# Why are Casimir energy differences so often finite?



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Abstract:
One of the very first applications of the quantum field theoretic vacuum state was in the development of the notion of Casimir energy. Now field theoretic Casimir energies, considered individually, are always infinite. But *differences* in Casimir energies are quite often finite — a fortunate circumstance which luckily made some of the early calculations, (for instance, for parallel plates and hollow spheres), tolerably tractable. We shall explore the extent to which this observation can be systematised. For instance: What are necessary and sufficient conditions for Casimir energy *differences* to be finite? When the Casimir energy *differences* are not finite, can anything useful be said? We shall see that it is the *difference* in the first few Seeley–DeWitt coefficients that is central to answering these questions. In particular, for any collection of conductors (perfect or imperfect) and/or dielectrics, as long as one merely moves them around without changing shape or volume, then the Casimir energy difference (and so the Casimir forces) are guaranteed finite.

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## Contents

## 1 Introduction

Quantum field theoretic Casimir energies (considered in isolation) are typically infinite, requiring both regularization and renormalization to extract mathematically sensible answers, this at the cost of sometimes obscuring the physics [1–5]. On the other hand Casimir energy *differences* are quite often finite, and have a much more direct physical interpretation [1, 2]. Additional background and general developments may be found in references [6–15]. In this article, I shall first argue (mathematically) that there are a large number of interesting physical situations where the Casimir energy differences, (and so the Casimir energy forces), are automatically known to be finite, even before starting specific computations. Secondly, I shall argue (mathematically) that one can often develop physically interesting "reference models" such that the Casimir energy difference between the physical system and the "reference model" is known to be finite, even before starting specific computations. (I will not actually calculate any Casimir energies — knowing that the result you are after is finite is often more than half the battle.)

I shall first start with a simple formal argument to get the discussion oriented, and then provide a more careful argument in terms of regularized (but not renormalized) Casimir energies.

## 2 Formal argument

The formal argument starts with the *exact* result that:

**Lemma 1 (Exact)**

$$\omega - \omega_* = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}} \left\{ e^{-\omega_*^2 t} - e^{-\omega^2 t} \right\}. \tag{2.1}$$

□

Now let $\omega_n$ and $(\omega_*)_n$ be two infinite sequences of numbers then, again as an exact result:

$$\sum_{n=1}^{N} \{\omega_n - (\omega_*)_n\} = \frac{1}{\sqrt{4\pi}} \sum_{n=1}^{N} \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}} \left\{ e^{-(\omega_*^2)_n t} - e^{-\omega_n^2 t} \right\}. \tag{2.2}$$

Taking $N \to \infty$ and formally interchanging integral and summation, (*and I will justify this much more carefully later on*):

$$\sum_n \{\omega_n - (\omega_*)_n\} = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}} \sum_n \left\{ e^{-(\omega_*)_n^2 t} - e^{-\omega_n^2 t} \right\}. \tag{2.3}$$

Then in terms of the heat kernel $K(t)$ defined by

$$K(t) = \sum_n e^{-\omega_n^2 t}, \tag{2.4}$$

we formally have:

$$\sum_n \{\omega_n - (\omega_*)_n\} = \int_0^\infty \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} \{K_*(t) - K(t)\}. \tag{2.5}$$

But, (now assuming that the $\omega_n^2$ and $(\omega_*)_n^2$ are in fact the eigenvalues of some second-order linear differential operators), by the standard Seeley–DeWitt expansion we have both

$$K(t) = (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{N} a_{i/2}\; t^{i/2} + \mathcal{O}\left(t^{(N+1)/2}\right) \right\}, \tag{2.6}$$

and

$$K_*(t) = (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{N} (a_*)_{i/2}\; t^{i/2} + \mathcal{O}\left(t^{(N+1)/2}\right) \right\}. \tag{2.7}$$

Note $d$ is the number of space dimensions. As will be discussed more fully below, the integer indexed $a_n$ have both bulk and boundary contributions, while the half-integer indexed $a_{n+\frac{1}{2}}$ have only boundary contributions. Then for the difference in heat kernels we have:

$$K_*(t) - K(t) = (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{N} \left\{ (a_*)_{i/2} - a_{i/2} \right\}\; t^{i/2} + \mathcal{O}\left(t^{(N+1)/2}\right) \right\}. \tag{2.8}$$

Now choose $N = d+1$, then formally

$$\sum_n \{\omega_n - (\omega_*)_n\} = \int_0^\infty \frac{\mathrm{d}t}{t} (4\pi t)^{-(d+1)/2} \left\{ \sum_{i=0}^{d+1} \left\{ (a_*)_{i/2} - a_{i/2} \right\}\; t^{i/2} \right\} + (\text{UV finite}). \tag{2.9}$$

Here the designation "UV finite" means that any remaining terms contributing to the "UV finite" piece are now guaranteed to not have any infinities coming from the $t \to 0$ region of integration. That is, taking $E_{\text{Casimir}} = \frac{1}{2}\hbar \sum_n \omega_n$, we have the formal result:

**Lemma 2 (Formal)**

$$\Delta(\text{Casimir Energy}) = -\frac{\hbar}{2}\int_0^\infty \frac{\mathrm{d}t}{t}(4\pi t)^{-(d+1)/2}\left\{\sum_{i=0}^{d+1}\Delta a_{i/2}\; t^{i/2}\right\} + (\text{UV finite}). \quad (2.10)$$

□

All of the potentially UV-divergent terms are now concentrated in the $d+2$ leading terms proportional to the $\Delta a_i$. The rest of the article will involve several refinements on this simple theme.

