the endpoitns of F are some vertex v°. It follows by Lemma 12.6 that there exists some $\phi \in \mathcal{I}_F$ such that the dynamic programming table contains a partial solution $\mathcal{S}$ at location (F, ϕ). Let $\phi = (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p)$. It follows by condition (T4) that for all $x \in B_{v^\circ}$ we have $\mathsf{in\text{-}degree}_{\mathcal{S}}(x) = \mathsf{out\text{-}degree}_{\mathcal{S}}(x)$. Thus by repeatedly merging pairs of walks that respectively terminate to and start from the same vertex, we obtain a collection $\mathcal{S}'$ of closed walks with $\mathsf{cost}_{\vec{G}}(\mathcal{S}') = \mathsf{cost}_{\vec{G}}(\mathcal{S})$ that visit all vertices in $V(\vec{H})$. By Lemma 11.3 we can assume that C is the trivial partition containing only one cluster, that is $C = \{\{B_{v^\circ}\}\}$. Since C is trivial, it follows by (T5) that all vertices in $V(\vec{H})$ are in the same weakly-connected component of $\bigcup_{W \in \mathcal{S}} W$. Thus all vertices in $V(\vec{H})$ are in the same strongly-connected component of $\bigcup_{W \in \mathcal{S}'} W$. Since all walks in $\mathcal{S}'$ are closed, by repeatedly shortcutting pairs of intersecting walks, we obtain a walk $\vec{W}$ with that visits all vertices in $V(\vec{H})$ with $\mathsf{cost}_{\vec{G}}(\vec{W}) = \mathsf{cost}_{\vec{G}}(\mathcal{S}') = \mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\vec{W}_{\mathsf{OPT}}) = \mathsf{OPT}_{\vec{G}}$.

The running time is polynomial in the size of the dynamic programming table, which is at most $|\mathcal{P}| \cdot \max_{P \in \mathcal{P}} |\mathcal{I}_P| = O(n^2) \cdot p^{O(p)} \cdot n^{O(p)} \cdot O(n^2) = n^{O(p)}$. □

13 The dynamic program for traversing a vortex in a bounded genus graph

For the remainder of this section let $\vec{G}$ be a n-vertex $(0, g, 1, p)$-nearly embeddable graph. Let $\vec{H}$ be the vortex in $\vec{G}$, attached to some face $\vec{F}$. Let $\vec{G}' = \vec{G} \setminus (V(\vec{H}) \setminus (\vec{F}))$ and fix some embedding ψ of $\vec{G}'$ on a surface S of genus g. Let F be the symmetrization of $\vec{F}$. Let $\vec{W}_{\mathsf{OPT}}$ be a closed walk in $\vec{G}$ that visits all vertices in $\vec{H}$ with minimum $\mathsf{cost}_{\vec{G}}(\vec{W})$. Fix a path-decomposition $\{B_v\}_{v \in V(F)}$ of $\vec{H}$ of width p. We present a similar algorithm as in Section 12 for computing a walk traversing all vertices in $V(\vec{H})$ based on dynamic programming. By Lemma 10.3 we may assume w.l.o.g. that $\vec{G}$ is facially normalized and cross normalized.

13.1 The dynamic program

Let $\mathcal{Q}$ be the set of all subpaths of F, where we allow a path to be closed. For every integer m, let

$$\mathcal{P}_m = \{A \subseteq \mathcal{Q} : |A| \leq m, \text{ for every } Q, Q' \in A \text{ we have } V(Q) \cap V(Q') = \emptyset\}.$$

And let

$$\mathcal{P}_\infty = \{A \subseteq \mathcal{Q} : \text{ For every } Q, Q' \in A \text{ we have } V(Q) \cap V(Q') = \emptyset\}.$$

For every $P = \{Q_1, \ldots, Q_m\} \in \mathcal{P}_\infty$, let $E(P) = \bigcup_{i=1}^m E(Q_i)$ and let $V(P) = \bigcup_{i=1}^m V(Q_i)$. For each $i \in \{1, \ldots, m\}$, let u_i and v_i be the endpoints of Q_i. Similar to the planar case, let

$$\vec{H}_P = \vec{H}\left[\bigcup_{x \in V(P)} B_x\right].$$

Let $\mathcal{B} = \bigcup_{i=1}^m (B_{u_i} \cup B_{v_i})$. Let $\mathcal{C}_P$ be the set of all possible partitions of $\mathcal{B}$. Let $\mathcal{D}_P^{\mathsf{in}} = \{0, \ldots, n\}^{\mathcal{B}}$, $\mathcal{D}_P^{\mathsf{out}} = \{0, \ldots, n\}^{\mathcal{B}}$, that is, every element of $\mathcal{D}_P^{\mathsf{in}} \cup \mathcal{D}_P^{\mathsf{out}}$ is a function $f : \mathcal{B} \to \{0, \ldots, n\}$.

13.1.1 The dynamic programming table.

Let $\mathcal{P} = \mathcal{P}_{324000g^4}$. With these definitions, the dynamic programming table is indexed the exact same way as in the planar case. Also, a *partial solution* is again a collection of walks in $\vec{G}$.

We say that a partial solution $\mathcal{S}$ is *compatible* with (P, ϕ) if the same conditions (T1)-(T5) as in the planar case are satisfied. The only difference here is that instead of $B_u \cup B_v$, we have $\mathcal{B}$.

Merging partial solutions. We follow a similar approach as in the planar case. Let $P = \{Q_1, \ldots, Q_m\}, P_1 = \{Q'_1, \ldots, Q'_{m'}\}, P_2 = \{Q''_1, \ldots, Q''_{m''}\} \in \mathcal{P}$ such that $E(P_1) \neq \emptyset$, $E(P_2) \neq \emptyset$, $E(P_1) \cap E(P_2) = \emptyset$, and $E(P) = E(P_1) \cup E(P_2)$. Let $\phi = (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p) \in \mathcal{I}_P$, $\phi_1 = (C_1, f_1^{\mathsf{in}}, f_1^{\mathsf{out}}, a_1, l_1, r_1, p_1) \in \mathcal{I}_{P_1}$, $\phi_2 = (C_2, f_2^{\mathsf{in}}, f_2^{\mathsf{out}}, a_2, l_2, r_2, p_2) \in \mathcal{I}_{P_2}$.

Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be partial solutions compatible with (P_1, ϕ_1) and (P_2, ϕ_2) respectively. Similar to the planar case, we compute a partial solution $\mathcal{S}$ compatible with (P, ϕ) as follows.

Merging phase 1: Joining the walks. For every $w \in V(P_1) \cap V(P_2)$, we check that for all $x \in B_w$ we have $f_1^{\mathsf{in}}(x) = f_2^{\mathsf{out}}(x)$ and $f_2^{\mathsf{in}}(x) = f_1^{\mathsf{out}}(x)$. If not then the merging procedure returns nil. Otherwise, for every $w \in V(P_1) \cap V(P_2)$ and every $x \in B_w$, we follow the exact same approach as in the planar case.

Merging phase 2: Updating the grip. This phase is exactly the same as in the planar case.

Merging phase 3: Checking connectivity. Similar to the planar case, we check that condition (T5) holds for $\mathcal{S}$ and we return nil if it does not.

13.1.2 Initializing the dynamic programming table.

For all $P \in \mathcal{P}_1$ with $|E(P)| \leq 1$, we follow the same approach as in the planar case.

13.1.3 Updating the dynamic programming table.

For all $P \in \mathcal{P}$ with $|E(P)| > 1$, and for all $P_1, P_2 \in \mathcal{P}$ with $E(P_1) \neq \emptyset$, $E(P_2) \neq \emptyset$, $E(P_1) \cap E(P_2) = \emptyset$ and $E(P_1) \cup E(P_2) = E(P)$, and for all $\phi_1 \in \mathcal{I}_{P_1}$ and $\phi_2 \in \mathcal{I}_{P_2}$ we proceed as follows. Suppose that for all $P' \in \mathcal{P}$ with $|E(P')| < |E(P)|$ and all $\phi' \in \mathcal{I}_{P'}$, we have computed the partial solutions in the dynamic programming table at (P', ϕ'). Now similar to the planar case, if there exists partial solutions $\mathcal{S}_1$ and $\mathcal{S}_2$ at (P_1, ϕ_1) and (P_2, ϕ_2) respectively, we call the merging process to (possibly) get a partial solution $\mathcal{S}$ at (P, ϕ) for some $\phi \in \mathcal{I}_P$. Now similar to the planar case, if there is no partial solution at (P, ϕ) then we store $\mathcal{S}$ at (P, ϕ). Otherwise if there there exists a partial solution $\mathcal{S}'$ stored at (P, ϕ) and $\mathsf{cost}_{\vec{G}}(\mathcal{S}) < \mathsf{cost}_{\vec{G}}(\mathcal{S}')$ then we replace $\mathcal{S}'$ with $\mathcal{S}$.

13.2 Analysis

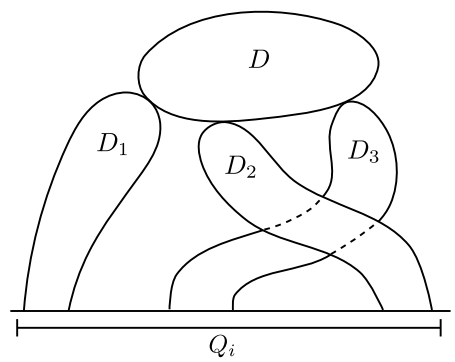
Let $\mathcal{W}$ be the collection of walks given by Lemma 11.2. Let $\mathcal{F}$ be the forest given by Lemma 11.3. Let $\mathcal{T}$ be a subtree of $\mathcal{F}$. We say that $\mathcal{T}$ is *trivial* if $\psi(F) \cup \bigcup_{D \in V(\mathcal{T})} \psi(D)$ is contractible. Otherwise, we say that $\mathcal{T}$ is *non-trivial*.

Let $Q \in \mathcal{Q}$, and let $\mathcal{T}$ be a subtree of some tree in $\mathcal{F}$. We define the terms Q *covers* $\mathcal{T}$ and Q *avoids* $\mathcal{T}$ the exact same way as in the planar case. Let $P = \{Q_1, \ldots, Q_m\} \in \mathcal{P}_\infty$. We say that P *covers* $\mathcal{T}$ if for all $D \in V(\mathcal{T})$ we have $V(D) \cap V(F) \subseteq V(Q_1 \cup \ldots \cup Q_m)$. We say that P *avoids* $\mathcal{T}$ if for all $Q_i \in P$ we have that Q_i avoids $\mathcal{T}$.

Definition 13.1 (Basic family of paths)**.** *Let $P = \{Q_1, \dots, Q_m\} \in \mathcal{P}_\infty$. For each $i \in \{1, \dots, m\}$, let u_i and v_i be the endpoints of Q_i. We say that P is* basic *(w.r.t. $\mathcal{W}$) if either $V(P) \setminus (\bigcup_{i=1}^m \{u_i, v_i\})$ does not intersect any of the walks in $\mathcal{W}$ (in which case we call it* empty basic*) or there exists $\mathcal{T} \in \mathcal{F}$ and $D \in V(\mathcal{T})$, with children $D_1, \dots, D_k$, intersecting D in this order along a traversal of D, such that the exact same conditions (1) & (2) as in Definition 12.2 hold, and $|P|$ is minimal subject to the following:*

(3) If P covers $\mathcal{T}_D$ and avoids $\mathcal{T} \setminus \mathcal{T}_D$, let $\mathcal{T}[P] = \mathcal{T}_D$, and otherwise let $\mathcal{T}[P] = \bigcup_{i=1}^j \mathcal{T}_{D_i}$. For every two disjoint subtrees $\mathcal{T}_1$ and $\mathcal{T}_2$ of $\mathcal{T}[P]$, the following holds. If there exists $Q_i \in P$ such that Q_i covers $\mathcal{T}_1 \cup \mathcal{T}_2$ and avoids $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$, then there exist edge-disjoint subpaths Q_i', Q_i'' of Q_i such that $Q_i = Q_i' \cup Q_i''$, Q_i' covers $\mathcal{T}_1$ and avoids $\mathcal{T} \setminus \mathcal{T}_1$, and Q_i'' covers $\mathcal{T}_2$ and avoids $\mathcal{T} \setminus \mathcal{T}_2$.

