COMBINATORIAL ASPECTS OF THE QUANTIZED UNIVERSAL ENVELOPING ALGEBRA OF $\mathfrak{sl}_{n+1}(\mathbb{C})$

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Abstract. There is increasing interest on the part of algebraic combinatorialists in the ideas behind the study of operator invariants of knots and in the associated algebras. A primary purpose of this article is explain how to construct certain crucial elements in this theory and how to work with expressions that arise in $\mathcal{U}_h(\mathfrak{sl}_2(\mathbb{C}))$.

A ribbon Hopf algebra was introduced by Drinfel'd in his construction of solutions to the Yang-Baxter Equation in his study of operator invariants of knots. This algebra is built upon the quantized universal enveloping algebra of the Lie algebra $\mathfrak{sl}_2$. In this paper, we use constructive, rather than inductive, methods to derive the necessary straightening formulae for $\mathcal{U}_h(\mathfrak{sl}_2(\mathbb{C}))$ and then construct Drinfeld's well-known solution to this equation from first principles. We then use these methods to explicitly present the ribbon Hopf algebra structure of $\mathcal{U}_h(\mathfrak{sl}_2(\mathbb{C}))$. Finally, we extend these techniques to the higher dimensional algebras $\mathcal{U}_h(\mathfrak{sl}_n(\mathbb{C}))$, $n \geq 2$.

The results contained in this paper are largely known. However, all previous works have relied on the Quantum Double method in performing the constructions we undertake here. Our methods, based on straightening, are different from the Quantum Double method and perhaps elucidates some of the rich algebra structure nestled within the objects constructed here.

Contents

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1. Introduction

The material presented here comes ultimately from the study of operator invariants of knots. We include information on the background to this area which we trust will provide sufficient motivation for the questions studied, without overburdening the Reader with detail.

1.1. **Background.**

1.1.1. *Knots.* A *knot* is an embedding of the unit circle into $\mathbb{R}^3$ and two knots $\mathfrak{a}$ and $\mathfrak{b}$ are *(ambient) isotopic* or, more briefly, *equivalent* if one may be transformed into the other by a smooth family of homeomorphisms $h_t\colon \mathbb{R}^3 \to \mathbb{R}^3$ indexed by a parameter $t \in [0,1]$ such that $h_0 = \mathrm{id}_{\mathbb{R}^3}$ and $h_1(\mathfrak{a}) = \mathfrak{b}$. This notion of equivalence makes precise the informal notion that two knots are equivalent if one may be transformed into the other smoothly: that is, without cutting and re-attaching the ends.

The study of knots has a long history. They attracted the interest of Gauss and Vandermonde, who compiled drawings of them, and later the interest of Kelvin in his efforts to explain certain physical phenomena. There was a resurgence of interest in the 1990's because of an indirect connexion to them through the Yang-Baxter Equation in mathematical physics. The same equation arises in knot theory because a knot may be represented as the closure of a braid, and one of the relations for the associated braid group underlies this equation. This period in the development of knot theory marked the beginning of what is now commonly termed *Modern Knot Theory.*

An essential question in knot theory is to construct a map $\theta\colon \mathcal{K} \to \mathcal{S}$, from the set $\mathcal{K}$ of all knots to a set $\mathcal{S}$ such that if $\mathfrak{a}$ and $\mathfrak{b}$ are knots, then $\theta(\mathfrak{a}) \neq \theta(\mathfrak{b})$ implies that $\mathfrak{a}$ and $\mathfrak{b}$ are *inequivalent* knots. The map θ is called a *knot invariant.* The set $\mathcal{S}$ is to be selected so that testing inequality of elements in it is "easier" than testing inequivalence of knots. For example, $\mathcal{S}$ may be selected to be a ring of polynomials, which clearly satisfies this criterion.

An oriented knot may be represented in the plane by its regular projection as a four regular graph, together with marks attached to each vertex to indicate a positive or negative crossing. Such an object is called a knot diagram, and two knot diagrams represent equivalent knots if and only if the diagrams are related by a finite sequence of Reidemeister Moves. A result of Alexander shows that a knot diagram may be viewed as the closure of a braid. This is done by selecting a base point in the plane of a knot diagram, and using the Reidemeister Moves in such a way that each segment of the diagram between successive vertices is directed in the anti-clockwise sense around the base point. The braid, in turn, can be expressed as a product of the braid generators $s_1, \dots, s_{n-1}$, for an n-stranded braid, where s_i is the transposition $(i, i+1)$, together with the braid relations, of which one is $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $i = 1, \dots, n-1$. This accounts for the appearance of (matrix) representations of the braid group. The Yang-Baxter Equation

$$(\mathbf{M} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{M})(\mathbf{M} \otimes \mathbf{I}) = (\mathbf{I} \otimes \mathbf{M})(\mathbf{M} \otimes \mathbf{I})(\mathbf{I} \otimes \mathbf{M}) \tag{1}$$

is the image of this relation in the matrix representation. Such a matrix $\mathbf{M}$ is called an *R-morphism.*

1.1.2. *Solutions of the Yang-Baxter Equation.* Solutions of the Yang-Baxter equation may be obtained through *Ribbon Hopf Algebras.* The remarkable work of Drinfeld and Jimbo in the late 1980's showed that every semisimple Lie algebra over $\mathbb{C}$ gives rise to such an algebra; throughout this article, all our algebras, unless otherwise mentioned, will be over the field $\mathbb{C}$ of complex numbers.

Let $(\mathcal{A}, m, \Delta, \varepsilon, \eta, \mathsf{S})$ be a *Hopf algebra* with product m, co-product Δ, unit η, co-unit (augmentation map) ε, and anti-homomorphism (antipode) S. Let R be a universal R-morphism, so $(\mathcal{A}, \mathsf{R})$ is a *quasi-triangular Hopf algebra.* Let $\mathbf{v}$ be an element (see Definition 6.1 and Definition 6.3) constructed from

$$\mathbf{v}^2 := \mathsf{S}(\mathbf{u}) \cdot \mathbf{u} \quad \text{where} \quad \mathbf{u} := \sum_{i \geq 0} \mathsf{S}(\beta_i) \cdot \alpha_i \quad \text{and} \quad \mathsf{R} := \sum_{i \geq 0} \alpha_i \otimes \beta_i. \tag{2}$$

Then $(\mathcal{A}, m, \Delta, \varepsilon, \eta, \mathsf{S}, \mathsf{R}, \mathbf{v})$ is a *Ribbon Hopf Algebra* provided four further conditions on $\mathbf{v}$ stated in Definition 6.3 are also satisfied.

1.2. **Historical comments.** The quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{sl}_2)$ was found by Kulish and Reshetikhin [KR81]. It may be regarded as the algebra $\mathcal{U}(\mathfrak{sl}_2)[[h]]$ of formal power series in h with coefficients in $\mathcal{U}(\mathfrak{sl}_2)$.

A Hopf structure on $\mathcal{U}_h(\mathfrak{sl}_2)$ was discovered by Skylanin [Skl85]. This algebraic structure is desirable because the additional structure gracefully handles tensor products and duals of representations of $\mathcal{U}_h(\mathfrak{sl}_2)$.

The Hopf $\mathcal{U}_h(\mathfrak{sl}_2)$ has the additional structure of a quasi-triangular Hopf algebra. The first instance of an invertible element R required for this was found in $\mathcal{U}_h(\mathfrak{sl}_2)$ by Drinfel'd [Dri85]. Independently, Jimbo[Jim85] constructed one in $\mathcal{U}_q$. Several R-morphisms have been found in $\mathcal{U}_h(\mathfrak{sl}_2)$: see, for example, Reshetikhin and Turaev [RT91].

1.3. **Organization of the paper.** We begin in Section 2 with general comments on the quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$. After a brief algebraic review, we discuss straightening in $\mathcal{U}_h(\mathfrak{sl}_2)$ and establish a few technical Lemmas which will be crucial to all that follows. Section 3 will apply these technical results to straighten the monomial $\mathsf{x}^a\mathsf{y}^b$ in $\mathcal{U}_h(\mathfrak{sl}_2)$ so that it is a sum of monomials of the form $\mathsf{y}^c\mathsf{x}^d$.