Note that for finiteness:

- In 3+1 dimensions we would want $\Delta a_0 = \Delta a_{1/2} = \Delta a_1 = \Delta a_{3/2} = \Delta a_2 = 0$.
- In 2+1 dimensions we would want $\Delta a_0 = \Delta a_{1/2} = \Delta a_1 = \Delta a_{3/2} = 0$.
- In 1+1 dimensions we would want $\Delta a_0 = \Delta a_{1/2} = \Delta a_1 = 0$.
- In 0+1 dimensions we would want $\Delta a_0 = \Delta a_{1/2} = 0$.

Generally, in $d$ space dimensions, if we are comparing two physical systems for which the first $d+2$ Seeley–DeWitt coefficients are equal, then the difference in Casimir energies will be finite.

## 3 Exact argument

Let us now regularize everything a little more carefully, to develop an exact rather than formal argument. (Initially we shall use the complementary error function [$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$] as a particularly simple and mathematically transparent reguator, but will subsequently show that physically almost any smooth cutoff function will do.) We have the *exact* result that:

**Lemma 3 (Exact)**

$$\omega\;\mathrm{erfc}(\omega/\Omega) = \frac{\Omega}{\sqrt{\pi}}\; e^{-\omega^2/\Omega^2} - \frac{1}{\sqrt{4\pi}}\int_{\Omega^{-2}}^\infty \frac{\mathrm{d}t}{t^{3/2}}\; e^{-\omega^2 t}\;. \quad (3.1)$$

□

This leads to the further exact result that:

$$\sum_n \omega_n\;\mathrm{erfc}(\omega_n/\Omega) = \frac{\Omega}{\sqrt{\pi}}\sum_n e^{-\omega_n^2/\Omega^2} - \frac{1}{\sqrt{4\pi}}\sum_n \int_{\Omega^{-2}}^\infty \frac{\mathrm{d}t}{t^{3/2}}\; e^{-\omega_n^2 t}\;. \quad (3.2)$$

But, because all the relevant quantities are guaranteed finite, we can now exchange sum and integral to obtain the exact (no longer just formal) result:

$$\sum_n \omega_n \,\mathrm{erfc}(\omega_n/\Omega) = \frac{\Omega}{\sqrt{\pi}} \sum_n e^{-\omega_n^2/\Omega^2} - \frac{1}{\sqrt{4\pi}} \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}} \sum_n e^{-\omega_n^2 t}\,. \tag{3.3}$$

Then in terms of the heat kernel:

$$\sum_n \omega_n \,\mathrm{erfc}(\omega_n/\Omega) = \frac{\Omega}{\sqrt{\pi}}\, K(\Omega^{-2}) - \frac{1}{\sqrt{4\pi}} \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}}\, K(t). \tag{3.4}$$

Now apply the Seeley–DeWitt expansion:

$$K(t) = (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{N} a_{i/2}\; t^{i/2} + \mathcal{O}\left(t^{(N+1)/2}\right) \right\}. \tag{3.5}$$

But then (now choosing $N = d$) for the heat kernel term we have:

$$\frac{\Omega}{\sqrt{\pi}} K(\Omega^{-2}) = 2 \left(\frac{\Omega}{\sqrt{4\pi}}\right)^{d+1} \left\{ \sum_{i=0}^{d} a_{i/2}\; \Omega^{-i} \right\} + (\text{finite as } \Omega \to \infty). \tag{3.6}$$

Working with the integral term is a little trickier. In the integral we instead choose $N = d+1$. Then, treating the logarithmic term separately, we have

$$\int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} K(t) = \frac{1}{\sqrt{4\pi}} \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}} (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{d} \left\{a_{i/2}\right\}\; t^{i/2} \right\}$$
$$+ \frac{a_{(d+1)/2}}{(4\pi)^{(d+1)/2}} \ln(\Omega^2) + (\text{finite as } \Omega \to \infty). \tag{3.7}$$

That the $a_{(d+1)/2}$ term leads to logarithmic term in the Casimir energy (and effective action) is well-known. See for instance references [5, 16, 17]. Performing the remaining integrals:

$$\int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} K(t) = -\frac{1}{(4\pi)^{(d+1)/2}} \left\{ \sum_{i=0}^{d} \frac{a_{i/2}\; \Omega^{d+1-i}}{d+1-i} \right\}$$
$$+ \frac{a_{(d+1)/2}}{(4\pi)^{(d+1)/2}} \ln(\Omega^2) + (\text{finite as } \Omega \to \infty). \tag{3.8}$$