Moreover if for each non-trivial tree $\mathcal{T}'$ in $\mathcal{F}$ with $\mathcal{T}' \neq \mathcal{T}$, we have that P avoids $\mathcal{T}'$, then we say that P is elementary*. Furthermore, if for each non-trivial tree $\mathcal{T}'$ in $\mathcal{F}$, we have that P avoids $\mathcal{T}'$, then we say that P is* trivial*.*



Definition 13.2 (Twins)**.** *Let $P, P' \in \mathcal{P}_\infty$. We say that P and P' are* twins *if for each subtree $\mathcal{T}$ of $\mathcal{F}$, P covers $\mathcal{T}$ if and only if P' covers $\mathcal{T}$, and P avoids $\mathcal{T}$ if and only if P' avoids $\mathcal{T}$.*

Definition 13.3 (Succinctness)**.** *Let $P = \{Q_1, \dots, Q_m\} \in \mathcal{P}_\infty$ be basic. For each $i \in \{1, \dots, m\}$, let u_i and v_i be the endpoints of Q_i, let $Q_i' = Q_i \setminus \{u_i\}$ and let $Q_i'' = Q_i \setminus \{v_i\}$. We say that P is* succinct *if for all $i \in \{1, \dots, m\}$, P and $\{Q_1, \dots, Q_{i-1}, Q_i', Q_{i+1}, \dots, Q_m\}$ are not twins, and moreover P and $\{Q_1, \dots, Q_{i-1}, Q_i'', Q_{i+1}, \dots, Q_m\}$ are not twins.*

Lemma 13.4. *Let $P \in \mathcal{P}_\infty$ be non-empty basic. Then there exists some succinct and basic $P' \in \mathcal{P}_\infty$ such that P and P' are twins with $E(P') \subseteq E(P)$.*

Proof. It is immediate by Definition 13.3. □

Lemma 13.5. *Let $P \in \mathcal{P}_\infty$ be non-empty basic elementary and succinct. Let $\mathcal{T}$, D, j, $D_1, \dots, D_j$ be as in Definition 13.1. Then there exist $P_1, P_2 \in \mathcal{P}_\infty$ satisfying the following conditions:*

(1) P_1 and P_2 are non-empty basic elementary and succinct.

(2) P_1 covers $\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i}$ and avoids $\mathcal{T} \setminus (\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i})$.

(3) P_2 covers $\mathcal{T}_{D_j}$ and avoids $\mathcal{T} \setminus \mathcal{T}_{D_j}$.

(4) $E(P_1) \subseteq E(P)$*,* $E(P_2) \subseteq E(P)$*, and* $E(P_1) \cap E(P_2) = \emptyset$*.*

Proof. We begin by defining auxiliary $Z_1, Z_2 \in \mathcal{P}_\infty$. The desired P_1 and P_2 will be succinct twins of Z_1 and Z_2. First we define Z_1. Initially, we set $Z_1 = P$ and we inductively modify Z_1 until it covers $C_1 = \bigcup_{i=1}^{j-1} \mathcal{T}_{D_i}$ and avoids $A_1 = \mathcal{T} \setminus (\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i})$ as follows: If Z_1 contains a path Q that does not intersect C_1 then we remove Q from Z_1. If Z_1 contains a path Q that intersects both C_1 and A_1 then we proceed as follows: let R be the collection of paths obtained from Q by deleting all vertices in $A_1 \cap Q$; let R' be the collection obtained from R by removing all paths that do not intersect C_1; let R'' be the collection obtained from R' by replacing each $Q' \in R'$ be the minimal subpath $Q'' \subseteq Q'$ with $V(Q'') \cap C_1 = V(Q') \cap C_1$. We repeat the above process until the resulting Z_1 covers $C_1 = \bigcup_{i=1}^{j-1} \mathcal{T}_{D_i}$ and avoids $A_1 = \mathcal{T} \setminus (\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i})$. In a similar fashion we define Z_2 that covers $C_2 = \mathcal{T}_{D_j}$ and avoids $A_2 = \mathcal{T} \setminus \mathcal{T}_{D_j}$. It is immediate by construction that $E(Z_1) \subset E(P)$, $E(Z_2) \subset E(P)$ and $E(Z_1) \cap E(Z_2) = \emptyset$.

Next we argue that Z_1 is basic. It is immediate that conditions (1) & (2) of Definition 13.1 are satisfied. It remains to establish condition (3) of Definition 13.1. Let $Q \in P_1$. By construction there exists $Q' \in P$ such that $Q \subseteq Q'$. Let $\mathcal{T}[Z_1]$ and $\mathcal{T}[P]$ be as in Definition 13.1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be disjoint subtrees of $\mathcal{T}[Z_1]$ such that Q covers $\mathcal{T}_1 \cup \mathcal{T}_2$ and avoids $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$. Let $\mathcal{T}_{Q'}$ be the minimal subtree of $\mathcal{T}[P]$ that contains all the nodes of $\mathcal{T}[P]$ that are covered by Q'. Let $\mathcal{T}' = \mathcal{T}_{Q'} \cup \mathcal{T}_1 \cup \mathcal{T}_2$. By definition we have that $\mathcal{T}'$ is a subtree of $\mathcal{T}[P]$. Therefore there exist disjoint subtrees $\mathcal{T}_1'$ and $\mathcal{T}_2'$ of $\mathcal{T}'$ such that $\mathcal{T}_1 \subseteq \mathcal{T}_1'$, $\mathcal{T}_2 \subseteq \mathcal{T}_2'$, and $\mathcal{T}' = \mathcal{T}_1' \cup \mathcal{T}_2'$. Since P is basic, it follows by condition (3) of Definition 13.1 that there exist edge-disjoint subpaths Q_1', Q_2' of Q' such that Q_1' covers $\mathcal{T}_1'$ and avoids $\mathcal{T} \setminus \mathcal{T}_1'$ and Q_2' covers $\mathcal{T}_2'$ and avoids $\mathcal{T} \setminus \mathcal{T}_2'$. Let $Q_1 = Q \cap Q_1'$ and $Q_2 = Q \cap Q_2'$. It now follows that Q_1 covers $\mathcal{T}_1$ and avoids $\mathcal{T} \setminus \mathcal{T}_1$ and Q_2 covers $\mathcal{T}_2$ and avoids $\mathcal{T} \setminus \mathcal{T}_2$, establishing Condition (3) of Definition 13.1.

It remains to show that $|P_1|$ is minimal subject to condition (3) of Definition 13.1. Suppose not. Then there are $Q, Q' \in P_1$ that are consecutive in F such that we can replace Q and Q' in P_1 by some subpath Q'' that contains Q and Q'. If Q and Q' are subpaths of the same path in P then this is a contradiction because by construction there must exist a vertex in $V(Q'') \setminus (V(Q) \cup V(Q'))$ that P_1 must avoid. Therefore there must exist distinct $R, R' \in P$ such that $Q \subseteq R$ and $Q' \subseteq R'$. However this means that we can replace R and R' in P by some subpath containing both R and R', contradicting the fact that P is basic. This establishes that $|P_1|$ is minimal subject to condition (3) of Definition 13.1. We have thus obtained that Z_1 is basic. It is also immediate by construction that since P is elementary, Z_1 is also elementary. By the exact same argument it follows that Z_2 is also basic and elementary.

For any $i \in \{1, 2\}$ let P_i be a succinct twin of Z_i obtained by Lemma 13.4. Since Z_i is basic and elementary, it follows that P_i is also basic and elementary, and thus condition (1) is satisfied. Since P_i is a twin of Z_i, it follows that conditions (2) & (3) are satisfied. Since $E(Z_i) \subset E(P)$ and $E(P_i) \subseteq E(Z_i)$, we get $E(P_i) \subset E(P)$. Finally, since $E(Z_1) \cap E(Z_2) = \emptyset$, we have $E(P_1) \cap E(P_2) = \emptyset$, and thus condition (4) is satisfied, which concludes the proof. □

Definition 13.6 (P-facial restriction)**.** *Let* $P = \{Q_1, \ldots, Q_m\} \in \mathcal{P}_\infty$ *be basic. We define the* P-facial restriction of $\mathcal{W}$ *the exact same way as in Definition 12.3. We remark that* P *is now a family of paths, while in the planar case* P *is a single path; the definition remains the same by replacing the notion of basic path given in Definition 12.2 by the notion of basic family of paths given in Definition 13.1.*

Lemma 13.7 (Malnič and Mohar [22]). *Let S be an either orientable or non-orientable surface of Euler genus g, and let $x \in S$. Let $\mathcal{X}$ be a collection of noncontractible curves. Suppose that at least one of the following holds: (i) The curves in $\mathcal{X}$ are disjoint and (freely) nonhomotopic. (ii) There exist $x \in S$ such that for every $C, C' \in \mathcal{X}$, we have $C \cap C' = x$, and the curves in $\mathcal{X}$ are nonhomotopic (in $\pi_1(S, x)$). Then,*

$$|\mathcal{X}| \leq \begin{cases} 0 & \text{if } S \text{ is the 2-sphere} \\ 1 & \text{if } S \text{ is the torus or the projective plane} \\ 3(g-1) & \text{otherwise} \end{cases}$$

We recall the following result on the genus of the complete bipartite graph [15].

Lemma 13.8. *For any $n, m \geq 1$, the Euler genus of $K_{m,n}$ is $\lceil (m-2)(n-2)/4 \rceil$.*

Lemma 13.9. *Let $P \in \mathcal{P}_\infty$ be elementary and succinct. Then $|P| \leq 18000g^3$.*

Proof. Let $\mathcal{T}[P]$ be as in Definition 13.1. If $\mathcal{T}[P]$ is trivial, we can find $Q \in \mathcal{Q}$ such that Q covers $\mathcal{T}[P]$ and avoids $\mathcal{T} \setminus \mathcal{T}[P]$, and thus $|P| = 1$ and we are done. Now suppose that $\mathcal{T}[P]$ is non-trivial. Let $m = |P|$ and suppose that $P = \{Q_1, \ldots, Q_m\}$ such that $Q_1, \ldots, Q_m$ are subpaths of F in this order along a traversal of F. Let $Z_1, \ldots, Z_{m'}$ be a sequence of disjoint subsets of P such that $P = \bigcup_{i=1}^{m'} Z_i$, and each Z_i is maximal subject to the following condition: Let Z_i' be the minimal subpath of F that contains all the paths in Z_i and does not intersect any other paths in P; then Z_i' avoids all non-trivial tress in $\mathcal{F} \setminus \mathcal{T}$. For every $j \in \{1, \ldots, m'-1\}$, let P_j be the subpath between Z_j' and Z_{j+1}' and let $\mathcal{P}' = \{P_1, \ldots, P_{m-1}\}$. Note that since P is basic, for every two consecutive Z_j' and Z_{j+1}', there exists a non-trivial tree $\mathcal{T}_j$ such that $\mathcal{T}_j$ intersects P_j.

We construct a set R of non-trivial trees as follows. We initially set $R = \{\mathcal{T}_1\}$. In each step $i > 1$, if there exists $\mathcal{T}' \in R$ such that $\mathcal{T}'$ intersects P_i, we continue to the next step. Otherwise, we let $\mathcal{T}_i'$ be some non-trivial tree that intersects P_i and we add $\mathcal{T}_i'$ to R and we continue to the next step. We argue that $|R| \leq 20g$. Suppose not. For each $\mathcal{T}' \in R$ where $\mathcal{T}'$ intersects some $P' \in \mathcal{P}'$ at some $x' \in V(P')$, we construct a path $\gamma_{\mathcal{T}'}$ in the surface, with both endpoints on $\psi(F)$, and such that after contracting $\psi(F)$ into single point, the loop resulting from $\gamma_{\mathcal{T}'}$ is non-contractible, as follows. First suppose that $\psi(\mathcal{T}')$ contains some non-contractible loop γ. Then let ζ be a path in $\psi(\mathcal{T}')$ between x' and some point in γ. We set $\gamma_{\mathcal{T}'}$ to be the path starting at x', traversing ζ, followed by γ, followed by the reversal of ζ, and terminating at x'. Otherwise, suppose that $\psi(cT')$ does not contain any non-contractible loops. Since $\mathcal{T}'$ is non-trivial, it follows that $\psi(\mathcal{T}')$ contains some path ξ with both endpoints in $\psi(F)$ such that after contracting $\psi(F)$ into a single point, the loop obtained from ξ is non-contractible. Let ξ' be some path in $\psi(\mathcal{T}')$ between x' and some point in ξ. By Thomassen's 3-path condition [23] it follows that $\xi \cup \xi'$ contains some path ξ'' with both endpoints in $\psi(F)$, such that one of these endpoints is x', and such that after contracting $\psi(F)$ into a single point, the loop resulting from ξ'' is non-contractible. We set $\gamma_{\mathcal{T}'}$ to be ξ''.