As a slight change of pace, Section 4 will discuss some q-identities in the context of inversions in permutations. There, we collect some identities and prove an extension (Lemma 4.5) of a classic identity by Cauchy; this identity will be crucial in our construction of the ribbon Hopf structure on $\mathcal{U}_h(\mathfrak{sl}_2)$.

We return in Section 5 to the structure of $\mathcal{U}_h(\mathfrak{sl}_2)$. This section discusses the notion of a quasi-triangular Hopf algebra and then proceed to constructively derive an R-morphism for this algebra. Following this, Section 6 explicitly constructs the associated ribbon Hopf structure on $\mathcal{U}_h(\mathfrak{sl}_2)$.

Having completed a basic study of $\mathcal{U}_h(\mathfrak{sl}_2)$, we generalize the techniques of Section 5 to derive an R-morphism for $\mathcal{U}_h(\mathfrak{sl}_3)$ in Section 7 and for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$, $n \geq 2$, in Section 8. These higher dimensional studies further clarifies the essential features of our technique. Although the resulting R-morphism is not new, the methods appear to be somewhat different from the standard techniques for constructing R.

We close this article in Section 9 with a few comments on how our method is related to the Quantum Double method and how this method might be extended to other Lie algebras.

2. Quantized Universal Enveloping Algebra of $\mathfrak{sl}_2$

In this section, we define the Drinfel'd-Jimbo quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{sl}_2)$ for the Lie algebra $\mathfrak{sl}_2$. For the convenience of the reader, we begin our discussion by recalling a few standard definitions.

Definition 2.1. *A (complex)* Lie algebra *is a vector space* $\mathfrak{g}$ *over* $\mathbb{C}$ *together with a bilinear map*

$$[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the Lie bracket*, such that for all* $x, y, z \in \mathfrak{g}$

(i) $[x, y] = -[y, x]$,
(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Condition (ii) in the Definition is called the Jacobi identity and is to be thought of as a weakened form of associativity for the "product" given by the Lie bracket. In general, if the Lie bracket were to be viewed as a multiplication on $\mathfrak{g}$, then $\mathfrak{g}$ would be a non-associative algebra. For many purposes, it is convenient to work with an associative algebra and that is the role of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. We give a concrete description of the universal enveloping algebra in the case that $\mathfrak{g}$ is a finite-dimensional Lie algebra with a basis $x_1, \dots, x_n$. Suppose also that, for each $1 \leq i < j \leq n$,

$$[x_i, x_j] = \sum_{k=1}^{n} c_{ij}^k x_k,$$

where the $c_{ij}^k \in \mathbb{C}$ are the *structure constants* of the Lie algebra $\mathfrak{g}$.

Definition 2.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra as above. The* universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ *of $\mathfrak{g}$ is the associative algebra generated by the x_i, subject to the relations*

$$x_i x_j - x_j x_i = \sum_{k=1}^{n} c_{ij}^k x_k. \tag{3}$$

2.1. **Poincaré-Birkhoff-Witt Bases.** An algebra, in particular $\mathcal{U}(\mathfrak{g})$, has an underlying vector space structure and thus concrete calculations can be done with a choice of basis. Although there is no canonical choice of basis in an arbitrary algebra, universal enveloping algebras admit a distinguished basis once an ordered basis for the associated Lie algebra is chosen (for instance, once an ordering is chosen on the generators of the Lie algebra). Suppose the generators of the Lie algebra $\mathfrak{g}$ are ordered $x_1 \prec \cdots \prec x_n$. Now, a linear generating set for $\mathcal{U}(\mathfrak{g})$ is, by definition, the set of monomials $\{x_{i_1} \cdots x_{i_m} : 1 \leq i_1, \dots, i_m \leq n, m \in \mathbb{Z}\}$. However, using the commutation relation (3), arbitrary monomials in the x_i can be *straightened*, which is to say that such monomials are expressed as a sum of terms in the form $x_1^{e_1} \cdots x_n^{e_n}$. This shows that the set

$$\mathcal{B} := \{\, x_1^{e_1} \cdots x_n^{e_n} : e_1, \dots, e_n \in \mathbb{Z}_{\geq 0} \,\}$$

is spanning. Linear independence is rather tedious to show, but can be done. This fact, that $\mathcal{B}$ is a basis for $\mathcal{U}(\mathfrak{g})$, is the content of the *Poincaré-Birkhoff-Witt Theorem*; unsurprisingly, $\mathcal{B}$ is therefore referred to as a *Poincaré-Birkhoff-Witt basis*, *PBW basis* for short, of $\mathcal{U}(\mathfrak{g})$ with respect to the ordering $x_1 \prec \cdots \prec x_n$.

2.2. **The Lie algebra $\mathfrak{sl}_2$.** The simplest Lie algebra is the *special linear algebra* $\mathfrak{sl}_2$. This Lie algebra abstractly consists of a three-dimensional complex vector space generated by elements x, y and h, and a Lie bracket structure given by

$$[\mathsf{h}, \mathsf{x}] = 2\mathsf{x}, \qquad [\mathsf{h}, \mathsf{y}] = -2\mathsf{y}, \qquad [\mathsf{x}, \mathsf{y}] = \mathsf{h}.$$

An explicit realization of $\mathfrak{sl}_2$ is obtained by considering the vector space of 2×2 traceless matrices over $\mathbb{C}$ equipped with the commutator as the Lie bracket. In terms of the abstract generators of $\mathfrak{sl}_2$, this matrix manifestation can be effected by mapping

$$\mathsf{x} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathsf{y} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathsf{h} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$ of $\mathfrak{sl}_2$ is the associative algebra with generators by x, y and h, subject to the relations

$$\mathsf{hx} - \mathsf{xh} = 2\mathsf{x}, \quad \mathsf{hy} - \mathsf{yh} = -2\mathsf{y}, \quad \mathsf{xy} - \mathsf{yx} = \mathsf{h}.$$

Imposing the ordering $\mathsf{h} \prec \mathsf{y} \prec \mathsf{x}$ on the linear generators, a PBW basis of $\mathcal{U}(\mathfrak{sl}_2)$ is the set of monomials $\{\mathsf{h}^a\mathsf{y}^b\mathsf{x}^c : a, b, c, \in \mathbb{Z}_{\geq 0}\}$.

2.3. **The Quantized Universal Enveloping Algebra $\mathcal{U}_h(\mathfrak{sl}_2)$.** A quantized universal enveloping algebra for a Lie algebra $\mathfrak{g}$ can be thought of as an associative algebra over the ring $\mathbb{C}[[h]]$ of formal power series in an indeterminate h such that the usual universal enveloping algebra is recovered when $h = 0$. In the case of $\mathfrak{sl}_2$, the *Drinfel'd-Jimbo quantized universal enveloping algebra* $\mathcal{U}_h(\mathfrak{sl}_2)$ is the $\mathbb{C}[[h]]$-algebra generated by x, y and h, subject to the relations

$$\mathsf{hx} - \mathsf{xh} = 2\mathsf{x}, \quad \mathsf{hy} - \mathsf{yh} = -2\mathsf{y}, \quad \mathsf{xy} - \mathsf{yx} = \frac{e^{\frac{h}{2}\mathsf{h}} - e^{-\frac{h}{2}\mathsf{h}}}{e^{\frac{h}{2}} - e^{-\frac{h}{2}}}. \tag{4}$$

Infinite linear combinations are permitted in $\mathcal{U}_h(\mathfrak{sl}_2)$, provided that they are graded in h and each coefficient in h is a finite linear combination of elements in $\mathcal{U}(\mathfrak{sl}_2)$. This can be made precise by introducing an appropriate topology from $\mathbb{C}[[h]]$ and taking a certain completion to form $\mathcal{U}_h(\mathfrak{sl}_2)$. For our purposes, it will be sufficient to think of $\mathcal{U}_h(\mathfrak{sl}_2) \cong \mathcal{U}(\mathfrak{sl}_2)[[h]]$.