Now assembling all the pieces:

$$\sum_n \omega_n \,\mathrm{erfc}(\omega_n/\Omega) = 2 \left(\frac{\Omega}{\sqrt{4\pi}}\right)^{d+1} \left\{ \sum_{i=0}^{d} \left\{a_{i/2}\right\}\; \Omega^{-i} \right\}$$
$$+ \frac{1}{(4\pi)^{(d+1)/2}} \left\{ \sum_{i=0}^{d} \frac{a_{i/2}\; \Omega^{d+1-i}}{d+1-i} \right\} + \frac{a_{(d+1)/2}}{(4\pi)^{(d+1)/2}} \ln(\Omega^2)$$
$$+ (\text{finite as } \Omega \to \infty). \tag{3.9}$$

We now have the exact result:

**Lemma 4 (Exact)**

$$\sum_n \omega_n \, \mathrm{erfc}(\omega_n/\Omega) = \left\{ \sum_{i=0}^{d} k_i \; a_{i/2} \; \Omega^{d+1-i} \right\} + k_{(d+1)/2} \; a_{(d+1)/2} \; \ln(\Omega^2)$$
$$+(\text{finite as } \Omega \to \infty). \tag{3.10}$$

*For our current purposes the specific values of the dimensionless coefficients $k_i$ are not important.* □

## 4 erfc-regularized Casimir Energy

Now define the erfc-regularized Casimir energy:

$$(\text{Casimir energy})_{\text{erfc-regularized}} = \frac{1}{2}\hbar \sum_n \omega_n \, \mathrm{erfc}(\omega_n/\Omega). \tag{4.1}$$

This is guaranteed to be finite as long as $\Omega$ is finite. Then:

**Theorem 1 (erfc-regularized Casimir energy)**

$$(\text{Casimir energy})_{\text{erfc-regularized}} = \frac{1}{2}\hbar \left\{ \sum_{i=0}^{d} k_i \; a_{i/2} \; \Omega^{d+1-i} \right\} + \frac{1}{2}\hbar k_{(d+1)/2} \; a_{(d+1)/2} \; \ln(\Omega^2)$$
$$+(\text{finite as } \Omega \to \infty). \tag{4.2}$$

□

Now take differences:

$$\Delta(\text{Casimir energy})_{\text{erfc-regularized}} = \frac{1}{2}\hbar \left\{ \sum_{i=0}^{d} k_i \; \Delta a_{i/2} \; \Omega^{d+1-i} \right\} + \frac{1}{2}\hbar k_{(d+1)/2} \; \Delta a_{(d+1)/2} \; \ln(\Omega^2)$$
$$+(\text{finite as } \Omega \to \infty). \tag{4.3}$$

Therefore, if the first $d+2$ Seeley–DeWitt coefficients, [from 0 to $(d+1)/2$], are unchanged,

$$\Delta a_0 = \Delta a_{1/2} = \cdots = \Delta a_{(d+1)/2} = 0, \tag{4.4}$$

then:

$$\Delta(\text{Casimir energy})_{\text{erfc-regularized}} = (\text{finite as } \Omega \to \infty). \tag{4.5}$$

*We can now safely take the limit as the cutoff is removed* $(\Omega \to \infty)$. We have:

**Theorem 2 (Casimir energy differences)**
*If we compare two systems where the first $d+2$ Seeley–DeWitt coefficients are equal,*

$$\Delta a_0 = \Delta a_{1/2} = \cdots = \Delta a_{(d+1)/2} = 0, \tag{4.6}$$

*then:*

$$\Delta(\text{Casimir energy}) = (\text{finite}). \tag{4.7}$$

□

This is a very nice mathematical theorem, but how relevant is it to real world physics? Just how general is this phenomenon?

## 5 Unchanging Seeley–DeWitt coefficients

Perhaps unexpectedly, there are *very many* physically interesting situations where the (first few) Seeley–DeWitt coefficients are unchanging. The pre-eminent cases are these:

- Parallel plates.
- Thin spherical shells.

In both of these cases an infra-red regulator is needed, and some subtle thought is still required. Much more radically:

- Take any collection of perfect conductors. Move them around relative to each other. (Without distorting their shapes and/or volumes.)
- Then the change in Casimir energy is finite.
- Then the Casimir forces are finite.

(Subsequently, we shall show that similar comments can be made for both imperfect conductors and dielectrics.)

To establish these results we note that for a region $\mathbf{V}$ with boundary $\partial\mathbf{V}$ we have the quite standard results that:

$$
\begin{aligned}
a_0 &\propto \int_{\mathbf{V}} 1 \, \sqrt{g_d} \, \mathrm{d}^d x = (\text{volume}); \\
a_{1/2} &\propto \int_{\partial\mathbf{V}} 1 \, \sqrt{g_{d-1}} \, \mathrm{d}^{d-1} x = (\text{surface area}); \\
a_1 &\propto \int_{\mathbf{V}} \{R, V\} \, \sqrt{g_d} \, \mathrm{d}^d x + \int_{\partial\mathbf{V}} \{K\} \, \sqrt{g_{d-1}} \, \mathrm{d}^{d-1} x. \\
a_{3/2} &\propto \int_{\partial\mathbf{V}} \{R, V, K^2, K_{ij}K^{ij}\} \, \sqrt{g_{d-1}} \, \mathrm{d}^{d-1} x. \\
a_2 &\propto \int_{\mathbf{V}} \{R^2, V^2, RV, \nabla^2 R, \nabla^2 V, R_{ab}R^{ab}, R_{abcd}R^{abcd}\} \, \sqrt{g_d} \, \mathrm{d}^d x \\
&\quad + \int_{\partial\mathbf{V}} \{R_{;n}, V_{;n}, K_{ii:jj}, K_{ij:ij}, VK, K^3, \mathrm{tr}(K^2)K, \mathrm{tr}(K^3)\} \, \sqrt{g_{d-1}} \, \mathrm{d}^{d-1} x \\
&\quad + \int_{\partial\mathbf{V}} \{RK, g^{ij}R_{ninj}K, R_{ninj}K^{ij}, g^{ik}R_{ijkl}K^{jl}\} \, \sqrt{g_{d-1}} \, \mathrm{d}^{d-1} x.
\end{aligned}
$$

Here the $\{\_,\_,\_\}$ denote various species-dependent linear combinations of the relevant terms. For current purposes we do not need to know the specific values of any of the dimensionless coefficients. (There are also contributions to the $a_i$ from kinks and corners; but let's stay with smooth boundaries for now.) Above we have retained terms due to both intrinsic and extrinsic curvature, plus a scalar potential $V(x)$. One could in principle obtain even more terms from background electromagnetic or gauge fields, but the terms retained above are sufficient for current purposes.

### 5.1 Parallel plates

Working with QED (so $V = 0$) in flat spacetime (Riemann tensor zero) with flat boundaries (extrinsic curvature zero):

$$
\begin{aligned}
a_0 &\propto (\text{volume}); \\
a_{1/2} &\propto (\text{surface area}); \\
a_1 &= 0; \\
a_{3/2} &= 0; \\
a_2 &= 0.
\end{aligned}
$$

So for finite Casimir energy differences one just needs to keep volume and surface area fixed. For example: Apply periodic boundary conditions in $d-1$ spatial directions, and apply conducting box boundary conditions in the remaining spatial direction.

Physically this means you put the Casimir plates inside a big box, of fixed size, with two faces parallel to the plates. Consider the situation where one varies the distance between the Casimir plates while keeping the size of the big box (the infra-red [IR] regulator) fixed. From the above, and with no further calculation required, we can at least deduce that the Casimir energy difference (and so the Casimir force between the plates) is finite.

### 5.2 Hollow spheres

We are now working with QED in flat spacetime with thin spherical boundaries. The idea is to understand as much as we can regarding Boyer's calculation [2], but without explicit computation. (We shall assume 3+1 dimensions.)

#### 5.2.1 Step I (QED in flat spacetime)

Using only the fact that we are working with QED ($V = 0$) in flat spacetime (Riemann tensor zero):

$$\begin{aligned} a_0 &\propto (\text{volume}); \\ a_{1/2} &\propto (\text{surface area}); \\ a_1 &\propto \int_{\partial \mathbf{V}} \{K\} \sqrt{g_2}\, \mathrm{d}^2x; \\ a_{3/2} &\propto \int_{\partial \mathbf{V}} \{K^2, K_{ij}K^{ij}\} \sqrt{g_2}\, \mathrm{d}^2x; \\ a_2 &\propto \int_{\partial \mathbf{V}} \{g^{ij}g^{kl}K_{ij:kl}, K^{ij}{}_{:ij}, K^3, \mathrm{tr}(K^2)K, \mathrm{tr}(K^3)\} \sqrt{g_2}\, \mathrm{d}^2x. \end{aligned}$$

Since the extrinsic curvature is non-zero, $K \neq 0$, keeping control of the higher $a_i$, the higher-order Seeley–DeWitt coefficients, is now a little trickier.

#### 5.2.2 Step II (thin boundaries)

As long as the boundaries are thin, then $K_{\text{inside}} = -K_{\text{outside}}$, leading to cancellations in both $a_1$ and $a_2$. Similarly the thin boundaries take up zero volume, so the total volume is held fixed. (The outermost boundary, the IR regulator, is always held fixed.) Then:

$$\begin{aligned} \Delta a_0 &\to 0; \\ \Delta a_{1/2} &\propto \Delta(\text{surface area}); \\ \Delta a_1 &\to 0; \\ \Delta a_{3/2} &\propto \Delta \int_{\partial \mathbf{V}} \{K^2, K_{ij}K^{ij}\} \sqrt{g_2}\, \mathrm{d}^2x; \\ \Delta a_2 &\to 0. \end{aligned}$$

### 5.2.3 Step III (rescaling — conformal invariance)

As long as the inner boundaries for the two situations we are considering are simply rescaled versions of each other, then $\int_{\partial \mathbf{V}} KK \sqrt{g_2}\, \mathrm{d}^2x$ is scale invariant, thus leading to a cancellation in $a_{3/2}$. (The outermost boundary, the IR regulator, is always held fixed.) Then:

$$\begin{aligned} \Delta a_0 &\to 0; \\ \Delta a_{1/2} &\propto \Delta(\text{surface area}); \\ \Delta a_1 &\to 0; \\ \Delta a_{3/2} &\to 0; \\ \Delta a_2 &\to 0. \end{aligned}$$

Note we still have to deal with $\Delta a_{1/2}$.