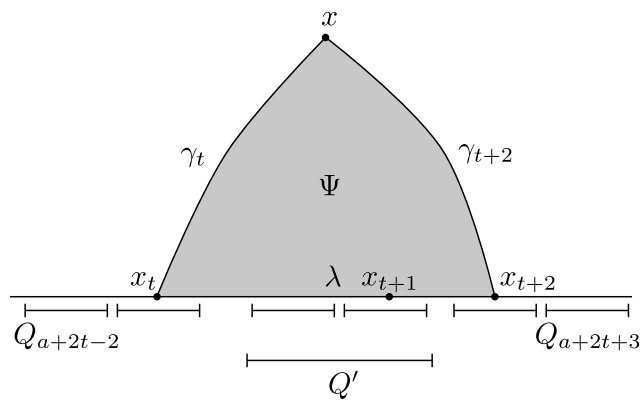
Let $\mathcal{L}$ be the set of all loops obtained from the paths $\gamma_{\mathcal{T}'}$ as follows. Pick a point x in the interior of the disk bounded by $\psi(F)$. Connect x to both endpoints of each $\gamma_{\mathcal{T}'}$ by paths such that all chosen paths are interior disjoint. During this process each path $\gamma_{\mathcal{T}'}$ gives rise to a non-contractible loop in $\mathcal{L}$ such that any two loops in $\mathcal{L}$ intersect only at x. Let $L', L'', L'' \in \mathcal{L}$ be distinct. We show that L', L'' and L''' can not be all homotopic. Suppose not. Since they are interior-disjoint, by removing them from the surface we obtain three connected components. Since $\psi(\mathcal{T})$ does not intersect any of L', L'' and L''', it has to be inside one of the three connected components completely. We may

assume w.l.o.g that $\mathcal{T}$ is inside the component which is bounded by L' and L''. Therefore, there is no path from T to L''' without crossing $L' \cup L''$, which is a contradiction.

Let $\mathcal{L}' \subseteq \mathcal{L}$ be a maximal subset such that for all $L', L'' \in \mathcal{L}'$ we have that L' and L'' are non-homotopic. Since $|\mathcal{L}| > 20g$ and for every three loops in $\mathcal{L}'$ we know that at most two of them are homotopic, we have that $|\mathcal{L}'| > 10g$, which contradicts Lemma 13.7. Therefore, we have that $|R| \leq 20g$ and thus there exists $\mathcal{T}_0 \in R$ such that $\mathcal{T}_0$ intersects at least $10g = 200g^2/20g$ elements of $\mathcal{P}'$. Let x_1 be a point inside the face. Let x_2 be a point in the root of $\mathcal{T}_0$ and let x_3 be a point in $\psi(D)$. There exists $P'_1, \ldots, P'_{10g} \in \mathcal{P}'$ such that for every $i \in \{1, \ldots, 10g\}$, $\mathcal{T}_0$ intersects P'_i. For every $i \in \{1, \ldots, 10g\}$, let y_i be a point on P'_i. By the construction, for every y_i we can find non-crossing paths to x_1, x_2 and x_3. Therefore, we get an embedding of $K_{3,10g}$ in a surface of genus g which contradicts Lemma 13.8. This establishes that $m' \leq 200g^2$.

Let $i \in \{1, \ldots, m'\}$. We next we bound $|Z_i|$. Suppose $Z_i = \{Q_a, Q_{a+1}, \ldots, Q_{a+\ell}\}$. Let x be an arbitrary point in $\psi(D)$. For each $j \in \{0, \ldots, \lfloor(\ell-1)/2\rfloor\}$ pick an arbitrary point $x_j \in \psi(Q_{a+2j}) \cap \psi(\mathcal{T})$; we define a path in the surface between x_j and x as follows: we start from the vertex D' of $\mathcal{T}$ containing x_j; if D' is a closed walk then we traverse D' clockwise until we reach the point that connects D' to its parent; otherwise we traverse the unique path between x_j and the point that connects D' to its parent. We continue in this fashion until we reach D, and we finally traverse D clockwise until we reach x. This completes the definition of the path γ_j. It is immediate that for all $j \neq j'$, the paths γ_j and $\gamma_{j'}$ are non-crossing.

Let S' be the surface obtained by contracting $\psi(F)$ into a single point y, and identifying x with y (note that S' has Euler genus at most $g+2$). For each $j \in \{0, \ldots, \lfloor(\ell-1)/2\rfloor\}$ let γ'_j be the loop in S' obtained from γ_j after the above contraction and identification. We argue that there can be at most 4 loops γ'_j that are pairwise homotopic. Suppose for the sake of contradiction that there are at least 5 such loops. It follows that there must exist $t \in \{0, \ldots, \lfloor(\ell-1)/2\rfloor\}$ such that γ'_t, γ'_{t+1}, and γ'_{t+2} are all pairwise homotopic. We are going to obtain a contradiction by arguing that the paths Q_{a+2t} and Q_{a+2t+1} violate the fact that P is basic. To that end, let Q' be the minimal subpath of F that contains both Q_{a+2t} and Q_{a+2t+1} and does not intersect any other paths in P. We will show that $(P \setminus \{Q_{a+2t}, Q_{a+2t+1}\}) \cup \{Q'\}$ is basic, thus violating the fact that $|P|$ is minimal subject to condition (3) of definition 13.1. Let $\mathcal{T}_1$, $\mathcal{T}_2$ be disjoint subtrees of $\mathcal{T}[P]$ such that Q' covers $\mathcal{T}_1 \cup \mathcal{T}_2$ and avoids $\mathcal{T} \setminus (\mathcal{T}_1 \cup \mathcal{T}_2)$. We need to show that there exist edge-disjoint subpaths Q'_1 and Q'_2 of Q' such that $Q' = Q'_1 \cup Q'_2$, Q'_1 covers $\mathcal{T}_1$ and avoids $\mathcal{T} \setminus \mathcal{T}_1$, and Q'_2 covers $\mathcal{T}_2$ and avoids $\mathcal{T} \setminus \mathcal{T}_2$. Let λ be the subpath of $\psi(F)$ in S between x_t and x_{t+1} that contains x_{t+1}. Since γ'_t, γ'_{t+1}, and γ'_{t+2} are homotopic, it follows that $\gamma_t \cup \lambda \cup \gamma_{t+2}$ bounds a disk Ψ in S. Each x_s is contained in the image of a unique vertex D'_s of $\mathcal{T}[P]$. Let B be the path in $\mathcal{T}[P]$ between D'_t and D'_{t+2}. For each $r \in \{1, 2\}$, $\mathcal{T}_r$ intersects B into some possibly empty subpath B_r. It follows that there exist disjoint disks $\Psi_1, \Psi_2 \subset \Psi$ such that for each $r \in \{1, 2\}$, $\psi(\mathcal{T}_r) \cap \Psi \subset \Psi_r$. Therefore there exist edge-disjoint subpaths Q'_1 and Q'_2 of Q' with $Q' = Q'_1 \cup Q'_2$ such that $\psi(Q') \cap \Psi_1 \subseteq \psi(Q'_1)$ and $\psi(Q') \cap \Psi_2 \subseteq \psi(Q'_2)$. It follows that Q'_1 covers $\mathcal{T}_1$ and avoids $\mathcal{T} \setminus \mathcal{T}_1$, and Q'_2 covers $\mathcal{T}_2$ and avoids $\mathcal{T} \setminus \mathcal{T}_2$. This contradicts the fact that P is basic, and concludes the proof that there are at most four loop γ'_t that are pairwise homotopic.



Pick $I \subseteq \{0, \ldots, \lfloor(\ell-1)/2\rfloor\}$, with $|I| \geq \lfloor(\ell-1)/10\rfloor$ such that for all $t \neq t' \in I$, we have that γ'_t and $\gamma'_{t'}$ are non-homotopic. Since the paths γ_t are non-crossing, we may assume w.l.o.g. that the paths γ'_t are interior disjoint after an infinitesimal perturbation. By Lemma 13.7 on S' it follows that $|I| \leq 3(g+1)$. Since $|I| \geq \ell/15$, we get $\ell \leq 45(g+1)$. Therefore $|Z_i| \leq 45(g+1)$.

We conclude that $m \leq \sum_{i=1}^{m'} |Z_i| \leq m' \cdot \max_{i\in\{1,\ldots,m'\}} |Z_i| \leq 9000g^2(g+1) \leq 18000g^3$. □

Lemma 13.10. *Let $P_1 = \{Q_1, \ldots, Q_m\} \in \mathcal{P}$ be succinct. Let $P_2 = \{Q'_1, \ldots, Q'_l\} \in \mathcal{P}$ such that P_1 and P_2 are twins, $V(P_1) \subseteq V(P_2)$ and $E(P_1) \subseteq E(P_2)$. For every $i \in \{1, \ldots, m\}$, let u_i and v_i be the endpoints of Q_i. Let $\mathcal{B}_1 = \bigcup_{i=1}^{m}(B_{u_i} \cup B_{v_i})$. Let $\mathcal{W}_{P_1}$ be the P_1-facial restriction of $\mathcal{W}$. Let $\Gamma_1 = \bigcup_{\vec{W}\in\mathcal{W}_{P_1}} \vec{W}$. Let C_1 be the partition of $\mathcal{B}_1$ that corresponds to the weakly-connected components of Γ_1. For any $x \in \mathcal{B}_1$ let $f_1^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_{P_1}}(x)$ and $f_1^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_{P_1}}(x)$. We similarly define $C_2, f_2^{\mathsf{in}}, f_2^{\mathsf{out}}$ and $\mathcal{W}_{P_2}$ for P_2. Suppose that there exists some $a_1 \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l_1, r_1, p_1 \in (V(\vec{G}) \cup \mathsf{nil})$ such that the dynamic programming table contains some partial solution $\mathcal{S}_1$ at location $(P_1, (C_1, f_1^{\mathsf{in}}, f_1^{\mathsf{out}}, a_1, l_1, r_1, p_1))$, with $\mathsf{cost}_{\vec{G}}(\mathcal{S}_1) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_1})$. Then there exists some $a_2 \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l_2, r_2, p_2 \in (V(\vec{G}) \cup \mathsf{nil})$ such that the dynamic programming table contains some partial solution $\mathcal{S}_2$ at location $(P_2, (C_2, f_2^{\mathsf{in}}, f_2^{\mathsf{out}}, a_2, l_2, r_2, p_2))$, with $\mathsf{cost}_{\vec{G}}(\mathcal{S}_2) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_{P_2})$.*

Proof. Let $E_1 = E(P_2) \setminus E(P_1)$. For every $e \in E_1$, let $Q_e \in \mathcal{Q}$ be the path containing a single edge e, and let $P_e = \{Q_e\}$. Note that P_e is an empty basic family of paths and $|E(P_e)| = 1$. Therefore for every $e \in E_1$, the initialization step of the dynamic programming, finds a partial solution $\mathcal{S}_e$ for P_e. By merging $\mathcal{S}_1$ with all these partial solutions sequentially in an arbitrary order, we get a partial solution $\mathcal{S}_2$ for P_2, as desired. □

Lemma 13.11. *Let $P = \{Q_1 \ldots, Q_m\} \in \mathcal{P}$ be basic and trivial (w.r.t. $\mathcal{W}$). For every $i \in \{1, \ldots, m\}$, let u_i and v_i be the endpoints of Q_i. Let $\mathcal{B} = \bigcup_{i=1}^{m}(B_{u_i} \cup B_{v_i})$. Let $\mathcal{W}_P$ be the P-facial restriction of $\mathcal{W}$. Let $\Gamma = \bigcup_{\vec{W}\in\mathcal{W}_P} \vec{W}$. Let C be the partition of $\mathcal{B}$ that corresponds to the weakly-connected components of Γ. For any $x \in \mathcal{B}$ let $f^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_P}(x)$ and $f^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_P}(x)$. Then there exists some $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l, r, p \in (V(\vec{G}) \cup \mathsf{nil})$ such that the dynamic programming table contains some partial solution $\mathcal{S}$ at location $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$, with $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$.*