2.3.1. *Notation.* In preparation for performing computations in this algebra, we introduce the notation

$$q := e^{\frac{h}{2}}, \quad \mathsf{k} := e^{\frac{h}{4}\mathsf{h}}, \quad \overline{q} := q^{-1}, \quad \overline{\mathsf{k}} := \mathsf{k}^{-1}, \quad [\mathsf{h} + n] := \frac{q^n\mathsf{k}^2 - \overline{q}^n\overline{\mathsf{k}}^2}{q - \overline{q}}. \tag{5}$$

With this notation, the commutation relation for x and y is simply expressed as

$$[\mathsf{x}, \mathsf{y}] := \mathsf{xy} - \mathsf{yx} = \frac{\mathsf{k}^2 - \overline{\mathsf{k}}^2}{q - \overline{q}} = [\mathsf{h}]. \tag{6}$$

2.4. **Straightening in $\mathcal{U}_h(\mathfrak{sl}_2)$.** As alluded to earlier, explicit computations in $\mathcal{U}_h(\mathfrak{sl}_2)$ are greatly simplified with the choice of a convention, such as $\mathsf{h} \prec \mathsf{y} \prec \mathsf{x}$, for the precedence of the generators h, x, y. The order is arbitrary and its choice is governed by the dictates of the question in hand. Having chosen a convention, a basis similar similar to the PBW bases of Section 2.1 can be constructed, conretely meaning that any formal series in the generators of $\mathcal{U}_h(\mathfrak{sl}_2)$ may be expressed as a sum of monomials of the form $\mathsf{h}^r\mathsf{y}^s\mathsf{x}^t$, with $r, s, t \in \mathbb{Z}_{\geq 0}$. This process shall be referred to as *straightening*.

Note that our current situation is slightly different from that considered in Section 2.1 as we are working with the *quantized* universal enveloping algebra $\mathcal{U}_h(\mathfrak{sl}_2)$ as opposed to the usual universal enveloping algebra $\mathcal{U}(\mathfrak{sl}_2)$. Nonetheless, either by using the PBW Theorem for $\mathcal{U}(\mathfrak{sl}_2)$ together with the isomorphism $\mathcal{U}_h(\mathfrak{sl}_2) \cong \mathcal{U}(\mathfrak{sl}_2)[[h]]$, or else directly, an analogue of the PBW Theorem holds for $\mathcal{U}_h(\mathfrak{sl}_2)$.

Theorem 2.3. [CP94, p.199] *The set* $\{\mathsf{h}^r\mathsf{y}^s\mathsf{x}^t : a, b, c \in \mathbb{Z}_{\geq 0}\}$ *is a basis for* $\mathcal{U}_h(\mathfrak{sl}_2)$.

As in the definition of $\mathcal{U}_h(\mathfrak{sl}_2)$, the term 'basis' here is to be interpreted so that infinite linear combinations are permitted, subject to the restrictions mentioned before.

This Theorem provides the theoretical justification for much of what will follow. Specifically, as straightened monomials form a basis, equality of two elements in $\mathcal{U}_h(\mathfrak{sl}_2)$ may be determined by first straightening then by comparing coefficients of corresponding terms. Straightening, while not necessarily the shortest route in demonstrating that two expressions are equal, has the signal advantage of being routine.

A main technical question that presents itself in this algebra is to express $\mathsf{x}^a\mathsf{y}^b$, where $a, b \in \mathbb{Z}_{\geq 0}$, as a formal sum of monomials of the form $\mathsf{h}^r\mathsf{y}^s\mathsf{x}^t$ where $r, s, t \in \mathbb{Z}_{\geq 0}$; the precedence order established is chosen to expedite subsequent computations. Were a straightened expression for $\mathsf{x}^a\mathsf{y}^b$ known, it would be a comparatively easy task to establish it by an inductive argument and, indeed, this we shall do later. However, the more general question we address here, for which the above is simply an example, is how to obtain a putative expansion in the first place. We therefore seek an approach, or a formalism, for deducing the straightened expansions.

2.4.1. *Further Notation.* It will be necessary to carry out explicit calculations in this algebra and to do so succinctly and, where possible, expeditiously. Together with the notation (5) earlier introduced, we shall also use the notation

$$[n]_q := \frac{q^n - \overline{q}^n}{q - \overline{q}}, \quad [n]!_q := [1]_q \cdot [2]_q \cdots [n]_q, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q\,[n-k]!_q} \tag{7}$$

for a *quantum integer*, the *quantum factorial function* and the *quantum binomial function*, where k is a non-negative integer. Here, the subscript q in each notation is to be thought of as the argument in its definition. That is, for any function $f(q)$ of q, we write

$$[n]_{f(q)} := \frac{f(q)^n - f(\overline{q})^n}{f(q) - f(\overline{q})}, \qquad [n]!_{f(q)} := [1]_{f(q)} \cdot [2]_{f(q)} \cdots [n]_{f(q)},$$

and similarly for the quantum binomial. It is convenient to adopt the convention that when an explicit subscript is omitted, it is to be understood that $f(q) = q$.

The *quantum lower factorial* is defined by

$$[n]_{q,i} := [n]_q \cdot [n-1]_q \cdot \cdots \cdot [n-i+1]_q.$$

Note that another collection of quantum analogues, led by $(n)_q$, will be encountered later and it is important to distinguish between these quantities.

A further item of notation extending $[\mathsf{h}+n]$ from (5) is the following:

$$[\mathsf{h}+n]_{(i)} := \prod_{r=0}^{i-1}[\mathsf{h}+n-r], \qquad \text{and} \qquad \begin{bmatrix} \mathsf{h}+n \\ i \end{bmatrix} := \frac{1}{[i]!}[\mathsf{h}+n]_{(i)} \tag{8}$$

where $i \in \mathbb{Z}_{\geq 0}$.

2.4.2. *Basic straightening rules.* First we establish some useful identities for quantum integers.

Lemma 2.4 (Separation Lemma)**.** *Let x, y, a be indeterminates. Then*

(i) $[-x] = -[x]$,

(ii) $[x]\,[y] - [x-a]\,[y+a] = [a]\,[y-x+a]$,
(iii) $[x]\,[y] + [a]\,[x+y+a] = [x+a]\,[y+a]$.

Proof. Identity (i) is immediate from definitions. For (ii), note that $(q-\overline{q})^2\,[x]\,[y] = \left(q^{x+y}+\overline{q}^{x+y}\right) - \left(q^{y-x}+\overline{q}^{y-x}\right)$. Applying this twice,

$$(q-\overline{q})^2\,\left([x]\,[y] - [x-a]\,[y+a]\right) = q^{y-x}\left(q^{2a}-1\right) + \overline{q}^{y-x}\left(\overline{q}^{2a}-1\right).$$

Finally, (iii) follows from (ii) upon replacing x by $-x$. □

Tracing through the proof, these identities are valid when either:

(a) both x and y are integers; or
(b) one of x or y is an integer and the other is an expression involving h.

Part (ii) of Lemma 2.4 may be viewed as a device for isolating the occurrences of x and y in $[x+y-a]$ into different quantum brackets.

The next lemma contains a collection of technical results that assist the straightening process and which will be used often.

Lemma 2.5 (Straightening)**.** *Let a be an integer, b and c be a non-negative integers and $f(x)$ be a formal power series in x. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_2)$.*

(i) $$[a+1]\,[\mathsf{h}+a] = [\mathsf{h}] + [a]\,[\mathsf{h}+a+1]$$

(ii) $$f(\mathsf{h})\mathsf{x}^b = \mathsf{x}^b\,f(\mathsf{h}+2b), \qquad f(\mathsf{h})\mathsf{y}^b = \mathsf{y}^b\,f(\mathsf{h}-2b)$$

(iii) $$\mathsf{k}^a\mathsf{x}^b = q^{ab}\mathsf{x}^b\mathsf{k}^a, \qquad \mathsf{k}^a\mathsf{y}^b = \overline{q}^{ab}\mathsf{y}^b\mathsf{k}^a$$

(iv) $$(\mathsf{k}\mathsf{x})^b = \overline{q}^{\frac{1}{2}b(b-1)}\mathsf{k}^b\mathsf{x}^b, \qquad (\mathsf{k}\mathsf{y})^b = q^{\frac{1}{2}b(b-1)}\mathsf{k}^b\mathsf{y}^b$$
$$(\overline{\mathsf{k}}\mathsf{x})^b = q^{\frac{1}{2}b(b-1)}\overline{\mathsf{k}}^b\mathsf{x}^b, \qquad (\overline{\mathsf{k}}\mathsf{y})^b = \overline{q}^{\frac{1}{2}b(b-1)}\overline{\mathsf{k}}^b\mathsf{y}^b$$