### 5.2.4 Step IV (TE and TM modes)

In spherical symmetry, one can easily define TE and TM modes. Note that they have equal and opposite contributions to $a_{1/2}$, again leading to a cancellation in $a_{1/2}$. (The outermost boundary is always held fixed.) Then:

$$\begin{aligned} \Delta a_0 &\to 0; \\ \Delta a_{1/2} &\to 0; \\ \Delta a_1 &\to 0; \\ \Delta a_{3/2} &\to 0; \\ \Delta a_2 &\to 0. \end{aligned}$$

This finally is enough to guarantee finiteness of the Casimir energy difference.

### 5.2.5 Step V (finiteness)

From the above we have

$$\Delta(\text{Casimir Energy}) = (\text{finite}). \tag{5.1}$$

This observation underlies the otherwise quite "miraculous cancellations" in Boyer's calculation of the Casimir energy of a hollow sphere [2]. Comparing two hollow spheres of radius $a$ and $b$; and letting the IR regulator (which is the same for each sphere) move out to infinity:

$$\Delta(\text{Casimir Energy}) = \hbar\, c\, B \left( \frac{1}{a} - \frac{1}{b} \right). \tag{5.2}$$

"All" one needs to do is "merely" to calculate the numerical coefficient $B$, which is now guaranteed to be finite. It is important to realize that if one has somehow determined

$$\Delta(\text{Casimir Energy}) = (\text{finite}), \tag{5.3}$$

then

$$\Delta(\text{Casimir Energy}) = \frac{1}{2}\hbar \left\{ \textit{any regular resummation technique} \right\} (\omega_n - (\omega_*)_n). \tag{5.4}$$

Boyer uses Riesz resummation, (the so-called "Riesz means"), which is justified only in hindsight. If you know the answer you want is finite, then any of the standard "regular" resummation techniques will do [18]. In contrast if you don't know beforehand that the answer you want is finite, then blindly calculating

$$\sum_n (\omega_n - (\omega_*)_n) \tag{5.5}$$

is asking for trouble.

### 5.3 Arbitrary arrangement of fixed-shape fixed-volume perfect conductors

Consider now any collection of fixed-shape fixed-volume perfect conductors in 3+1 dimensions. We are working with QED ($V = 0$) in flat spacetime (Riemann tensor zero). Then:

$$\begin{aligned}
a_0 &\propto (\text{volume}); \\
a_{1/2} &\propto (\text{surface area}); \\
a_1 &\propto \int_{\partial \mathbf{V}} \{K\} \sqrt{g_2}\, \mathrm{d}^2 x; \\
a_{3/2} &\propto \int_{\partial \mathbf{V}} \{K^2, K_{ij} K^{ij}\} \sqrt{g_2}\, \mathrm{d}^2 x; \\
a_2 &\propto \int_{\partial \mathbf{V}} \{g^{ij} g^{kl} K_{ij:kl}, K^{ij}{}_{:ij}, K^3, \mathrm{tr}(K^2) K, \mathrm{tr}(K^3)\} \sqrt{g_2}\, \mathrm{d}^2 x.
\end{aligned}$$

Fixed-shape fixed-volume implies fixed extrinsic curvature, so all the $\Delta a_i \equiv 0$.
That is:

- Take any collection of perfect conductors. Move them around relative to each other. (Without distorting their shapes and/or volumes.)
- Then the change in Casimir energy, and the Casimir forces, are finite.

We shall subsequently see how to generalize this result to imperfect conductors and/or dielectrics.

# 6 Reference models

Consider now a non-zero potential ($V \neq 0$), in flat spacetime (Riemann tensor zero), with periodic boundary conditions (so that there is no boundary). We have:

$$\begin{aligned}
a_0 &\propto \text{(volume)}; \\
a_{1/2} &= 0; \\
a_1 &\propto \int_{\mathbf{V}} \{V\} \sqrt{g_d}\, \mathrm{d}^d x; \\
a_{3/2} &= 0; \\
a_2 &\propto \int_{\mathbf{V}} \{V^2\} \sqrt{g_d}\, \mathrm{d}^d x.
\end{aligned}$$

So for finiteness we "*just*" need to keep $a_0$, $a_1$, and $a_2$ fixed.

## 6.1 1+1 dimensions

In (1+1) dimensions let us define the spatial average

$$\overline{V} = \frac{\int_0^L V(x)\, \mathrm{d}x}{L}. \tag{6.1}$$

Compare the two situations:

- $D = -\nabla^2 + V(x)$; eigenvalues $\omega_n^2$.
- $\overline{D} = -\nabla^2 + \overline{V}$; eigenvalues $\overline{\omega}_n^2$.