Proof. Since P is trivial, we have that $P \in \mathcal{P}_1$, and thus the exact same argument as in Lemma 12.6 applies here. □

Lemma 13.12. *Let $P = \{Q_1 \dots, Q_m\} \in \mathcal{P}$ be succinct and elementary (w.r.t. $\mathcal{W}$). For every $i \in \{1, \dots, m\}$, let u_i and v_i be the endpoints of Q_i. Let $\mathcal{B} = \bigcup_{i=1}^{m}(B_{u_i} \cup B_{v_i})$. Let $\mathcal{W}_P$ be the P-facial restriction of $\mathcal{W}$. Let $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$. Let C be the partition of $\mathcal{B}$ that corresponds to the weakly-connected components of Γ. For any $x \in \mathcal{B}$ let $f^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_P}(x)$ and $f^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_P}(x)$. Then there exists some $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l, r, p \in (V(\vec{G}) \cup \mathsf{nil})$ such that the dynamic programming table contains some partial solution $\mathcal{S}$ at location $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$, with $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$.*

Proof. Let $\mathcal{T}$, D, k, $D_1, \dots, D_k$ and j be as in Definition 13.1. We prove the assertion by induction on $\mathcal{T}$. For the base case, where D is a leaf of $\mathcal{T}$, the same argument as in Lemma 12.6 applies here. Suppose that D is non-leaf. In this case, we prove the assertion by another induction on j. For the base case, where $j = 1$, the same argument as in Lemma 12.6 applies here. Now suppose that we have proved the assertion for all $j' < j$. Let $P_1 \in \mathcal{P}_\infty$ such that $V(P_1) \subseteq V(P)$, $E(P_1) \subseteq E(P)$, P_1 covers $\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i}$ and avoids $\mathcal{T} \setminus (\bigcup_{i=1}^{j-1} \mathcal{T}_{D_i})$. Let $P_2 \in \mathcal{P}_\infty$ such that $E(P_2) = E(P) \setminus E(P_1)$. By the construction, P_1 and P_2 are elementary. Therefore, by Lemma 13.4, there exists elementary and succinct $P_1' \in \mathcal{P}_\infty$ and $P_2' \in \mathcal{P}_\infty$ for P_1 and P_2 respectively. Moreover, by Lemma 13.5, we have that $E(P_1') \subseteq E(P)$, $E(P_2') \subseteq E(P)$ and $E(P_1') \cap E(P_2') = \emptyset$. By Lemma 13.9 we have that $P, P_1', P_2' \in \mathcal{P}_{18000g^3}$. We next define some $P_1'' \in \mathcal{P}_\infty$. Let Γ be the graph obtained as the union of all paths in P, that is $\Gamma = \bigcup_{Q \in P} Q$. Similarly, let $\Gamma' = \bigcup_{Q \in P_2'} Q$, let $\Gamma'' = \Gamma \setminus E(\Gamma')$, and let Γ''' be the graph obtained from Γ'' by deleting all isolated vertices. We define P_1'' to be the set of connected components in Γ'''. Note that Γ''' has at most $36000g^3$ connected components, and each such component is a path. Thus $P_1'' \in \mathcal{P}_{36000g^3}$, and $E(P_1') \subseteq E(P_1'')$. By the induction hypothesis, there exists a partial solution $\mathcal{S}_1'$ for P_1', and thus by Lemma 13.10, there exists a partial solution $\mathcal{S}_1''$ for P_1''. Also, by the induction hypothesis, there exists a partial solution $\mathcal{S}_2'$ for P_2'. By merging $\mathcal{S}_1''$ and $\mathcal{S}_2'$, we get a partial solution $\mathcal{S}$ for P, as desired. □

We define the following three types of non-trivial trees in $\mathcal{F}$:

(1) We say that a non-trivial tree $\mathcal{T}$ is of the *first type*, if there exists a non-leaf $D \in V(\mathcal{T})$, such that D is a closed walk, with $V(D) \cap V(F) = \emptyset$, such that $\psi(D)$ is non-contractible.

(2) We say that a non-trivial tree $\mathcal{T}$ is of the *second type*, if it is not of the first type and there exists a leaf $D \in V(\mathcal{T})$ such that $\psi(D) \cup \psi(F)$ is non-contractible.

(3) We say that a non-trivial tree $\mathcal{T}$ is of the *third type*, if it is not of the first type nor of the second type.

We say that a non-trivial tree $\mathcal{T}$ is *good*, if at least one of the following conditions holds:

(1) $\mathcal{T}$ is of the second type, and for every $D_1, D_2 \in V(\mathcal{T})$ where $\psi(D_1) \cup \psi(F)$ and $\psi(D_1) \cup \psi(F)$ are non-contractible, we have that the loops obtained from $\psi(D_1)$ and $\psi(D_1)$ by contracting $\psi(F)$ into a single poit x are homotopic in $\pi_1(S, x)$. Let $D \in V(\mathcal{T})$ such that $\psi(D) \cup \psi(F)$ is non-contractible. We let $\beta(\mathcal{T})$ to be the homotopy class of the loop $\psi(D)$ in the surface obtained after contracting $\psi(F)$ into a single point.

(2) $\mathcal{T}$ is of the third type and the following holds. Let $X = \psi(F) \cup \bigcup_{D \in V(\mathcal{T})} \psi(D)$ and let X' be the image of X after contracting $\psi(F)$ into a single point x. Then and all non-contractible loops in X' are homotopic in $\pi_1(S, x)$. We let $\beta(\mathcal{T})$ be the homotopy class in $\pi_1(S, x)$ of all

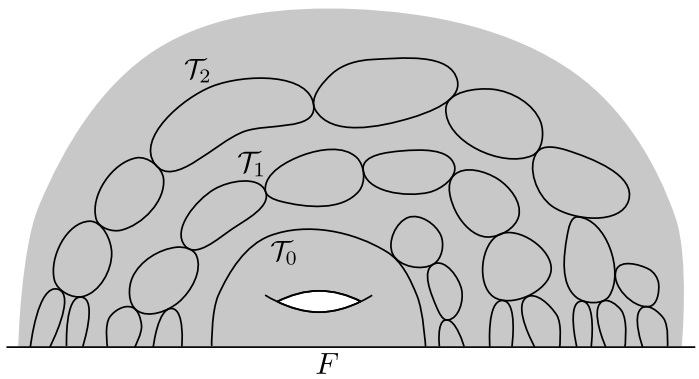


Figure 3: Example of good non-trivial trees $\mathcal{T}_0$, $\mathcal{T}_1$, and $\mathcal{T}_2$ that are pairwise friends; note that $\mathcal{T}_0$ is of the second type while $\mathcal{T}_1$ and $\mathcal{T}_2$ are of the third type.

non-contractible loops in X'; note that we may always take a non-contractible loop in X' that contains the basepoint x since X is connected.

Otherwise, we say that $\mathcal{T}$ is a *bad* tree.

Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be non-trivial good trees in $\mathcal{F}$. We say that $\mathcal{T}_0$ is a *friend* of $\mathcal{T}_1$ if $\beta(\mathcal{T}_0) = \beta(\mathcal{T}_1)$ (see Figure 3 for an example).

Definition 13.13 (Friendly)**.** *Let $P \in \mathcal{P}_\infty$. We say that P is* friendly *if the following holds.*

(1) P avoids all non-trivial bad trees.

(3) For any two non-trivial good trees $\mathcal{T}_0$ and $\mathcal{T}_1$ in $\mathcal{F}$ that P covers, we have that $\beta(\mathcal{T}_0) = \beta(\mathcal{T}_1)$.

(3) If P covers a non-trivial good tree $\mathcal{T}_0$, then P covers all non-trivial good trees $\mathcal{T}_1$ with $\beta(\mathcal{T}_0) = \beta(\mathcal{T}_1)$.

Lemma 13.14. *There exists at most $12g$ bad trees in $\mathcal{F}$.*

Proof. We partition all bad trees in $\mathcal{F}$ into three sets:

(1) $F_1 = \{\mathcal{T} \in \mathcal{F} : \mathcal{T}$ is a bad tree of the first type$\}$.

(2) $F_2 = \{\mathcal{T} \in \mathcal{F} : \mathcal{T}$ is a bad tree of the second type$\}$.

(3) $F_3 = \{\mathcal{T} \in \mathcal{F} : \mathcal{T}$ is a bad tree of the third type$\}$.

We first bound $|F_1|$. Let $\mathcal{T} \in F_1$. We let $\beta(\mathcal{T})$ to be the homotopy class of some non-leaf $D \in V(T)$, such that D is a closed walk, with $V(D) \cap V(F) = \emptyset$, and $\psi(D)$ is non-contractible. Note that by the definition of trees of the first type, such a non-leaf $D \in V(\mathcal{T})$ exists. Let $\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3 \in F_1$ be distinct. We show that $\beta(\mathcal{T}_1) = \beta(\mathcal{T}_2) = \beta(\mathcal{T}_3)$ cannot happen. Suppose not, and we have that $\beta(\mathcal{T}_1) = \beta(\mathcal{T}_2) = \beta(\mathcal{T}_3)$. For any $i \in \{1, 2, 3\}$, let $D_i \in V(\mathcal{T}_i)$ such that $\beta(\mathcal{T}_i)$ is the homotopy class of D_i. By removing $\psi(D_1)$, $\psi(D_2)$ and $\psi(D_3)$ from the surface we obtain three connected components. We may assume w.l.o.g that $\psi(D_2)$ is inside an annulus bounded by $\psi(D_1)$ and $\psi(D_3)$. Therefore, there is no path in the surface from $\psi(D_2)$ to $\psi(F)$, that does not intersect $\psi(D_1 \cup D_3)$, which is a contradiction. Therefore by Lemma 13.7 we have that $|F_1| \leq 6g$.

Next we bound $|F_2|$ and $|F_3|$. Let $\mathcal{T} \in F_2$. By the construction, there exist $D_1, D_2 \in V(\mathcal{T})$ where $\psi(D_1) \cup \psi(F)$ and $\psi(D_2) \cup \psi(F)$ are non-contractible, and the loops obtained from $\psi(D_1)$ and $\psi(D_2)$ after contracting $\psi(F)$ into a single point y are non-homotopic in $\pi_1(S, y)$. Let x be a point in the interior of the disk bounded by $\psi(F)$. For every $\mathcal{T} \in F_2$, similarly to Lemma 13.9, we construct two non-homotopic loops $\gamma_{\mathcal{T}'}$ and $\gamma'_{\mathcal{T}'}$ in the surface, corresponding to D_1 and D_2 respectively, such that they only intersect at x. Let $\mathcal{L}$ be the set of all these loops. Let $L', L'', L''' \in \mathcal{L}$ be distinct. Similarly to Lemma 13.9, we show that L', L'' and L''' can not be all homotopic. Suppose not. We may assume w.l.o.g. that L'' is inside the disk bounded by L' and L'''. Let $\mathcal{T}'' \in F_2$ be the tree corresponding to L''. Now note that L'' is inside the disk bounded by L' and L''', and thus all non-contractible loops in $\psi(F) \cup \bigcup_{D \in V(\mathcal{T})} \psi(D)$ are in the same homotopy class as L''. This contradicts the fact that $\psi(D_1) \cup \psi(F)$ and $\psi(D_2) \cup \psi(F)$ are non-homotopic. Therefore, by Lemma 13.7 we have that $|F_2| \leq 3g$. Using a similar argument, we can show that $|F_3| \leq 3g$.