(v) $$\mathsf{x}\,\mathsf{y}^b = \mathsf{y}^b\mathsf{x} + [b]\,[\mathsf{h}+b-1]\,\mathsf{y}^{b-1}, \qquad \mathsf{x}^b\mathsf{y} = \mathsf{y}\,\mathsf{x}^b + [b]\,[\mathsf{h}-b+1]\,\mathsf{x}^{b-1}$$

(vi) $$(\mathsf{k}\mathsf{x})^b f(\mathsf{h}) = f(\mathsf{h}-2b)\,(\mathsf{k}\mathsf{x})^b, \qquad (\overline{\mathsf{k}}\mathsf{y})^b f(\mathsf{h}) = f(\mathsf{h}+2b)\,(\overline{\mathsf{k}}\mathsf{y})^b$$

(vii) $$\mathsf{x}^b(\mathsf{k}\mathsf{x})^c = \overline{q}^{\frac{1}{2}(c^2+2bc-c)}\mathsf{k}^c\mathsf{x}^{b+c}, \qquad \mathsf{y}^b(\mathsf{k}\mathsf{y})^c = q^{\frac{1}{2}(c^2+2bc-c)}\mathsf{k}^c\mathsf{y}^{b+c}$$

Proof. If a result is one of a pair, only the first is proved since the second follows by a similar argument.

Item (i) follows immediately by first setting $a=1$ in (ii) of Lemma 2.4 and then setting $x = a+1$ and $y = \mathsf{h}+a$.

Identity (ii) follows from the defining relations relations (4). Indeed, rearranging the commutator gives $\mathsf{h}\mathsf{x} = \mathsf{x}(\mathsf{h}+2)$. Iterating this gives $\mathsf{h}^a\mathsf{x} = \mathsf{x}(\mathsf{h}+2)^a$ so, by linearity, $f(\mathsf{h})\mathsf{x} = \mathsf{x}f(\mathsf{h}+2)$, which when iterated for X, gives the result. Setting $f(x) := e^{\frac{h}{4}ax}$ and recalling the definition of k in (5), (iii) follows from Result (ii).

We now prove (iv). Let $r>0$ be an integer. Using Result (iii) with $a=b=1$, $(\mathsf{k}\mathsf{x})^r = \mathsf{k}(\mathsf{x}\mathsf{k})^{r-1}\mathsf{x} = \overline{q}^{r-1}\mathsf{k}(\mathsf{k}\mathsf{x})^{r-1}\mathsf{x}$. The result follows by iterating this recursion for $(\mathsf{k}\mathsf{x})^r$. The three other results are proved similarly.

For (v), let $A_b := \mathsf{x}\mathsf{y}^b$. Then, from (6) and part (ii),

(9) $$A_b = (\mathsf{x}\mathsf{y})\mathsf{y}^{b-1} = \mathsf{y}A_{b-1} + [\mathsf{h}]\mathsf{y}^{b-1} = \mathsf{y}A_{b-1} + [\mathsf{h}]\mathsf{y}^{b-1}.$$

Iterating this gives

(10) $$A_b = \mathsf{y}^b\mathsf{x} + f_b(\mathsf{h})\mathsf{y}^{b-1} \quad \text{where} \quad f_0(\mathsf{h}) = 0$$

which, when substituted into (9) and Result (ii) is applied, gives $f_b(\mathsf{h}) = f_{b-1}(\mathsf{h}+2) + [\mathsf{h}]$. Thus

$$f_b(\mathsf{h}) = \sum_{i=0}^{b-1}[\mathsf{h}+2i] = f_{b-1}(\mathsf{h}) + [\mathsf{h}+2b-2].$$

But, from (iii) of Lemma 2.4 with $a=1$, $x=b-2$, and $y=\mathsf{h}+b-2$, we have $[\mathsf{h}+2b-2] = [b]\,[\mathsf{h}+b-1] - [b-1]\,[\mathsf{h}+b-2]$ so

$$f_b(\mathsf{h}) - [b]\,[\mathsf{h}+b-1] = f_{b-1}(\mathsf{h}) - [b-1]\,[\mathsf{h}+b-2] = c$$

where c is therefore independent of b. Setting $b=0$ in the left hand side gives $c = f_0(\mathsf{h}) = 0$, so $f_b(\mathsf{h}) = [b]\,[\mathsf{h}+b-1]$ and the result follows from (10).

To see (vi), apply Results (iv) and (ii) to obtain

$$(\mathsf{kx})^b\mathsf{h} = \overline{q}^{\frac{1}{2}b(b-1)}\mathsf{k}^b\mathsf{x}^b\mathsf{h} = \overline{q}^{\frac{1}{2}b(b-1)}\mathsf{k}^b(\mathsf{h}-2b)\mathsf{x}^b.$$

By Result (iv), this is equal to $(\mathsf{h}-2b)(\mathsf{kx})^b$ since k and h commute. Iterating this, we have $(\mathsf{kx})^b\mathsf{h}^n = (\mathsf{h}-2b)^n(\mathsf{kx})^b$ for $n \geq 0$, and the result follows by linearity.

Finally, (vii) follows immediately from Results (iii) and (iv). □

When constructing an R-morphism in Section 5.2, a term $e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}}$ will come into play. The following result will be useful for commuting terms past this exponential.

Lemma 2.6. *Let $f(x)$ be a formal power series in x. Then, for any integer m,*

(i) $f(1\otimes\mathsf{k}^m\mathsf{x}) \cdot e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} \cdot f(\overline{\mathsf{k}}^2\otimes\mathsf{k}^m\mathsf{x})$;
(ii) $f(1\otimes\mathsf{k}^m\mathsf{y}) \cdot e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} \cdot f(\mathsf{k}^2\otimes\mathsf{k}^m\mathsf{y})$.

Proof. Let $f = \sum_{n\geq 0} f_n x^n$. For Part (i), we have by (ii) of Lemma (2.5),

$$f(1\otimes\mathsf{k}^m\mathsf{x}) \cdot e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} = \sum_{n\geq 0} f_n(1\otimes(\mathsf{k}^m\mathsf{x})^n)\cdot e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} = \sum_{n\geq 0} f_n e^{\frac{h}{4}\mathsf{h}\otimes(\mathsf{h}-2n)}\cdot(1\otimes(\mathsf{k}^m\mathsf{x})^n).$$

But $e^{\frac{h}{4}\mathsf{h}\otimes(\mathsf{h}-2n)} = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}}e^{-\frac{nh}{2}\mathsf{h}\otimes 1}$ and $e^{-\frac{nh}{2}\mathsf{h}\otimes 1}(1\otimes(\mathsf{k}^m\mathsf{x})^n) = (\overline{\mathsf{k}}^2\otimes\mathsf{k}^m\mathsf{x})^n$, from which the result follows. Part (ii) can be proved similarly. Alternatively, (ii) can be obtained from (i) by applying the algebra automorphism defined on the generators of $\mathcal{U}_h(\mathfrak{sl}_2)$ by $\mathsf{h}\mapsto -\mathsf{h}$, $\mathsf{x}\mapsto\mathsf{y}$ and $\mathsf{y}\mapsto\mathsf{x}$. □

3. Straightening of $\mathsf{x}^a\mathsf{y}^b$

We propose to straighten $\mathsf{x}^a\mathsf{y}^b$, $a,b\in\mathbb{Z}_{\geq 0}$, with respect to the precedence order $\prec$ by a constructive method.