Then:

$$\sum_n \{\omega_n \operatorname{erfc}(\omega_n/\Omega) - \overline{\omega}_n \operatorname{erfc}(\overline{\omega}_n/\Omega)\} = (\text{finite as } \Omega \to \infty). \tag{6.2}$$

That is:

$$(\text{Casimir energy of } D) - (\text{Casimir energy of } \overline{D}) = (\text{finite}). \tag{6.3}$$

In fact in this situation the reference eigenvalues $\bar{\omega}_n$ can be written down explicitly as

$$\overline{\omega}_n = \sqrt{\frac{(2\pi n)^2}{L^2} + \overline{V}}. \tag{6.4}$$

Thence

$$\sum_n \left( \omega_n - \sqrt{\frac{(2\pi n)^2}{L^2} + \overline{V}} \right) = (\text{finite}). \tag{6.5}$$

The $\omega_n$ depend on $V(x)$ and can be quite messy; the difference between the $\omega_n$ and the reference problem $\overline{\omega}_n$ is however well behaved.

### 6.2 3+1 dimensions

In (3+1) dimensions define the two spatially averaged quantities:

$$\overline{V} = \frac{\int_{\mathbf{V}} V(x)\ \mathrm{d}^3 x}{\text{volume}(\mathbf{V})}; \qquad \overline{V^2} = \frac{\int_{\mathbf{V}} V(x)^2\ \mathrm{d}^3 x}{\text{volume}(\mathbf{V})}. \tag{6.6}$$

Now solve

$$m_1^2 + m_2^2 = 2\overline{V}; \qquad m_1^4 + m_2^4 = 2\overline{V^2} \tag{6.7}$$

Now compare the three situations:

- $D = -\nabla^2 + V(x)$; eigenvalues $\omega_n^2$.
- $\overline{D_1} = -\nabla^2 + m_1^2$; eigenvalues $\overline{(\omega_1)}_n^2$.
- $\overline{D_2} = -\nabla^2 + m_2^2$; eigenvalues $\overline{(\omega_2)}_n^2$.

Then:

$$\sum_n \left\{ \omega_n \operatorname{erfc}(\omega_n/\Omega) - \frac{1}{2}\,\overline{(\omega_1)}_n \operatorname{erfc}\left(\overline{(\omega_1)}_n/\Omega\right) - \frac{1}{2}\,\overline{(\omega_2)}_n \operatorname{erfc}\left(\overline{(\omega_2)}_n/\Omega\right)\right\}$$
$$= (\text{finite as } \Omega \to \infty). \tag{6.8}$$

This implies:

$$(\text{Casimir energy of } D) - \frac{1}{2}\ \left(\text{Casimir energy of } \overline{D_1}\right) - \frac{1}{2}\ \left(\text{Casimir energy of } \overline{D_2}\right)$$
$$= (\text{finite}). \tag{6.9}$$

While this argument does not calculate the finite piece for you, it at least guarantees that you are looking for a finite answer. (And since the answer you are looking for us guaranteed finite, almost any way of manipulating the series and thereby getting to that finite answer will give the correct answer.)

But the two comparison models $\overline{D_1}$ and $\overline{D_2}$ are now sufficiently simple that one can invoke analytic techniques, (zeta functions and the like [5, 19]), and simply define

$$(\text{Casimir energy of } D) = \frac{1}{2}\left(\text{Casimir energy of } \overline{D_1}\right) + \frac{1}{2}\left(\text{Casimir energy of } \overline{D_2}\right)$$
$$+(\text{finite}). \tag{6.10}$$

Of course this does not calculate the "finite piece" for you, but it gives you some confidence regarding what to aim for before you start calculating.

## 7 What if Casimir energy differences are not finite?

Now there are certainly (mathematical) situations where the $\Delta a_i \neq 0$ and the Casimir energy difference is not naively finite. This merely means one has to be more careful thinking about the physics. For instance:

- Real metals and real dielectrics are transparent in the UV.
- The UV cutoff $\Omega$ is then merely a stand-in for all the complicated physics.

For real metals and real dielectrics the cutoff represents real physics. See for instance the discussion in references [20–22] and compare with the discussion in [23–26]. Note that the discussion regarding real metals and real dielectrics has often lead to some considerable disagreement regarding interpretation [27–29]. (My own view, as should be clear from the current article, is that Casimir energies are ultimately determined by looking at differences in zero-point energies, summed over all relevant modes.)

### 7.1 General class of cutoff functions

Let us write a general class of cutoff functions as

$$f\left(\frac{\omega}{\Omega}\right) = \int_0^\infty g(\xi)\, \mathrm{erfc}\left(\frac{\omega}{\xi\Omega}\right)\, \mathrm{d}\xi; \qquad \int_0^\infty g(\xi)\, \mathrm{d}\xi = 1. \tag{7.1}$$

Note $f(0) = 1$, while $f(\infty) = 0$, and $f(\omega/\Omega)$ is monotone decreasing.