Therefore, we have that $|F_1| + |F_2| + |F_3| \leq 12g$, completing the proof. □

Lemma 13.15. *Let $P = \{Q_1 \ldots, Q_m\} \in \mathcal{P}_\infty$ be friendly, succinct and basic (w.r.t. $\mathcal{W}$). For every $i \in \{1, \ldots, m\}$, let u_i and v_i be the endpoints of Q_i. Let $\mathcal{B} = \bigcup_{i=1}^m (B_{u_i} \cup B_{v_i})$. Let $\mathcal{W}_P$ be the P-facial restriction of $\mathcal{W}$. Let $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$. Let C be the partition of $\mathcal{B}$ that corresponds to the weakly-connected components of Γ. For any $x \in \mathcal{B}$ let $f^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_P}(x)$ and $f^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_P}(x)$. Then there exists some $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l, r, p \in (V(\vec{G}) \cup \mathsf{nil})$ such that the dynamic programming table contains some partial solution $\mathcal{S}$ at location $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$, with $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$.*

Proof. If P does not cover any non-trivial trees in $\mathcal{F}$, then P is trivial and thus we have that $P \in \mathcal{P}_1$ and by Lemma 13.11 there exists a partial solution $\mathcal{S}$ for P. Otherwise suppose that P covers a non-trivial good tree $\mathcal{T}$ in $\mathcal{F}$. By the definition of friendly, for every non-trivial good tree $\mathcal{T}'$ in $\mathcal{F}$ with $\beta(\mathcal{T}) = \beta(\mathcal{T}')$, we have that P covers $\mathcal{T}'$. Let $A = \{\mathcal{T}' \in \mathcal{F} : \beta(\mathcal{T}) = \beta(\mathcal{T}')\}$. Suppose that $A = \{\mathcal{T}_0, \ldots, \mathcal{T}_m\}$ such that they intersect F in this order along a traversal of F. For any $i \in \{0, \ldots, m\}$, let $P_i \in \mathcal{P}_\infty$ be elementary and succinct such that P_i covers $\mathcal{T}_i$ and avoids any other non-trivial trees in $\mathcal{F}$. By Lemma 13.9 we have that $P_i \in \mathcal{P}_{18000g^3}$. Moreover, by Lemma 13.12 we have that there exists a partial solution $\mathcal{S}_i$ for P_i. For any $i \in \{0, \ldots, m\}$, let $P'_i \in \mathcal{P}_\infty$ be succinct and basic, such that P'_i covers $\bigcup_{j=0}^i \mathcal{T}_j$ and avoids any other non-trivial trees in $\mathcal{F}$. By the construction, each P'_i can be obtained by merging P'_{i-1} with P_i and some trivial and empty basic family of paths, and thus we have that $P'_i \in \mathcal{P}_{36000g^3}$. Therefore, by induction and Lemma 13.11, there exists a partial solution $\mathcal{S}'_i$ for each P'_i, and thus there exists a partial for solution $\mathcal{S}'_m$ for P'_m. Note that $P'_m = P$. Therefore, we get a partial solution $\mathcal{S}$ for P, as desired. □

Lemma 13.16. *Let $P = \{F\}$. Let $v^\circ \in V(F)$. Let $\mathcal{W}_P$ be the P-facial restriction of $\mathcal{W}$. Let $\Gamma = \bigcup_{\vec{W} \in \mathcal{W}_P} \vec{W}$. Let C be the partition of B_{v° that corresponds to the weakly-connected components of Γ. For any $x \in B_{v^\circ}$ let $f^{\mathsf{in}}(x) = \mathsf{in\text{-}degree}_{\mathcal{W}_P}(x)$ and $f^{\mathsf{out}}(x) = \mathsf{out\text{-}degree}_{\mathcal{W}_P}(x)$. Then there exists some $a \in \mathcal{A} \cup (\mathcal{A} \times \mathcal{A}) \cup \mathsf{nil}$ and $l, r, p \in (V(\vec{G}) \cup \mathsf{nil})$ such that the dynamic programming table contains some partial solution $\mathcal{S}$ at location $(P, (C, f^{\mathsf{in}}, f^{\mathsf{out}}, a, l, r, p))$, with $\mathsf{cost}_{\vec{G}}(\mathcal{S}) \leq \mathsf{cost}_{\vec{G}}(\mathcal{W}_P)$.*

Proof. We first partition F to the sets of basic families of paths as follows:

(1) $F_1 = \{P' \in \mathcal{P}_\infty : P'$ is elementary and succinct, P' covers some bad tree $\mathcal{T}\}$.

(2) $F_2 = \{P' \in \mathcal{P}_\infty : P'$ is basic, succinct and friendly$\}$.

(3) $F_3 = \{P' \in \mathcal{P}_\infty : P'$ is basic, succinct and trivial$\}$.

(4) $F_4 = \{P' \in \mathcal{P}_\infty : P'$ is basic, succinct and empty$\}$.

For every bad tree $\mathcal{T}$ in $\mathcal{F}$, let $P_1 \in F_1$ such that P_1 covers $\mathcal{T}$ and avoids any other non-trivial tree in $\mathcal{F}$. Since P_1 is elementary and succinct, by Lemma 13.9 we have that $|P_1| \leq 18000g^3$. Moreover by Lemma 13.12 there exists a partial solution $\mathcal{S}_1$ for P_1. Also by Lemma 13.14, there exist at most $12g$ bad trees in $\mathcal{F}$, and thus $|F_1| \leq 12g$. Therefore there exists $P_1' \in \mathcal{P}_{216000g^4}$, such that for any $\mathcal{T}' \in \mathcal{F}$, P_1' covers (resp. avoids) $\mathcal{T}'$ if and only if there exists $P_1 \in F_1$ such that P_1 covers (resp. avoids) $\mathcal{T}'$. Moreover the dynamic programming table contains a partial solution $\mathcal{S}_1'$ for P_1'. Now we will show that $|F_2| \leq 3g$. For every $P_2 \in F_2$, let $\mathcal{T}_2$ be some non-trivial good tree that P_2 covers. Let x be a point in the interior of the disk bounded by $\psi(F)$. Similar to Lemma 13.9 we construct a non-contractible loop γ_{P_2} that contains x, corresponding to $\mathcal{T}_2$. Let L be the set of all these interior-disjoint loops. For every $P', P'' \in F_2$, since P' and P'' are succinct and friendly, $\gamma_{P'}$ and $\gamma_{P''}$ are not homotopic. Therefore by Lemma 13.7 we have that $|L| \leq 3g$, and thus $|F_2| \leq 3g$. Furthermore for every $P_2 \in F_2$, by Lemma 13.15, we have that $P_2 \in \mathcal{P}_{36000g^3}$, and there exists a partial solution $\mathcal{S}_2$ for P_2. Therefore there exists $P_2' \in \mathcal{P}_{108000g^4}$, such that for any $\mathcal{T}' \in \mathcal{F}$, P_2' covers (resp. avoids) $\mathcal{T}'$ if and only if there exists $P_2 \in F_2$ such that P_2 covers (resp. avoids) $\mathcal{T}'$. Moreover the dynamic programming table contains a partial solution $\mathcal{S}_2'$ for P_2'. Let $P_{1,2}' \in \mathcal{P}_{324000g^4}$ such that for any $\mathcal{T}' \in \mathcal{F}$, $P_{1,2}'$ covers (resp. avoids) $\mathcal{T}'$ if and only if there exists $P' \in F_1 \cup F_2$ such that P' covers (resp. avoids) $\mathcal{T}'$. By merging $\mathcal{S}_1'$ and $\mathcal{S}_2'$ we get a partial solution $\mathcal{S}_{1,2}'$ for $P_{1,2}'$.

Let

$$F_5 = \{\{Q\} \in F_3 \cup F_4 : Q \text{ is maximal subject to } E(Q) \cap E(P_{1,2}') = \emptyset\}.$$

For every $P_3 \in F_3$, by Lemma 13.11 we have that $P_3 \in \mathcal{P}_1$ and there exists a partial solution $\mathcal{S}_3$ for P_3. Finally, for every $P_4 \in F_4$, we have that $P_4 \in \mathcal{P}_1$ and also the dynamic programming table contains a partial solution $\mathcal{S}_4$ for P_4. Therefore for every $\{Q\} \in F_5$, the dynamic programming table contains a partial solution for $\{Q\}$. By merging these partial solutions with $P_{1,2}'$ in an arbitrary order, we get a partial solution $\mathcal{S}$, as desired. We remark that after merging the current partial solution with the partial solution for some $\{Q\} \in F_5$, we obtain a new partial solution for some $P \in P_\infty$ with fewer paths. Therefore for all intermediate $P \in P_\infty$ we have $P \in P_{324000g^4}$, concluding the proof. □

Theorem 13.17. *Let $\vec{G}$ be an n-vertex $(0, g, 1, p)$-nearly embeddable graph and let $\vec{H}$ be the vortex of $\vec{G}$. Then there exists an algorithm which computes a walk $\vec{W}$ visiting all vertices in $V(\vec{H})$ of total length $\mathsf{OPT}_{\vec{G}}(V(\vec{H}))$ in time $n^{O(pg^4)}$.*

Proof. Let $P = \{F\}$. By Lemma 13.16, there exists a partial solution for P. Now the exact same argument as in Theorem 12.7 applies here. □

14 The algorithm for traversing a vortex in a nearly-embeddable graph

In this Section we obtain an exact algorithm for computing a closed walk that traverses all the vertices in the single vortex of a nearly-embeddable graph.

Proof of Theorem 5.1. The algorithm is as follows. Let $A \subset V(\vec{G})$ be the set of apices and let $\vec{F}$ be the face on which the vortex is attached. For each $A' \subseteq A$ we construct a $(0, g, 1, p+a)$-nearly embeddable graph $\vec{G}_{A'}$ as follows: Initially we set $\vec{G}_{A'} = \vec{G} \setminus A$. We then add A' to $V(\vec{G}_{A'})$ and for every $u \in A'$ and every $v \in V(\vec{F})$ we add edges (u, v) and (v, u) of length $d_{\vec{G}}(u, v)$ and $d_{\vec{G}}(v, u)$ respectively; let $E_{A'}$ be the set of all these new edges. We consider A' as being part of the vortex and we modify the path decomposition of the vortex by adding A' to each one of its bubbles; it is immediate that the result is a path decomposition of width at most $a+p$. We then run the algorithm of Theorem 13.17 on $\vec{G}_{A'}$ and find an optimal closed walk visiting all vertices in $V(\vec{H}) \cup A'$. Let $\vec{W}_{A'}$ be the resulting walk in $\vec{G}_{A'}$. We obtain a walk $\vec{W}'_{A'}$ in $\vec{G}$ visiting all vertices in $V(\vec{H})$ with $\mathsf{cost}_{\vec{G}}(\vec{W}'_{A'}) = \mathsf{cost}_{\vec{G}_{A'}}(\vec{W}_{A'})$ by replacing every edge in $(u, v) \in E_{A'}$ by a shortest path from u to v in $\vec{G}$. After considering all subsets $A' \subseteq A$, we output the walk $\vec{W}'_{A'}$ of minimum cost that we find. This completes the description of the algorithm.

It is immediate that the running time is $O(2^a n^{(a+p)g^4}) = O(n^{(a+p)g^4})$. It remains to show that the algorithm computes an optimum walk. Let $\vec{R}$ be the walk computed by the algorithm. Let $\vec{W}_{\mathsf{OPT}}$ be a walk of minimum cost in $\vec{G}$ visiting all vertices in $V(\vec{H})$. Let A^* be the set of apices visited by $\vec{W}_{\mathsf{OPT}}$, that is $A^* = A \cap V(\vec{W}_{\mathsf{OPT}})$. Let $\vec{G}'$ be the genus-g piece of $\vec{G}$. We can construct a walk $\vec{Z}$ in $\vec{G}_{A^*}$ that visits all the vertices in $V(\vec{H}) \cup A^*$ of cost $\mathsf{cost}_{\vec{G}}(\vec{W}_{\mathsf{OPT}})$ as follows: we replace every sub-walk of $\vec{W}_{\mathsf{OPT}}$ that is contained in $\vec{G}'$, has endpoints $u, v \in V(\vec{F})$, and visits some apex $w \in A^*$, by the path u, w, v in $\vec{G}_{A^*}$. It is immediate that the resulting walk does not traverse any apices and therefore it is contained in $\vec{G}_{A^*}$. Thus we have

$$\mathsf{cost}_{\vec{G}}(\vec{R}) \leq \mathsf{cost}_{\vec{G}}(\vec{W}'_{A^*}) = \mathsf{cost}_{\vec{G}_{A^*}}(\vec{W}_{A^*}) \leq \mathsf{cost}_{\vec{G}}(\vec{W}_{\mathsf{OPT}}) \leq \mathsf{OPT}_{\vec{G}},$$

which concludes the proof. □

15 The lower bound for graphs of bounded pathwidth

In this section we present the proof of Theorem 1.2. This is done via the following chain of reductions:

15.1 Edge Balancing

Let D be a directed graph and let $\chi : E(D) \to \mathbb{Z}^+$ be an assignment of integers to the edges. We can extend χ to a set $F \subseteq E(D)$ of edges in the obvious way by defining $\chi(F) = \sum_{e \in F} \chi(e)$. Let $\delta^+_D(v)$ and $\delta^-_D(v)$ be the set of outgoing and incoming edges of v, respectively. We say that χ is *balanced* at $v \in V(D)$ if $\sum_{e \in \delta^+_D(v)} \chi(e) = \sum_{e \in \delta^-_D(v)} \chi(e)$ holds and we say that χ is *balanced* if it is balanced at every $v \in V(D)$.