From (v) of Lemma (2.5), the straightening of the premultiplication of y^b by x

$$\mathsf{x}\,\mathsf{y}^b = \mathsf{y}^b\mathsf{x} + [b]\,[\mathsf{h}+b-1]\,\mathsf{y}^{b-1}.$$

Iterating this a times, and noting that x may be moved through quantum brackets containing only h by means of Lemma (2.5) (ii),

$$\mathsf{x}^a\mathsf{y}^b = \sum_{0\leq i,j\leq\min(a,b)} F_{a,b,i,j}(\mathsf{h})\,\mathsf{y}^i\mathsf{x}^j. \tag{11}$$

But $e^{\mathsf{h}}\,\mathsf{x}^a\mathsf{y}^b = \mathsf{x}^a\mathsf{y}^b\,e^{\mathsf{h}-2a+2b}$ by Lemma (2.5) (ii) and so, applying this to (11), gives

$$\mathsf{x}^a\mathsf{y}^b e^{\mathsf{h}-2a+2b} = \sum_{0\leq i,j\leq\min(a,b)} F_{a,b,i,j}(\mathsf{h})\,\mathsf{y}^i\mathsf{x}^j\,e^{\mathsf{h}-2j+2i}.$$

Inverting the right-hand exponential,

$$\mathsf{x}^a\mathsf{y}^b = \sum_{0\le i,j\le \min(a,b)} F_{a,b,i,j}(\mathsf{h})\,\mathsf{y}^i\mathsf{x}^j e^{2(a-j)-2(b-i)}.$$

Equating coefficients of $\mathsf{y}^i\mathsf{x}^j$ on the right hand side of this and (11), we have $i = b-k$ and $j = a-k$ for some non-negative integer k, whence we conclude that

$$\mathsf{x}^a\mathsf{y}^b = \sum_{0\le k\le \min(a,b)} G_{a,b,k}(\mathsf{h})\,\mathsf{y}^{b-k}\mathsf{x}^{a-k} \tag{12}$$

where $G_{a,b,k}(\mathsf{h}) := F_{a,b,a-k,b-k}(\mathsf{h})$. Since commuting x from the left of y^b yields a single term of top degree with coefficient 1, the boundary condition is

$$G_{a,b,0}(\mathsf{h}) = 1. \tag{13}$$

A recursion for $G_{a,b,k}$ is obtained from the identity $\mathsf{x}^a\mathsf{y}^b = \mathsf{x}^{a-1}\left(\mathsf{x}\,\mathsf{y}^b\right)$. First, from Lemma (2.5) (ii) and (v),

$$\mathsf{x}^a\mathsf{y}^b = \left(\mathsf{x}^{a-1}\mathsf{y}^{b-1}\right)(\mathsf{y}\mathsf{x}) + [b]\,[\mathsf{h}+b-2a+1]\,\left(\mathsf{x}^{a-1}\mathsf{y}^{b-1}\right).$$

Then, substituting (12) into this,

$$\sum_{k\ge 0} G_{a,b,k}\mathsf{y}^{b-k}\mathsf{x}^{a-k} = \sum_{k\ge 0} G_{a-1,b-1,k}\Big(\mathsf{y}^{b-k-1}\mathsf{x}^{a-k-1}(\mathsf{x}\mathsf{y}) + [b]\,[\mathsf{h}+b-2a+1]\mathsf{y}^{b-k-1}\mathsf{x}^{a-k-1}\Big).$$

But, from Part (v) of Lemma (2.5),

$$\mathsf{x}^{a-k-1}\mathsf{y} = \mathsf{y}\,\mathsf{x}^{a-k-1} + [a-k-1]\,[\mathsf{h}-a+k+2]\,\mathsf{x}^{a-k-2}.$$

Substituting this into the above, and then equating the coefficients of $\mathsf{y}^{b-k}\mathsf{x}^{a-k}$ gives the recurrence equation

$$G_{a,b,k} = G_{a-1,b-1,k} + \Big([a-k]\,[\mathsf{h}+2b-a-k+1] + [b]\,[\mathsf{h}+b-2a+1]\Big)G_{a-1,b-1,k-1}.$$

Then from (ii) of Lemma 2.4, with $x \mapsto b$, $y \mapsto \mathsf{h}+b-2a+1$ and $a \mapsto a-k$,

$$G_{a,b,k} = G_{a-1,b-1,k} + [a+b-k]\,[\mathsf{h}+b-a-k+1]\,G_{a-1,b-1,k-1} \tag{14}$$

Each instance of $G_{i,j,\cdot}$ in this recurrence equation satisfies $i-j = a-b$. The only term that does not contain $a-b$ is $[a+b-k]$. This suggests using $[k]\,[a+b-k] = [a]\,[b] - [a-k]\,[b-k]$ from the Separation Lemma 2.4 to separate a and b in this quantum bracket and then transforming $G_{a,b,k}$ to form a new recurrence equation in which $a-b$ is an invariant. Let

$$G_{a,b,k} =: \frac{[a]_k\,[b]_k}{[k]!}\,B_{a,b,k}. \tag{15}$$

Then, substituting (15) into (14) gives

$$\begin{aligned}[a]\,[b]\,B_{a,b,k} &- [a-k]\,[b-k]\,B_{a-1,b-1,k} \\ &= \Big([a]\,[b] - [a-k]\,[b-k]\Big)\,[\mathsf{h}+b-a-k+1]\,B_{a-1,b-1,k-1}.\end{aligned} \tag{16}$$

Suppose now that the $B_{a,b,k}$ depend only on the difference $b-a$. Then $B_{a,b,k} = B_{a-1,b-1,k}$ and (16) becomes

$$B_{a,b,k} = [\mathsf{h}+b-a-k+1]\,B_{a,b,k-1}$$

for $k \geq 1$ and $B_{a,b,0} = 1$ from (13). This suggests the solution $B_{a,b,k} = [\mathsf{h} + b - a]_k$. Indeed, it is readily checked that this does indeed satisfy (16), from which we have

$$G_{a,b,k} = [a]_k\,[b]_k \begin{bmatrix} \mathsf{h} + b - a \\ k \end{bmatrix}.$$

So, from (12), we have therefore (both derived and) proved the following lemma.

Lemma 3.1. *Let a and b be non-negative integers. Then*

$$\frac{\mathsf{x}^a}{[a]!}\,\frac{\mathsf{y}^b}{[b]!} = \sum_{i \geq 0} \begin{bmatrix} \mathsf{h} + b - a \\ i \end{bmatrix} \frac{\mathsf{y}^{b-i}}{[b-i]!}\,\frac{\mathsf{x}^{a-i}}{[a-i]!}.$$

An important special case of Lemma 3.1 is when the powers of x and y coincide.

Lemma 3.2. *Let n be a non-negative integer. Then*

$$\mathsf{x}^n\mathsf{y}^n = \sum_{i=0}^{n} [n]_i^2 \begin{bmatrix} \mathsf{h} \\ i \end{bmatrix} \mathsf{y}^{n-i}\mathsf{x}^{n-i}.$$

We remark that Lemma 3.1 may be rewritten immediately in the following form, as a straightening result for the commutator $[\mathsf{x}^a, \mathsf{y}^b]$.

Corollary 3.3. *Let a and b be non-negative integers. Then*

$$[\mathsf{x}^a, \mathsf{y}^b] = \sum_{i \geq 1} [a]_i\,[b]_i \begin{bmatrix} \mathsf{h} + b - a \\ i \end{bmatrix} \mathsf{y}^{b-i}\mathsf{x}^{a-i}.$$

4. Inversions in permutations

A discussion of the universal R-morphism and certain elements (conventionally denoted by **u** and **v**) of the Ribbon Hopf Algebra may be made more natural by using properties of certain q-series. The properties that will be needed may be obtained succinctly by considering combinatorial properties of inversions in permutations.

4.1. The q-factorial function.

The *q-factorial* function $(n)!_q$ and the *q-binomial coefficient* $\binom{n}{k}_q$ in an indeterminate q are defined by

$$(n)_q := \frac{1 - q^n}{1 - q}, \quad (n)!_q := (1)_q \cdot (2)_q \cdots (n)_q, \quad \binom{n}{k}_q := \frac{(n)!_q}{(k)!_q\,(n-k)!_q}.$$

The q-factorial function is also known as the *Gaussian coefficient.* These are functions encountered in the theory of basic hypergeometric series, and also in enumerative combinatorics. The *q-exponential series* is defined by

$$\exp_q(x) := \sum_{n \geq 0} \frac{x^n}{(n)!_q} \in \mathbb{Q}(q)\,[[x]]$$

as a formal power series in x with coefficients that are rational functions of q. The next result gives a multiplicative property of this series and is stated in terms of two indeterminats a and b that satisfy the *quantum commutation condition*

$$ab = q\,ba.$$

The proof uses the observation that q is associated with a combinatorial property of sets, as follows. An *ordered bipartition* $\{1, \ldots, n\}$ of *type* $(r, n-r)$ is (α, β), where α and β are disjoint subsets of $\{1,\dots,n\}$ of size r and $n-r$, respectively. A *between-set inversion* of (α,β) is a pair $(i,j)\in\alpha\times\beta$ such that $i>j$. An *inversion* in a permutation $\pi\in\mathfrak{S}_n$ is a pair (i,j) with $1\le i<j\le n$ such that $\pi(i)>j$.