To see just how general this class of cutoff functions is, we proceed by noting that

$$\mathrm{erfc}\left(\frac{\omega}{\xi\Omega}\right) = \frac{2}{\Omega\xi\sqrt{\pi}} \int_\omega^\infty \exp\left(-\frac{x^2}{\Omega^2\xi^2}\right) \mathrm{d}x. \tag{7.2}$$

So we see

$$f'\left(\frac{\omega}{\Omega}\right) = -\frac{2}{\sqrt{\pi}} \int_0^\infty \frac{g(\xi)}{\xi}\; \exp\left(-\frac{\omega^2}{\Omega^2\xi^2}\right)\, \mathrm{d}\xi. \tag{7.3}$$

Substituting $\chi = 1/\xi^2$ we obtain

$$f'\left(\frac{\omega}{\Omega}\right) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{g(\chi^{-1/2})}{\chi}\; \exp\left(-\frac{\omega^2}{\Omega^2}\,\chi\right)\, \mathrm{d}\chi. \tag{7.4}$$

But this is just the Laplace transform of $g(\chi^{-1/2})/\chi$, evaluated at the point $s = \omega^2/\Omega^2$. Consequently, as long as the inverse Laplace transform of $f'(s^{1/2})$ exists, which is a relatively mild condition on the cutoff function $f(s^{1/2})$, then we can determine $g(\xi)$ in terms of $f(\omega/\Omega)$.

Indeed, there is a little-known algorithm due to Post [30], see also Bryan [31], and reference [32], that allows for inversion of Laplace transforms by taking arbitrarily high derivatives. Specifically, if $G(s)$ is the Laplace transform of $g(z)$ then

$$g(z) = \lim_{n\to\infty} \frac{(-1)^n}{n!} \left(\frac{n}{z}\right)^{n+1} G^{(n)}\left(\frac{n}{z}\right). \tag{7.5}$$

This algorithm may not always be practical, since one needs arbitrarily high derivatives. Even if not always practical, it again settles an important issue of principle — knowledge of the cutoff $f(\omega/\Omega)$ in principle allows one to reconstruct an equivalent weighting $g(\xi)$.

The point is that almost any cutoff function $f(\omega/\Omega)$ can be cast in this "weighted integral over erf-functions" form. (In particular we could rephrase all of the preceding discussion concerning erf-regularization in terms of this more general $f$-regularization, but when $\Delta a_i = 0$ nothing new is obtained. It is only when $\Delta a_i \neq 0$ that general $f$-regularization becomes at all interesting.)

### 7.2 $f$-regularized Casimir energy

Let us now consider a generic regularized sum of eigen-frequencies:

$$\sum_n \omega_n\, f\left(\frac{\omega_n}{\Omega}\right). \tag{7.6}$$

Then our previous result

$$\sum_n \omega_n \operatorname{erfc}(\omega_n/\Omega) = \left\{ \sum_{i=0}^{d} k_i\, a_{i/2}\, \Omega^{d+1-i} \right\} + k_{(d+1)/2}\, a_{(d+1)/2}\, \ln(\Omega^2) \\ + (\text{finite as } \Omega \to \infty), \tag{7.7}$$

becomes

$$\sum_n \omega_n\, f\left(\frac{\omega_n}{\Omega}\right) = \left\{ \sum_{i=0}^{d} k_i \left( \int_0^\infty g(\xi)\, \xi^{d+1-i}\, \mathrm{d}\xi \right) a_{i/2}\, \Omega^{d+1-i} \right\} \\ + k_{(d+1)/2}\, a_{(d+1)/2} \left\{ \ln(\Omega^2) + 2\int_0^\infty g(\xi)\, \ln\xi\, \mathrm{d}\xi \right\} \\ + (\text{finite as } \Omega \to \infty). \tag{7.8}$$

The integrals over $g(\xi)$ can be absorbed into redefining the dimensionless constants $k_i$ in a $f$-dependent manner. That is:

**Theorem 3 (Physical cutoff)**
*For a general cutoff $f(\omega/\Omega)$ one has*

$$\sum_n \omega_n \; f\left(\frac{\omega_n}{\Omega}\right) = \left\{ \sum_{i=0}^{d} [k(f)]_i \;\; a_{i/2} \;\; \Omega^{d+1-i} \right\} + k_{(d+1)/2} \;\; a_{(d+1)/2} \;\; \ln(\Omega^2)$$
$$+(\text{finite as } \Omega \to \infty). \qquad (7.9)$$

*The $[k(f)]_i$ are dimensionless phenomenological parameters that depend on the detailed physics of the specific cutoff function $f(\omega/\Omega)$. However $k_{(d+1)/2}$ is cutoff independent.* The $\Omega$ dependence represents real physics. Live with it! □

**Definition 1 (Physically regularized Casimir energy)**
*For a general cutoff $f(\omega/\Omega)$ one has*

$$\Delta(\text{Casimir energies})_{f-\text{regularized}} = \frac{1}{2}\hbar \; \Delta \left( \sum_n \omega_n \; f\left(\frac{\omega_n}{\Omega}\right) \right). \qquad (7.10)$$

□

**Theorem 4 (Physically regularized Casimir energy)**
*For a general cutoff $f(\omega/\Omega)$ one has*

$$\Delta(\text{Casimir energies})_{f-\text{regularized}} = \frac{1}{2}\hbar \left\{ \sum_{i=0}^{d} [k(f)]_i \; \Delta a_{i/2} \; \Omega^{d+1-i} \right\}$$
$$+\frac{1}{2}\hbar \; k_{(d+1)/2} \; \Delta a_{(d+1)/2} \;\; \ln(\Omega^2)$$
$$+(\text{finite as } \Omega \to \infty).$$