Edge Balancing: Given a directed graph D and a set X_e of positive integers for every edge $e \in E(D)$, the task is to select a $\chi(e) \in X_e$ for every edge $e \in E(D)$ such that χ is balanced.

An easy counting argument shows that it is sufficient to require that χ balanced at all but one vertex:

Proposition 15.1. *If χ is balanced at every vertex of $V(D) \setminus \{v\}$, then it is also balanced at v.*

We give a lower bound for Edge Balancing by a parameterized reduction from Multicolored Biclique.

Multicolored Biclique: Given a graph G with a partition $(V_1, \ldots, V_{2k})$ of vertices, find a vertex $v_i \in V_i$ for every $1 \le i \le 2k$ such that v_{i_1} and v_{i_2} are adjacent for every $1 \le i_1 \le k$ and $k+1 \le i_2 \le 2k$.

There is a very simple folklore reduction from Clique to Multicolored Biclique (see, e.g., [7]) where the output parameter equals the input one, hence the lower bound of Chen et al. [6] for Clique can be transferred to Multicolored Biclique.

Theorem 15.2. *Assuming ETH,* Multicolored Biclique *cannot be solved in time $f(k)n^{o(k)}$ for any computable function f.*

The reduction from Multicolored Biclique to Edge Balancing uses the technique of k-non-averaging sets to encode vertices [11, 18]. We say that a set X of positive integers is *k-non-averaging* if for every choice of k (not necessarily distinct) integers $x_1, \ldots, x_k \in X$, their average is in X only if $x_1 = x_2 = \cdots = x_k$. For example, $X = \{(k+1)^i \mid 1 \le i \le n\}$ is certainly a k-non-averaging set of size n. However, somewhat surprisingly, it is possible to construct much denser k-non-averaging sets where each integer is polynomially bounded in k and the size n of the set.

Lemma 15.3 (Jansen *et al.* [18])**.** *For every k and n there exists a k-non-averaging set X of n positive integers such that the largest element of X has value at most $32p^2n^2$. Furthermore, X can be constructed in time $O(p^2n^3)$ time.*

Lemma 15.4. *Assuming ETH,* Edge Balancing *has no $f(k)n^{o(k)}$ time algorithm for any computable function f, where k is the number of vertices of D.*

Proof. The proof is by reduction from Multicolored Biclique. Let G be an undirected graph with a partition $V_1, \ldots, V_{2k}$ of the vertices in $V(G)$. By padding the instance with isolated vertices, we may assume without loss of generality that each V_i has the same number n of vertices; let $v_{i,j}$ be the j-th vertex of V_i. We construct an instance of Edge Balancing on a directed graph D having $2k+1$ vertices w, w_1, w_2, $\ldots$, w_{2k}. Let us use Lemma 15.3 to construct a k-non-averaging set $X = \{x_1, \ldots, x_n\}$ of n positive integers such that the maximum value in X is $M = O(k^2n^2)$. Let $B = 2kM$. The edges of D and the sets of integers on them are constructed the following way (see Figure 4). For every $1 \le i_1 \le k$ and $k+1 \le i_2 \le 2k$, we introduce the edge (w_{i_1}, w_{i_2}) into D and we define the set $X_{(w_{i_1}, w_{i_2})}$ the following way: for every edge $v_{i_1,j_1}v_{i_2,j_2}$ between V_{i_1} and V_{i_2}, let us introduce the positive integer $x_{j_1} + Bx_{j_2}$ into $X_{(w_{i_1}, w_{i_2})}$. Next, for every $1 \le i \le k$, we introduce the edge (w, w_i) and let $X_{(w,w_i)} = \{kx + By \mid x \in X, 1 \le y \le kM\}$. Finally, for every $k+1 \le i \le 2k$, we introduce the edge (w_i, w) and let $X_{(w_i,w)} = \{y + Bkx \mid x \in X, 1 \le y \le kM\}$. This completes the description of the reduction.

Biclique $\Rightarrow$ balanced assignment χ. Suppose that $v_{i,j_i} \in V_i$ for $1 \le i \le 2k$ form a solution of Multicolored Biclique. We define $\chi : E(D) \to \mathbb{Z}^+$ the following way. For every $1 \le i_1 \le k$ and $k+1 \le i_2 \le 2k$, we set (see the values on the edges in Figure 4)

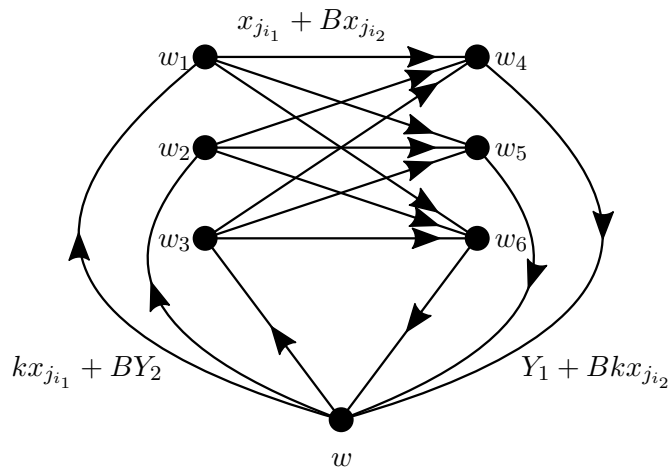


Figure 4: The instance of Edge Balancing constructed in the proof of Lemma 15.4. The values on the edges indicate the value of χ corresponding to a solution $(v_{i,j_i})_{i=1,\ldots,2k}$ of the Multicolored Biclique instance (we have $Y_1 = \sum_{1\le i\le k} x_{j_i}$ and $Y_2 = \sum_{k+1\le i\le 2k} x_{j_i}$).

- $\chi((w_{i_1}, w_{i_2})) = x_{j_{i_1}} + Bx_{j_{i_2}}$,
- $\chi((w, w_{i_1})) = kx_{j_{i_1}} + BY_2$, where $Y_2 = \sum_{k+1\le i\le 2k} x_{j_i}$, and
- $\chi((w_{i_1}, w)) = Y_1 + Bkx_{j_{i_2}}$, where $Y_1 = \sum_{1\le i\le k} x_{j_i}$.

Note that $\chi(e) \in X_e$ holds for every edge $e \in E(D)$. Let us verify that χ is balanced. For any $1 \le i_1 \le k$, we have

$$\chi(\delta_D^+(w_{i_1})) = \sum_{k+1\le i_2\le 2k} (x_{j_{i_1}} + Bx_{j_{i_2}}) = kx_{j_{i_1}} + BY_2 = \chi((w, w_{i_1})) = \chi(\delta_D^-(w_{i_1})),$$

as required. Similarly, for for any $k + 1 \le i_2 \le 2k$, we have

$$\chi(\delta_D^-(w_{i_2})) = \sum_{1\le i_1\le k} (x_{j_{i_1}} + Bx_{j_{i_2}}) = Y_1 + Bkx_{j_{i_2}} = \chi((w_{i_2}, w)) = \chi(\delta_D^+(w_{i_2})).$$

Thus we have shown that χ is balanced at $w_1, \ldots, w_{2k}$ and it follows by Proposition 15.1 that χ is balanced also at w.

Balanced assignment $\chi \Rightarrow$ biclique. For the reverse direction of the equivalence, suppose that $\chi : E(D) \to \mathbb{Z}^+$ is a balanced assignment with $\chi(e) \in X_e$ for every $e \in E(D)$. For every $1 \le i_1 \le k$, the definition of $X_{(w,w_{i_1})}$ implies that $\chi((w, w_{i_1}))$ is of the form $kx_{j_{i_1}} + By_{i_1}$ where $x_{j_{i_1}} \in X$ for some $1 \le j_{i_1} \le n$ and y_{i_1} is a positive integer. As $kx_{j_{i_1}} \le kM < B$ follows from $x_{j_{i_1}} \in X$, the value of $\chi((w, w_{i_1}))$ uniquely determines j_{i_1} and y_{i_1}. Similarly, for every $k + 1 \le i_2 \le 2k$, we have that $\chi((w_{i_2}, w))$ is of the form $y_{i_2} + Bkx_{j_{i_2}}$ for uniquely determined positive integers $1 \le j_{i_2} \le n$ and y_{i_2}.

Having defined the values $j_1, \ldots, j_{2k}$, we show that $\chi((w_{i_1}, w_{i_2})) = x_{j_{i_1}} + Bx_{j_{i_2}}$ for every $1 \le i_1 \le k$ and $k + 1 \le i_2 \le 2k$. If this is true, then the vertices $v_{i,j_i} \in V_i$ for $1 \le i \le 2k$ form

a solution of Multicolored Biclique: the fact that $x_{j_{i_1}} + Bx_{j_{i_2}}$ was introduced into $X_{(w_{i_1},w_{i_2})}$ implies that there is an edge between $v_{i_1,j_{i_1}} \in V_{i_1}$ and $v_{i_2,j_{i_2}} \in V_{i_2}$.

As the balance requirement holds at vertex w_{i_1}, it also holds if we count modulo B. We have that $\chi((w, w_{i_1}))$ modulo B is exactly $kx_{j_{i_1}} < B$ (here we use that $kM < B$). For every $k+1 \le i_2 \le 2k$, the value $\chi((w_{i_1}, w_{i_2}))$ modulo B is an integer from X and the value $\chi(\delta_D^+(w_{i_1}))$ modulo B is exactly the sum of these k integers from X (as again by $kM < B$, this sum cannot reach B). Therefore, we have that the sum of k integers from X is exactly $kx_{j_{i_1}}$. Since X is a k-non-averaging set, this is only possible if these k integers are all equal to $x_{j_{i_1}}$. Thus we have shown that $\chi((w_{i_1}, w_{i_2})) = x_{j_{i_1}}$ modulo B for every $1 \le i_1 \le k$ and $k+1 \le i_2 \le 2k$.

Let us consider now a vertex w_{i_2} for $k+1 \le i_2 \le 2k$. The balance requirement in particular implies that $\lfloor \chi(\delta_D^+(w_{i_2}))/B \rfloor = \lfloor \chi(\delta_D^-(w_{i_2}))/B \rfloor$. First, we have $\lfloor \chi(\delta_D^+(w_{i_2}))/B \rfloor = \lfloor \chi((w_{i_2}, w))/B \rfloor = kx_{j_{i_2}}$. Therefore, the fact that χ is balanced at w_{i_2} implies

$$kx_{j_{i_2}} = \lfloor \chi(\delta_D^-(w_{i_2}))/B \rfloor = \left\lfloor \sum_{1 \le i_1 \le k} \chi((w_{i_1}, w_{i_2}))/B \right\rfloor = \sum_{1 \le i_1 \le k} \lfloor \chi((w_{i_1}, w_{i_2}))/B \rfloor,$$

where the third equality holds because we have $\chi((w_{i_1}, w_{i_2}))/B - \lfloor \chi((w_{i_1}, w_{i_2}))/B \rfloor \le M/B < 1/k$. The definition of $X_{(w_{i_1},w_{i_2})}$ implies that $\lfloor \chi((w_{i_1}, w_{i_2}))/B \rfloor$ is an integer from X. Therefore, the equation above states the the sum of k integers from X is exactly $kx_{j_{i_2}}$. Since X is a k-non-averaging set, this is only possible if these k integers are all equal to $x_{j_{i_2}}$. Therefore, we have shown that $\chi((w_{i_1}, w_{i_2}))$ modulo B is exactly $x_{i_{j_1}}$ and $\lfloor \chi((w_{i_1}, w_{i_2}))/B \rfloor$ is exactly $x_{j_{i_2}}$, proving that $\chi((w_{i_1}, w_{i_2})) = x_{j_{i_1}} + Bx_{j_{i_2}}$ indeed holds. □

15.2 Constrained Closed Walk and ATSP

To make the hardness proof for ATSP cleaner, we first prove hardness for the variant of the problem, where instead of optimizing the length of the tour, the only constraint is that certain vertices cannot be visited more than once.