Lemma 4.1. *Let a,b be such that $ab=q\,ba$. Then*

$$\exp_q(a+b)=\exp_q(a)\cdot\exp_q(b).$$

Proof. There is clearly an expression for $(a+b)^n$ of the form

$$(a+b)^n=\sum_{r=0}^{n} f_{r,n-r}(q)a^r b^{n-r} \tag{17}$$

where $f_{r,n-r}(q)$ is a polynomial in q. Then $[q^k]f_{r,n-r}(q)$ is immediately identified as the number of ordered bi-partitions of $\{1,\dots,n\}$ of type $(r,n-r)$ with precisely k between-set inversions.

We determine the generating series $g_n(q)$, where $[q^k]g_n(q)$ is the number of inversions in $\pi\in\mathfrak{S}_n$, in two different ways. First, by considering the contribution to inversions by the symbol n in π, we have the recursion

$$g_n(q)=g_{n-1}(q)\,(1+q+\cdots+q^{n-1})\qquad\text{for } n\ge 1$$

with $g_0(q)=1$, so $g_n(q)=n!_q$. On the other hand, by considering a fixed bipartition (α,β) of type $(r,n-r)$, we have $(n!)_q=g_r(q)\,g_{n-r}(q)\,f_{r,n-r}(q)$ since each inversion of π occurs within α, or within β, or between α and β. Then $f_{r,n-r}(q)=n!_q/(r!_q\,(n-r)!_q)$ and the result then follows immediately from (17). □

Theorem 4.2 (Bimodal Permutation [GJ83])**.** *Let q,w,x,y,z be indeterminates, let n be a non-negative integer and let $Q_n(x,y):=\prod_{i=0}^{n-1}\left(y+xq^i\right)$. Then*

$$\sum_{k=0}^{n}\binom{n}{k}_q Q_k(x,y)\,Q_{n-k}(w,z)=\sum_{k=0}^{n}\binom{n}{k}_q Q_k(w,y)\,Q_{n-k}(x,z).$$

The q-analogue of the Binomial Theorem along with several classical q-identities are now easily obtained. For more, see, for example, [GJ04].

Corollary 4.3 (q-analogue of the Binomial Theorem)**.** *Let q,x,y,z be indeterminates and let n be a non-negative integer. Then*

$$Q_n(-x,z)=\sum_{k=0}^{n}\binom{n}{k}_q Q_k(-x,y)\,Q_{n-k}(-y,z).$$

An immediate consequence of this is a finite product identity due to Cauchy [Cau09], and its inverse.

Lemma 4.4. *Let z and q be indeterminates. Then*

(i) $z^n=\sum_{k=0}^{n}\binom{n}{k}_q\prod_{i=0}^{k-1}\left(z-q^i\right),$

(ii) $\prod_{i=0}^{n-1}\left(z-q^i\right)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}_q q^{\frac{1}{2}k(k-1)}\,z^{n-k}.$

Proof. Part (i) follows from Corollary 4.3 by setting $y=1$ and $x=0$; for part (ii), set $x=1$ and $y=0$. □

Taking z to be an exponential related to q, Cauchy's finite product identity can be extended to an identity of certain formal power series.

Lemma 4.5. *Let h, t and x be indeterminates such that $q = e^{\frac{h}{2}}$. Then, in $\mathbb{Q}[x,t][[h]]$,*

$$e^{xht} = \sum_{k=0}^{\infty} \binom{x}{k}_{q^2} \prod_{i=0}^{k-1} (e^{ht} - q^{2i}). \tag{18}$$

Proof. It is clear that the coefficient of h^m is a polynomial in t and x on the left hand side. For the right hand side, first note that

$$\mathrm{val}_h \left(\prod_{i=0}^{k-1} (e^{ht} - q^{2i}) \right) = \mathrm{val}_h \left(\prod_{i=0}^{k-1} (t-i)h \right) = k,$$

where the h-valuation val_h of an element $p(h) \in R[[h]]$, R some coefficient ring, is

$$\mathrm{val}_h\, p(h) := \min\{\ell \in \mathbb{Z} : [h^\ell]\, p(h) \neq 0\}.$$

Thus nonzero contributions to the coefficient of h^m come from the finitely many indices $0 \le k \le m$. Next, the power series expansion of $\binom{x}{k}_{q^2}$ in h also has coefficients which are polynomial in x. Combining these contributions, we conclude that each coefficient of h^m on the right hand side of (18) is polynomial in t and x.

In particular, it follows that the coefficient of $h^m t^n$ is a polynomial in x on both sides. By ((i)) of Lemma 4.4, these polynomials in x agree for each positive integer and thus they must be equal as polynomials. □

4.2. **The quantum exponential function.** It is readily seen that the functions defined in (7) and the q-factorial and the q-binomial functions are related through

$$[n]_q = \overline{q}^{n-1}(n)_{q^2}, \quad [n]!_q = \overline{q}^{\frac{1}{2}n(n-1)}(n)!_{q^2}, \quad \begin{bmatrix} n \\ k \end{bmatrix}_q = \overline{q}^{k(n-k)} \binom{n}{k}_{q^2}. \tag{19}$$

The *quantum exponential function* defined by

$$\mathrm{Exp}_q(x) := \sum_{n \ge 0} \frac{q^{\frac{1}{2}n(n-1)}}{[n]!_q} x^n \tag{20}$$

enjoys the same multiplicative property as the q-exponential series under quantum commutation.

Lemma 4.6. *Let a and b be such that $ab = q^2\, ba$. Then*

$$\mathrm{Exp}_q(a+b) = \mathrm{Exp}_q(a)\,\mathrm{Exp}_q(b).$$

Proof. Since $[n]!$ is unchanged by replacing q by $\overline{q}$, so from (19), $[n]! = q^{\frac{1}{2}n(n-1)}(n)!_{\overline{q}^2}$. Then $\mathrm{Exp}_q(x) = \exp_{\overline{q}^2}(x)$. Thus, from Lemma 4.1, $\mathrm{Exp}_q(a+b) = \exp_{\overline{q}^2}(a+b) = \exp_{\overline{q}^2}(a)\, \exp_{\overline{q}^2}(b)$, and the result follows. □

5. An R-morphism

Quasi-triangular Hopf algebras are Hopf algebras $\mathfrak{A}$ equipped with an element $\mathsf{R} \in \mathfrak{A}^{\otimes 2}$, called an R-morphism. This structure provides a systematic means to producing solutions to the Yang-Baxter equation (1).

5.1. **Quasi-Triangular Hopf Algebras.** Hopf algebras are algebras with the additional structure of a coproduct and an antipode. Quasi-triangular Hopf algebras are those which posses an element, typically denoted R, satisfying properties akin to the Yang-Baxter equation. Indeed, as was mentioned in the Introduction, the element R can be used to systematically construct solutions to the Yang-Baxter equation. We begin by recalling some algebraic definitions.

5.1.1. *Hopf structure.* Let $\mathcal{A}$ be a vector space over $\mathbb{C}$ and let $\mathfrak{A} := (\mathcal{A}, m, \Delta, \eta, \varepsilon, \mathsf{S})$ be a bi-algebra with product m, a co-product Δ, a unit η, a co-unit (augmentation map) ε and an anti-homomorphism $\mathsf{S} : \mathfrak{A} \to \mathfrak{A}$ (so $\mathsf{S}(ab) = \mathsf{S}(b) \cdot \mathsf{S}(a)$ for $a, b \in \mathfrak{A}$). Thus Δ and ε are algebra morphisms, and m and η are co-algebra morphisms.