*The $[k(f)]_i$ are dimensionless phenomenological parameters that depend on the detailed physics of the specific cutoff function $f(\omega/\Omega)$. However $k_{(d+1)/2}$ is cutoff independent.*

The $\Omega$ dependence represents real physics. Live with it! □

Part of the reason it was never worthwhile to keep explicit track of the $k_i$ is that, once the $f$-cutoff is introduced, the $k_i$ would in any case be replaced by the purely phenomenological and cutoff dependent $[k(f)]_i$.

Furthermore, if the first $d+2$ of the $\Delta a_i$ are zero, then the cutoff dependence drops out of the calculation. That is, even for imperfect conductors and dielectrics, if one is comparing two situations where the conductors/dielectrics have merely been moved around, (without changing shape and/or volume), then the difference in Casimir energies (and so the Casimir forces) are guaranteed finite.

## 8 Forcing finiteness?

Can one *force* the Casimir energy difference to be finite?

By hook or by crook find a number of "simple" problems $\overline{D_i}$ such that

$$a_{i/2}(D) = \frac{1}{m}\sum_{i=1}^{m} a_{i/2}(\overline{D_i}); \qquad i \in \{0, 1, 2, \dots, d+1\}. \tag{8.1}$$

Then it is certainly safe to say

$$(\text{Casimir energy of } D) - \frac{1}{m}\sum_{i=1}^{m}\left(\text{Casimir energy of } \overline{D_i}\right) = (\text{finite}). \tag{8.2}$$

Of course this does not calculate the "finite piece" for you, but it gives you some confidence regarding what to aim for before you start calculating. More formally, if the $\overline{D_i}$ are sufficiently simple one might apply analytic techniques (such as zeta functions [5, 19] or the like) to argue that it might make sense to define:

$$(\text{Casimir energy of } D) = \frac{1}{m}\sum_{i=1}^{m}\left(\text{Casimir energy of } \overline{D_i}\right) + (\text{finite}). \tag{8.3}$$

A somewhat safer statement is to compare two systems and assert

$$\Delta(\text{Casimir energy of } D) = \frac{1}{m}\sum_{i=1}^{m}\Delta\left(\text{Casimir energy of } \overline{D_i}\right) + (\text{finite}). \tag{8.4}$$

Only if the two sets of "reference problems" $\overline{D_i}$ are the same (or at least sum to the same $\sum_{i=1}^{m} a_j\left[\,\overline{D_i}\,\right]$) does this process make any real physical sense, in which case it reduces to our previous result

$$\Delta(\text{Casimir energy of } D) = (\text{finite}). \tag{8.5}$$

Otherwise the sum

$$\frac{1}{m}\sum_{i=1}^{m}\Delta\left(\text{Casimir energy of } \overline{D_i}\right), \tag{8.6}$$

while analytically continued to be finite, is purely formal. It need not be a physical energy difference. In short, one should seek at all times to calculate Casimir energy differences between clearly defined and specified physical systems. This might, at a pinch, involve differences between linear combinations of physical systems, but to get a physically meaningful Casimir energy one must either enforce $\Delta\left(\sum_{i=1}^{m} a_j\left[\,\overline{D_i}\,\right]\right) = 0$, or develop an explicit physical model for the cutoff $f(\omega/\Omega)$.

## 9 Conclusions

In $(d+1)$ dimensions, iff the first $d+2$ Seeley–DeWitt coefficients agree,

$$\Delta a_0 = \Delta a_{1/2} = \dots \Delta a_{(d+1)/2} = 0, \tag{9.1}$$

then the difference in Casimir energies is guaranteed finite. This is an extremely useful thing to check before you start explicitly calculating. The erfc function, in the form $\mathrm{erfc}(\omega/\Omega)$, is a perhaps unexpectedly useful regulator

$$\mathrm{erfc}(0) = 1; \qquad \mathrm{erfc}(\infty) = 0. \tag{9.2}$$

More generally, any reasonably smooth function $f(\omega/\Omega)$ satisfying

$$f(0) = 1; \qquad f(\infty) = 0; \tag{9.3}$$

will do. (Roughly speaking, as long as $f(\omega/\Omega)$ has an inverse Laplace transform.) For real metals and real dielectrics, which become transparent in the UV, the cutoff is physical, and its influence on the Casimir energy is encoded in a small number of dimensionless parameters $[k(f)]_i$ and an overall cutoff scale $\Omega$. Various generalizations of this argument, (such as counting differences in eigenstates, or calculating differences of sums of powers of eigenvalues), are also possible. Similar arguments, regarding differences in Seeley–DeWitt coefficients, can also be applied to the one-loop effective action [33]. Finally, I should emphasise that I have not *renormalized* anything anywhere in this article, the worst I have done is to temporarily *regularize* some infinite series, to allow some otherwise formal manipulations to be mathematically well-defined.

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