> Constrained Closed Walk: Given an unweighted directed graph G and set $U \subseteq V(G)$ of vertices, find a closed walk (of any length) that visits each vertex at least once and visits each vertex in U exactly once.

There is a simple reduction from Constrained Closed Walk to ATSP that preserves treewidth.

Lemma 15.5. *An instance of* Constrained Closed Walk *on an unweighted directed graph D can be reduced in polynomial time to an instance of* ATSP *with polynomially bounded positive integer weights on an edge-weighted version D^* of D.*

Proof. It is easy to see that if we assign weight 1 to every edge (u, v) with $v \in U$ and weight 0 to every other edge, then the Constrained Closed Walk instance has a solution if and only if the resulting weighted graph has closed walk of length at most $|U|$ (or, equiavelently, less than $|U|+1$) visiting every vertex. To ensure that every weight is positive, let us replace every weight 0 with weight $\epsilon := 1/(2n^2)$. As a minimum solution of ATSP contains at most n^2 edges, this modification increases the minimum cost by at most $1/2$. Thus it remains true that the Constrained Closed Walk instance has a solution if and only if there is a closed walk of length less than $|U|+1$ visiting every vertex. Finally, to ensure that every cost is integer, we multiply each of them by $2n^2$. □

The rest of the section is devoted to giving a lower bound for Constrained Closed Walk. The lower bound proof uses certain gadgets in the construction of the instances. Formally, we define a *gadget* to be a graph with a set of distinguished vertices called *external vertices;* every other vertex is *internal.* To avoid degenerate situations, we always require that the external vertices of a gadget are independent and each external vertex has either indegree 0 or outdegree 0 in the gadget; in particular, this implies that a path between two external vertices contains no other external vertex. Also, this implies that there is no closed walk containing an external vertex.

We say that a set $\mathcal{P}$ of paths of the gadget *satisfies* a gadget if (1) both endpoints of each path are external vertices and (2) every internal vertex of the gadge tis visited by exactly one path in $\mathcal{P}$. If a path $P \in \mathcal{P}$ connects two external vertices of a gadget, then we define the *type* of P to be the (ordered) pair of its endpoints. If $\mathcal{P}$ satisfies the gadget, then we define the *type* of $\mathcal{P}$ to be the multiset of the types of the paths in $\mathcal{P}$. For brevity, we use notation such as $a \times (v_1, v_2) + b \times (v_3, v_4)$ to denote the type that contains a times the pair (v_1, v_2) and b times the pair (v_3, v_4). For a gadget H, we let the set $\mathcal{T}(H)$ contain every possible type of a set $\mathcal{P}$ of paths satisfying H.

We construct gadgets where we can exactly tell the type of the collections of paths that can satisfy the gadget, that is, the set $\mathcal{T}(H)$ is of a certain form. In the first gadget, we have a simple choice between one path or a specified number of paths.

Lemma 15.6. *For every $s \geq 1$, we can construct in time polynomial in n a gadget H_s with the following properties:*

1. *H_s has four external vertices a_{in}, a_{out}, b_{in}, and b_{out}.*
2. *H_s minus its external vertices has constant pathwidth.*
3. *$\mathcal{T}(H_s)$ contains exactly two types: the type $(b_{\text{in}}, b_{\text{out}})$ and the type $s \times (a_{\text{in}}, a_{\text{out}})$ (in other words, the gadget can be satisfied by a path from b_{in} to b_{out}), can be satisfied by a collection of s paths from a_{in} to a_{out}), but cannot be satisfied by any other type of collection of paths).*

Proof. The gadget H_s has $6s$ internal vertices v_j^i ($1 \leq i \leq 6$, $1 \leq j \leq s$) connected as shown in Figure 5(a). Additionally, we introduce the edges (b_{in}, v_1^1), $(v_s^6, b_{\text{out}}$, and for every $1 \leq j \leq s$, the edges (a_{in}, v_j^3) and (v_j^4, a_{out}). It is clear that statement (2) holds: H_s minus its vertices is a graph with constant pathwidth.

This gadget can be satisfied by a path from b_{in} to b_{out} (see Figure 5(b)) and also by a collection of s paths where the j-th path is a_{in}, v_j^3, v_j^2, v_j^1, v_j^6, v_j^5, v_j^4, a_{out} (see Figure 5(c)). To complete the proof of statement (3), we need to show that if $\mathcal{P}$ satisfies H_s, then $\mathcal{P}$ is one of these two types. The basic observation is that if a path in $\mathcal{P}$ contains v_j^2, then it has to contain v_j^1 and v_j^3 as well (as each internal vertex is visited exactly once), hence the three vertices v_j^1, v_j^2, v_j^3 have to appear on the same path of $\mathcal{P}$. The same is true for the vertices v_j^4, v_j^5, v_j^6. Suppose that $\mathcal{P}$ contains a path P starting at b_{in}. Then its next vertex is v_1^1, which should be followed by v_1^2 and v_1^3 by the argument above. The next vertex is v_1^4 (the only outneighbor of v_j^3 not yet visited), which is followed by v_1^5 and v_1^6. Now the next vertex is v_2^1, the only outneighbor of v_1^6 not yet visited. With similar arguments, we can show that P is exactly of the form shown in Figure 5(b), hence $\mathcal{P}$ contains only this path, and $\mathcal{P}$ is of type $\{(b_{\text{in}}, b_{\text{out}})\}$.

Suppose now that $\mathcal{P}$ does not contain a path starting at b_{in}. Then the only way to reach vertex v_1^1 is with a path starting as a_{in}, v_1^3, v_1^2, v_1^1. This has to be followed by the unique outneighbor v_1^6 of v_1^1 that was not yet visited. This means that the path contains also v_1^5 and v_1^4, which has to be

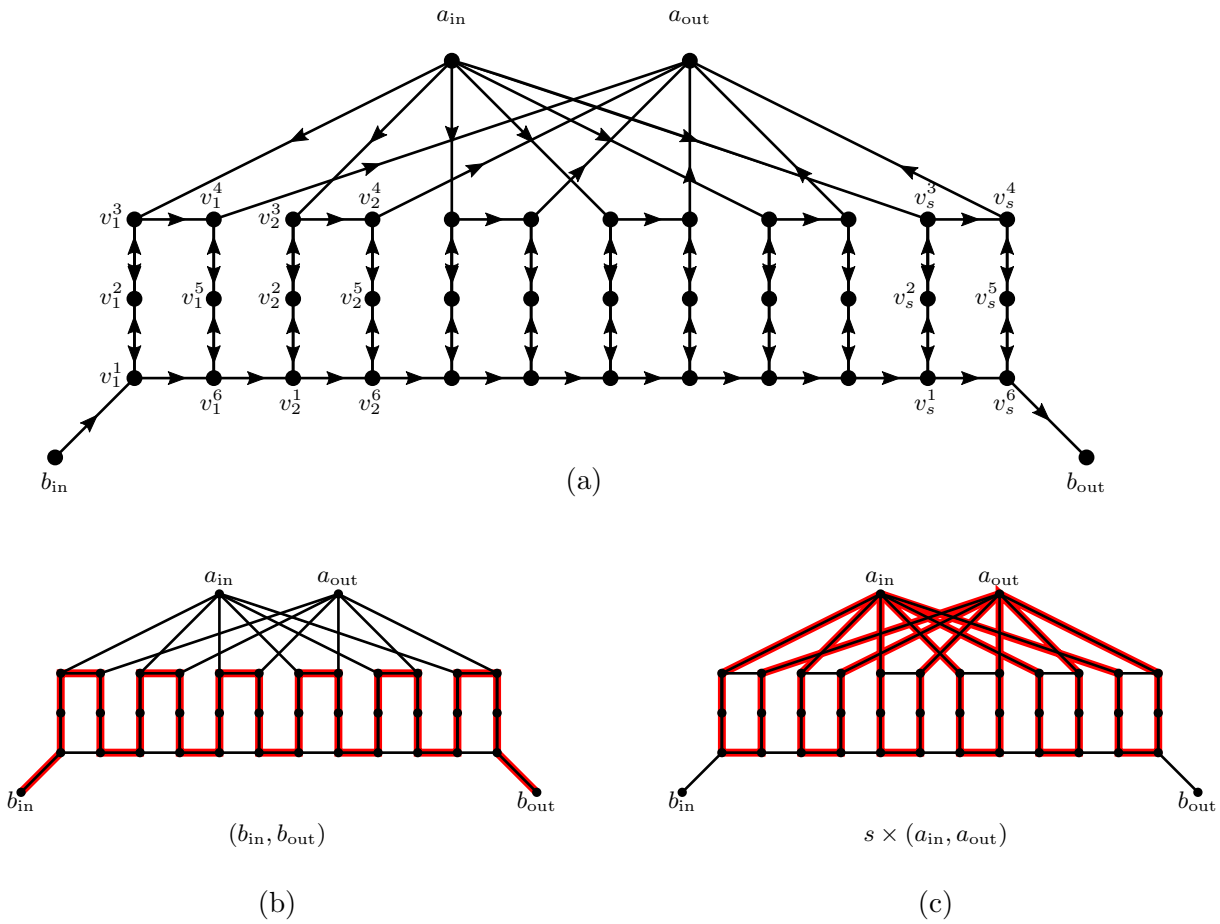


Figure 5: The gadget of Lemma 15.6 with two collections of paths satisfying it.

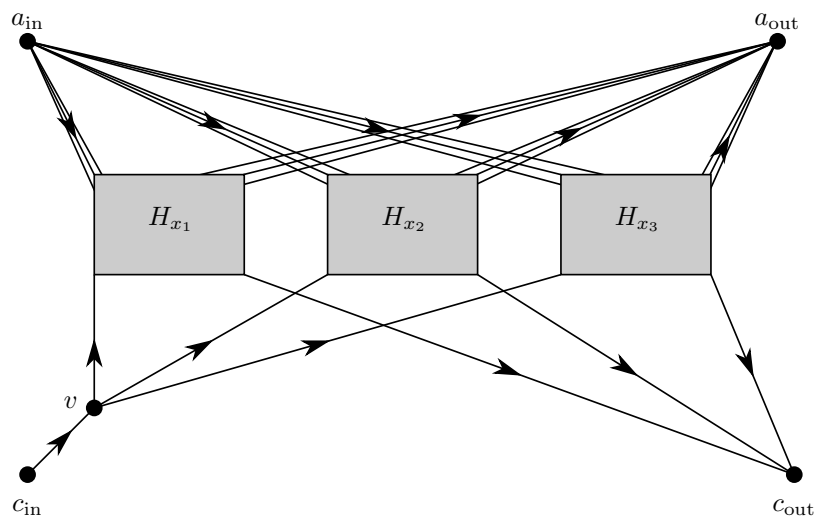


Figure 6: The gadget H_X of Lemma 15.7 for a set $X = \{x_1, x_2, x_3\}$ of three integers. The gray rectangles represent the *internal* vertices of the three gadgets H_{x_1}, H_{x_2}, and H_{x_3}.

followed by a_{out}. Then with similar arguments, we can show for every $j \geq 2$ that v_j^1 is visited by the path a_{in}, v_j^3, v_j^2, v_j^1, v_j^6, v_j^5, v_j^4, a_{out}. This means that $|\mathcal{P}| = s$ and the type of $\mathcal{P}$ is $s \times (a_{\text{in}}, a_{\text{out}})$. $\square$

Lemma 15.7. *Let X be a set of n positive integers, each at most M and let $S = \sum_{x \in X} x$. In time polynomial in n and M, we can construct a gadget H_X with the following properties:*

1. *H_X has four external vertices a_{in}, a_{out}, c_{in}, and c_{out}.*
2. *H_X minus its external vertices has constant pathwidth.*
3. *$\mathcal{T}(H_s)$ contains exactly $|X|$ types: for every $x \in X$, it contains the type $(c_{\text{in}}, c_{\text{out}}) + (S - x) \times (a_{\text{in}}, a_{\text{out}})$.*

Proof. The gadget H_X is constructed the following way (see Figure 6). Let us introduce an internal vertex v and the edge (c_{in}, v). For every $x \in X$, let us introduce a copy of H_x defined by Lemma 15.6 where a_{in}, a_{out}, v, c_{out} of H_X play the role of a_{in}, a_{in}, b_{in}, b_{out}, respectively. If we remove the four external vertices of H_X, then we get a graph with constant pathwidth: if we remove one more vertex, v, then we get the disjoint union of internal vertices of the gadgets H_x's, which have constant pathwidth by Lemma 15.6.