To show that $\mathfrak{A}$ is a Hopf algebra with antipode S, it is necessary to check that the following hold:

<u>Product $m : \mathfrak{A} \otimes \mathfrak{A} \to \mathfrak{A}$</u>: The product is *associative*:

$$m \circ (\mathrm{id} \otimes m) = (m \otimes \mathrm{id}) \circ m,$$

and possesses a *unit*, *i.e.* a map $\eta : \mathbb{K} \to \mathfrak{A}$ such that

$$m(a \otimes \eta(1_{\mathbb{K}})) = a = m(\eta(1_{\mathbb{K}}) \otimes a) \quad \text{for all } a \in \mathfrak{A}.$$

The element $\eta(1_{\mathbb{K}}) \in \mathfrak{A}$ will be denoted $1_{\mathfrak{A}}$ and will be referred to as the *multiplicative unit.*

<u>Co-product $\Delta \colon \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$</u>: The co-product is *co-associative*:

$$(\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta,$$

and possesses a *co-unit*, *i.e.* a map $\varepsilon \colon \mathfrak{A} \to \mathbb{K}$ such that

$$(\varepsilon \otimes \mathrm{id}) \circ \Delta(a) = 1_{\mathbb{K}} \otimes a \quad \text{and} \quad (\mathrm{id} \otimes \varepsilon) \circ \Delta(a) = a \otimes 1_{\mathbb{K}} \quad \text{for all } a \in \mathfrak{A}$$

<u>Compatibility relations</u>: The coproduct Δ and co-unit ε need to be algebra morphisms, that is:

$$\Delta \circ m = (m_{13} \otimes m_{24}) \circ (\Delta \otimes \Delta), \quad \varepsilon \circ m = m_{\mathbb{K}} \circ (\varepsilon \otimes \varepsilon)$$

where $m_{13} \otimes m_{24} : (\mathfrak{A} \otimes \mathfrak{A}) \otimes (\mathfrak{A} \otimes \mathfrak{A}) \to \mathfrak{A} \otimes \mathfrak{A}$ acts on a pure tensor by $a_1 \otimes a_2 \otimes a_3 \otimes a_4 \mapsto a_1 a_3 \otimes a_2 a_4$, and $m_{\mathbb{K}} : \mathbb{K} \otimes \mathbb{K} \to \mathbb{K}$ is the product on the underlying field. Note that, consequently, the product and unit are coalgebra morphisms.

Finally, the *antipode* S must be an algebra antihomomorphism, *i.e.*, for all $a, b \in \mathfrak{A}$, $(\mathsf{S} \circ m)(a \otimes b) = m(\mathsf{S}(b) \otimes \mathsf{S}(a))$, and it must satisfy

$$m \circ (\mathsf{S} \otimes \mathrm{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\mathsf{S} \otimes \mathrm{id}) \circ \Delta.$$

5.1.2. *Hopf Structure on $\mathcal{U}_h(\mathfrak{sl}_2)$.* The next result, which is due to Sklyanin [Skl85], shows how $\mathcal{U}_h(\mathfrak{sl}_2)$ may be endowed with Hopf structure.

Theorem 5.1. *The quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{sl}_2)$ is a Hopf algebra with unit ϵ given by $\epsilon(1) = 1$ and co-multiplication Δ, antipode S, co-unit ε defined on generators by*

$$\begin{array}{lll}
\Delta : \mathcal{U}_h(\mathfrak{sl}_2) \to \mathcal{U}_h(\mathfrak{sl}_2) \otimes \mathcal{U}_h(\mathfrak{sl}_2), & \mathsf{S} : \mathcal{U}_h(\mathfrak{sl}_2) \to \mathcal{U}_h(\mathfrak{sl}_2), & \varepsilon : \mathcal{U}_h(\mathfrak{sl}_2) \to \mathbb{C}, \\
\mathsf{x} \mapsto \mathsf{x} \otimes \mathsf{k} + \overline{\mathsf{k}} \otimes \mathsf{x}, & \mathsf{x} \mapsto -q\mathsf{x}, & \mathsf{x} \mapsto 0, \\
\mathsf{y} \mapsto \mathsf{y} \otimes \mathsf{k} + \overline{\mathsf{k}} \otimes \mathsf{y}, & \mathsf{y} \mapsto -\overline{q}\mathsf{y}, & \mathsf{y} \mapsto 0, \\
\mathsf{h} \mapsto \mathsf{h} \otimes 1 + 1 \otimes \mathsf{h}, & \mathsf{h} \mapsto -\mathsf{h}, & \mathsf{h} \mapsto 0.
\end{array}$$

where Δ and ε are extended as algebra morphisms, and m and η are extended as co-algebra morphisms.

5.1.3. *Action on* k. We determine the action of Δ, S and ε on k. For $\Delta(\mathsf{k})$, we have, by routine calculation using the fact that Δ is an algebra morphism,

$$\begin{aligned}\Delta(\mathsf{k}) &= e^{\frac{h}{4}\Delta(\mathsf{h})} = e^{\frac{h}{4}(\mathsf{h}\otimes 1+1\otimes\mathsf{h})} = \left(e^{\frac{h}{4}(\mathsf{h}\otimes 1)}\right)\left(e^{\frac{h}{4}(1\otimes\mathsf{h})}\right) = \left(e^{\frac{h}{4}\mathsf{h}}\otimes 1\right)\left(1\otimes e^{\frac{h}{4}\mathsf{h}}\right)\\ &= e^{\frac{h}{4}\mathsf{h}}\otimes e^{\frac{h}{4}\mathsf{h}} = \mathsf{k}\otimes\mathsf{k}\end{aligned}$$

since $\mathsf{h}\otimes 1$ and $1\otimes\mathsf{h}$ commute. For $\mathsf{S}(\mathsf{k})$, we have

$$\mathsf{S}(\mathsf{k}) = \mathsf{S}\left(e^{\frac{h}{4}\mathsf{h}}\right) = e^{\frac{h}{4}\mathsf{S}(\mathsf{h})} = e^{-\frac{h}{4}\mathsf{h}} = \overline{\mathsf{k}}$$

using the definition of k from (5) and since S is an algebra morphism. For $\varepsilon(\mathsf{k})$, we have

$$\varepsilon(\mathsf{k}) = \varepsilon\left(e^{\frac{h}{4}\mathsf{h}}\right) = e^{\frac{h}{4}\varepsilon(\mathsf{h})} = e^0 = 1.$$

Collecting these evaluations:

$$\Delta(\mathsf{k}) = \mathsf{k}\otimes\mathsf{k}, \qquad \mathsf{S}(\mathsf{k}) = \overline{\mathsf{k}}, \qquad \varepsilon(\mathsf{k}) = 1. \tag{21}$$

5.1.4. *Quasi-triangular Hopf algebras.*

Definition 5.2 (Quasi-triangular Hopf algebra)**.** *A* quasi-triangular Hopf algebra *is a pair $(\mathfrak{A},\mathsf{R})$ where $(\mathfrak{A},m,\Delta,\varepsilon,\eta,\mathsf{S})$ is a Hopf algebra, with an invertible antipode S, and $\mathsf{R}\in\mathfrak{A}\otimes\mathfrak{A}$ is an invertible element satisfying the conditions*

(i) $(\tau\circ\Delta)(a) = \mathsf{R}\cdot\Delta(a)\cdot\mathsf{R}^{-1}$ *for every* $a\in\mathfrak{A}$,
(ii) $(\Delta\otimes\mathrm{id}_{\mathfrak{A}})(\mathsf{R}) = \mathsf{R}_{13}\cdot\mathsf{R}_{23}$,
(iii) $(\mathrm{id}_{\mathfrak{A}}\otimes\Delta)(\mathsf{R}) = \mathsf{R}_{13}\cdot\mathsf{R}_{12}$,

where τ is the twist map, $\tau(a\otimes b) := b\otimes a$, and in which R_{12}, R_{23} and R_{13} are defined in terms of R by

(iv) $\mathsf{R} := \sum_i \alpha_i\otimes\beta_i$ *for* $\alpha_i,\beta_i\in\mathfrak{A}$,
(v) $\mathsf{R}_{12} := \sum_i \alpha_i\otimes\beta_i\otimes 1_{\mathfrak{A}} = \mathsf{R}\otimes 1_{\mathfrak{A}}$,
(vi) $\mathsf{R}_{13} := \sum_i \alpha_i\otimes 1_{\mathfrak{A}}\otimes\beta_i = (\mathrm{id}_{\mathfrak{A}}\otimes\tau)\circ(\mathsf{R}\otimes 1_{\mathfrak{A}})$,
(vii) $\mathsf{R}_{23} := \sum_i 1_{\mathfrak{A}}\otimes\alpha_i\otimes\beta_i = 1_{\mathfrak{A}}\otimes\mathsf{R}$.