For every $x \in X$, the gadget can be satisfied by the following collection of paths. The copy of H_x in H_X can be satisfied by a path from v to c_{out}, which can be extended with the edge (c_{in}, v) to a path from c_{in} to c_{out}. For every $x' \in X$, $x' \neq x$, we can satisfy the copy of $H_{x'}$ in H_X by a collection of x' paths from a_{in} to a_{out}. This way, we constructed a collection $\mathcal{P}$ of paths satisfying H_X that consists of a single path of type $(c_{\text{in}}, c_{\text{out}})$ and exactly $\sum_{x' \in X \setminus \{x\}} x' = S - x$ paths of type $(a_{\text{in}}, a_{\text{out}})$.

To complete the proof of statement (3), consider a collection $\mathcal{P}$ of paths satisfying H_X. Let P be the unique path of $\mathcal{P}$ visiting vertex v. The vertex of P after v is has to be an internal vertex of the copy of H_x for some $x \in X$ (here we use that the external vertices of the gadget H_x are

independent, hence v cannot be followed by any of a_{in}, a_{out}, and c_{out}). As v was identified with vertex b_{in} of H_x, Lemma 15.6 implies that P visits every internal vertex of this copy of H_x and leaves H_x at its vertex b_{out}, which was identified with c_{out}. Consider now some $x' \in X$ with $x' \neq x$. Vertex b_{in} of $H_{x'}$ was identified with v, path P is the only path of $\mathcal{P}$ visiting v, and P does not visit any internal vertex of $H_{x'}$. Therefore, by Lemma 15.6, the internal vertices of $H_{x'}$ are visited by exactly x' paths of type $(a_{\text{in}}, a_{\text{out}})$. Thus $\mathcal{P}$ contains one path of type $(c_{\text{in}}, c_{\text{out}})$ and exactly $\sum_{x' \in X \setminus \{x\}} x' = S - x$ paths of type $(a_{\text{in}}, a_{\text{out}})$. □

Lemma 15.8. *Assuming ETH, there is no $f(p)n^{o(p)}$ time algorithm for* Constrained Closed Walk *on graphs of pathwidth at most p for any computable function f.*

Proof. The proof is by reduction from Edge Balancing on a directed graph D with k vertices $w_1, \ldots, w_k$. We construct a Constrained Closed Walk instance on a directed graph D^* the following way. First, let us introduce the vertices $w_1, \ldots, w_k$ into D^*, as well as two auxiliary vertices c_{in} and c_{out}. For every edge $e = (w_{i_1}, w_{i_2}) \in E(D)$ with a set X_e of integers associated to it in the Edge Balancing instance, we construct a copy of the gadget H_{X_e} defined by Lemma 15.7 and identify external vertices a_{in}, a_{out}, c_{in}, c_{out} of the gadget H_{X_e} with vertices w_{i_1}, w_{i_2}, c_{in}, c_{out} of D^*, respectively. Let $S_e = \sum_{x \in X_e} x$ for every edge $e \in V(D)$, let $S_i^+ = \sum_{e \in \delta_D^+(w_i)} S_s$, let $S_i^- = \sum_{e \in \delta_D^-(w_i)} S_s$, and let $S^* = \sum_{e \in E(D)} X_e = \sum_{i=1}^k S_i^+ = \sum_{i=1}^k S_i^-$. We further extend D^* the following way.

1. For every $1 \leq i \leq k$, we introduce a set $\mathcal{P}_i^+$ of S_i^+ paths of length two from c_{in} to w_i (that is, each of these paths consists of vertex c_{in}, vertex w_i, and one extra newly introduced vertex).
2. For every $1 \leq i \leq k$, we introduce a set $\mathcal{P}_i^-$ of S_i^- paths of length two from w_i to c_{out}.
3. We introduce a set $\mathcal{P}^*$ of $S^* + |E(D)|$ paths of length two from c_{out} to c_{in}.

Let $Z := \{w_1, \ldots, w_k, c_{\text{in}}, c_{\text{out}}\}$; note that Z form an independent set in G^* (as the external vertices of each gadget are independent). We define $U := V(D^*) \setminus Z$ to be the set of vertices that have to be visited exactly once. This completes the description of the reduction.

Observe that if we remove Z from D^*, then what remains is the disjoint union of the internal vertices of the gadgets H_{X_e}, which have constant pathwidth by Lemma 15.7. As removing a vertex can decrease pathwidth at most by one, it follows that D^* has pathwidth $|Z| + O(1) = O(k)$. Thus if we are able to show that the constructed instance D^* of Constrained Closed Walk is a yes-instance if and only if D is a yes-instance of Edge Balancing, then this implies that an $f(p)n^{o(p)}$ time algorithm for Constrained Closed Walk on graphs of pathwidth p can be used to solve Edge Balancingon k vertex graphs in time $(k)n^{o(k)}$, which would contradict ETH by Lemma 15.4.

Balanced assignment $\chi \Rightarrow$ closed walk. Suppose that balanced assignment $\chi : E(D) \to \mathbb{Z}^+$ is a solution to the Edge Balancing instance. For every $e = (w_{i_1}, w_{i_2}) \in E(D)$, the construction of the gadget H_{X_e} implies that H_{X_e} can be satisfied by a collection $\mathcal{P}_e$ of paths having type $(c_{\text{in}}, c_{\text{out}}) + (S_e - \chi(e)) \times (w_{i_1}, w_{i_2})$. Let $\mathcal{P}$ be a collection of paths that is the union of the set $\mathcal{P}^*$, the sets $\mathcal{P}_i^+$ and $\mathcal{P}_i^-$ for $1 \leq i \leq k$, and the set $\mathcal{P}_e$ for $e \in E(G)$. Observe that every vertex of U is contained in exactly one path in $\mathcal{P}$ and the paths in $\mathcal{P}$ are edge disjoint. Let H^* be the subgraph of D^* formed by the union of every path in $\mathcal{P}$. It is easy to see that H^* is connected: every path in $\mathcal{P}$ has endpoints in Z and the paths in $\mathcal{P}^*$, $\mathcal{P}_i^-$, $\mathcal{P}_i^+$ ensure that every vertex of Z is in the same

component of H^*. It is also clear that every vertex of U has indegree and outdegree exactly 1, as each vertex in U is visited by exactly one path in $\mathcal{P}$. We show below that every vertex of Z is balanced in H^* (its indegree equals its outdegree). If this is true, then H^* has a closed Eulerian walk, which gives a closed walk in G^* visiting every vertex at least once and every vertex in U exactly once, what we had to show.

The endpoints of every path in $\mathcal{P}$ are in Z, hence every vertex of U is balanced in H^* (in particular has indegree and outdegree exactly 1). Consider now a vertex w_i.

- For every $e \in \delta_D^+(w_i)$, the set $\mathcal{P}_e$ contains $S_e - \chi(e)$ paths starting at w_i.
- For every $e \in \delta_D^-(w_i)$, the set $\mathcal{P}_e$ contains $S_e - \chi(e)$ paths ending at w_i.
- The set $\mathcal{P}_i^+$ contains S_i^+ paths ending at w_i.
- The set $\mathcal{P}_i^-$ contains S_i^- paths starting at w_i.

As these paths are edge disjoint, the difference between the outdegree and the indegree of w_i in H^* is

$$(S_i^- + \sum_{e \in \delta_D^+(w_i)} (S_e - \chi(e))) - (S_i^+ + \sum_{e \in \delta_D^-(w_i)} (S_e - \chi(e))) = (S_i^- + S_i^+ - \chi(\delta_D^+(w_i))) - (S_i^+ + S_i^- - \chi(\delta_D^-(w_i))) = 0,$$

since χ is balanced at w_i. Consider now vertex c_{in}.

- For every $1 \leq i \leq k$, the set $\mathcal{P}_i^+$ contains S_i^+ paths starting at c_{in}.
- For every $e \in E(D)$, the set $\mathcal{P}_e$ contains one path starting at c_{in}.
- The set $\mathcal{P}^*$ contains $S^* + |E(D)|$ paths ending at c_{in}.

It follows that c_{in} is balanced in H^* with indegree and outdegree exactly $S^* + |E(D)| = \sum_{i=1}^k S_i^+ + |E(D)|$ and a similar argument shows the same for c_{out}. Thus we have shown that the Constrained Closed Walk instance has a solution.

Closed walk $\Rightarrow$ balanced assignment χ. For the reverse direction, suppose that the constructed Constrained Closed Walk instance has a solution (a closed walk W). The closed walk can be split into a collection $\mathcal{P}$ of walks with endpoints in Z and every internal vertex in U. In fact, these walks are paths: (1) as each vertex of U is visited only once, the internal vertices of each walk are distinct, (2) the walk cannot be a cycle, since we have stated earlier that no gadget has a cycle through an external vertex. When defining the sets $\mathcal{P}^*$, $\mathcal{P}_i^+$, $\mathcal{P}_i^-$, we introduced a large number of vertices into D^* with indegree and outdegree 1. The fact that these vertices are visited implies that $\mathcal{P}$ has to contain the set $\mathcal{P}^*$ and the sets $\mathcal{P}_i^+$ and $\mathcal{P}_i^-$ for every $1 \leq i \leq k$. Moreover, every path of $\mathcal{P}$ not in these sets contains an internal vertex of some gadget H_{X_e} (here we use that Z is independent) and a path of $\mathcal{P}$ cannot contain the internal vertices of two gadgets (as this would imply that it has an internal vertex in Z). Therefore, the remaining paths can be partitioned into sets $\mathcal{P}_e$ for $e \in E(D)$ such that the internal vertices of H_{X_e} are used only by the paths in $\mathcal{P}_e$. This means that the set $\mathcal{P}_e$ satisfies gadget H_{X_e}. If $e = (w_{i_1}, w_{i_2})$, then it follows by Lemma 15.7 that $\mathcal{P}_e$ has type $(c_{\text{in}}, c_{\text{out}}) + (S_e - \chi(e))(w_{i_1}, w_{i_2})$ for some integer $\chi(e) \in X_e$. In particular, this means that $\mathcal{P}_e$ contains $S_e - \chi(e)$ paths starting at w_{i_1} and the same number of paths ending at w_{i_2}.

We claim that χ form a solution of the Edge Balancing problem. Consider a vertex w_i. Taking into account the contribution of the paths in $\mathcal{P}_i^-$ and $\mathcal{P}_e$ for $e \in \delta_D^+(w_i)$, we have that the outdegree of w_i in the walk W is exactly

$$S_i^- + \sum_{e \in \delta_D^+(w_i)} (S_e - \chi(e)) = S_i^+ + S_i^- - \chi(\delta_D^+(w_i)).$$

Taking into account the contribution of the paths in $\mathcal{P}_i^+$ and $\mathcal{P}_e$ for $e \in \delta_D^-(w_i)$, we have that the indegree of w_i in the walk W is exactly

$$S_i^+ + \sum_{e \in \delta_D^-(w_i)} (S_e - \chi(e)) = S_i^- + S_i^+ - \chi(\delta_D^-(w_i)).$$

As the indegree of w_i in W is clearly the same as its outdegree, these two values have to be equal. This is only possible if $\chi(\delta_D^+(w_i)) = \chi(\delta_D^-(w_i))$, that is χ is balanced at w_i. As this is true for every $1 \leq i \leq k$, it follows that the Edge Balancing instance has a solution. □

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