The element R is called a universal R-morphism *or an* R-morphism*.*

Condition (i) of this Definition is related to co-commutativity: if, for instance, $\mathsf{R} = 1_{\mathfrak{A}}\otimes 1_{\mathfrak{A}}$ then this would be precisely expressing co-commutativity of $\mathfrak{A}$. Conditions (ii) and (iii) of the Definition can be explicitly written using the notation of (iv), (v), (vi), (vii) as:

(ii)' $\sum_i\sum_{(\alpha_i)} \alpha_i'\otimes\alpha_i''\otimes\beta_i = \sum_i\sum_j \alpha_i\otimes\alpha_j\otimes\beta_i\beta_j$,
(iii)' $\sum_i\sum_{(\beta_i)} \alpha_i\otimes\beta_i'\otimes\beta_i'' = \sum_i\sum_j \alpha_i\alpha_j\otimes\beta_j\otimes\beta_i$.

These conditions are induced from the desire that R induces solutions to the Yang-Baxter equation on modules of $\mathfrak{A}$ [Dri87]. More precisely, suppose $\mathcal{V}$ is a vector space carrying a representation $\rho : \mathfrak{A} \to \mathrm{End}(\mathcal{V})$ of $\mathfrak{A}$. Then the element $\tau \circ (\rho \otimes \rho)(\mathsf{R}) \in \mathrm{End}(\mathcal{V} \otimes \mathcal{V})$ is a solution of the Yang-Baxter equation (1). In this way, we may regard these conditions, as a *universal Yang-Baxter equation*, and the Definition as a purely algebraic context in which to work with it.

5.2. **Constructing an R-morphism.** We now propose to construct a universal R-morphism for $\mathcal{U}_h(\mathfrak{sl}_2)$, thereby endowing the Hopf algebra with a quasi-triangular structure. This will be done by first considering the general form of dependency of R on x, y and h in order to come up with a skeletal form for the R-morphism, with coefficients and parameters to be determined. Conditions (i), (ii) and (iii) of Definition 5.2 will then be used to determine the parameters.

5.2.1. *An* ansatz *for* R. We suppose that R is a sum of terms, whose general term is to be denoted by T. Since $T \in \mathcal{A} \otimes \mathcal{A}$, the dependency of T on x and y after straightening in each tensor component is therefore $\mathsf{x}^m\mathsf{y}^t \otimes \mathsf{x}^u\mathsf{y}^n$, where m, t, u and n are non-negative integers. As a byproduct of straightening the x and y terms, Lemma 3.1 shows that factors of h are introduced into T through powers of k and $\overline{\mathsf{k}}$. This suggests that the h dependence in T is through powers of $\mathsf{k} \otimes 1 = e^{\frac{h}{4}\mathsf{h}\otimes 1}$ and $1 \otimes \mathsf{k} = e^{\frac{h}{4}1\otimes\mathsf{h}}$. Indeed, this suggestion is further supported by the conditions (ii) and (iii) of Definition 5.2, as the action of Δ on x and y, from Theorem 5.1, and on k, from (21), further introduces dependency on h through powers of k on a single tensor component.

Preliminary calculations suggest that the *ansatz* for R constructed thus far is inadequate. Indeed, both calculations and inspection of Condition (i) in Definition 5.2 suggest the presence of powers of $e^{\frac{h}{4}h\otimes h}$ in T. To see this, rewrite Condition (i) as

$$(\tau \circ \Delta)(a) \cdot \mathsf{R} = \mathsf{R} \cdot \Delta(a)$$

and note that it suffices to check this condition for the generators of $\mathcal{U}_h(\mathfrak{sl}_2)$. Checking this condition on x (or y), the asymmetry of the co-product $\Delta(\mathsf{x}) = \mathsf{x}\otimes\mathsf{k}+\overline{\mathsf{k}}\otimes\mathsf{x}$ in powers of k and $\overline{\mathsf{k}}$ affects powers of $\mathsf{k}\otimes 1$ and $1\otimes\mathsf{k}$ in T in significantly different ways in each of the two sides of the relation. This suggests the need for a device which can introduce additional powers of $\mathsf{k}\otimes 1$ and $1\otimes\mathsf{k}$ *via* straightening. A natural solution for this is to include powers of $e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}}$, as straightening with expressions involving x and y creates terms of the form $e^{\frac{h}{4}(\mathsf{h}+a)\otimes(\mathsf{h}+b)}$ through (ii) of Lemma 2.5. Thus, we argue that the dependence of T on h is through $e^{\frac{h}{4}(k\,\mathsf{h}\otimes\mathsf{h}+r\,\mathsf{h}\otimes 1+s\,1\otimes\mathsf{h})}$ where k, r and s are integers.

The above argument leads to

$$\mathsf{R} = \sum_{k,m,n,r,s,t,u} a_{k,r,s,m,n,t,u}(q)\, e^{\frac{h}{4}(k\,\mathsf{h}\otimes\mathsf{h}+r\,\mathsf{h}\otimes 1+s\,1\otimes\mathsf{h})}\mathsf{x}^m\mathsf{y}^t \otimes \mathsf{x}^u\mathsf{y}^n. \tag{22}$$

as a conjectural form for R, where $a_{k,r,s,m,n,t,u}(q)$ is a function of q.

Finally, to simplify further, observe that Conditions (ii)' and (iii)' of Definition 5.2 suggest the two tensor components R are essentially independent of one another. We therefore begin by considering the simpler form in which not both x and y appear in each of the tensor components. We confine the occurrence of x and y exclusively to the first and second tensor components, respectively, by setting

$t = 0$ and $u = 0$ and, with minor abuse of notation, denoting $a_{k,r,s,m,n,0,0}(q)$ by $a_{k,r,s,m,n}(q)$, to obtain

$$\mathsf{R} = \sum_{k,m,n,r,s} a_{k,r,s,m,n}(q)\, e^{\frac{h}{4}(k\,\mathsf{h}\otimes\mathsf{h}+r\,\mathsf{h}\otimes 1+s\,1\otimes\mathsf{h})}\mathsf{x}^m \otimes \mathsf{y}^n.$$

Thus the proposed *ansatz* for R is

$$\mathsf{R} = \sum_{k,m,n,r,s} a_{k,r,s,m,n}(q)\, e^{\frac{h}{4}k(\mathsf{h}\otimes\mathsf{h})}\mathsf{k}^r\mathsf{x}^m \otimes \mathsf{k}^s\mathsf{y}^n. \tag{23}$$

For brevity, explicit mention of the dependence of $a_{k,r,s,m,n}(q)$ on q will henceforth be suppressed.

5.2.2. *Condition* (i) *of Definition 5.2.* Since Δ is an algebra morphism it is sufficient to show that this condition holds for the generators h, x and y.

For the generator h: The condition asserts that $(\tau\circ\Delta)(\mathsf{h})\cdot\mathsf{R} = \mathsf{R}\cdot\Delta(\mathsf{h})$, so

$$(\tau\circ\Delta)(\mathsf{h})\cdot\mathsf{R} - \mathsf{R}\cdot\Delta(\mathsf{h}) = (\mathsf{h}\otimes 1 + 1\otimes\mathsf{h})\cdot\mathsf{R} - \mathsf{R}\cdot(\mathsf{h}\otimes 1+1\otimes\mathsf{h}) = 0.$$

Since $e^{\frac{h}{4}(k\,\mathsf{h}\otimes\mathsf{h}+r\,\mathsf{h}\otimes 1+s\,1\otimes\mathsf{h})}$ commutes with $\mathsf{h}\otimes 1+1\otimes\mathsf{h}$, the condition is equivalent to $(\mathsf{h}\otimes 1+1\otimes\mathsf{h})(\mathsf{k}^r\mathsf{x}^m\otimes\mathsf{k}^s\mathsf{y}^n) = (\mathsf{k}^r\mathsf{x}^m\otimes\mathsf{k}^s\mathsf{y}^n)(\mathsf{h}\otimes 1+1\otimes\mathsf{h})$. But, by (ii) of Lemma 2.5, the left hand side is equal to

$$(\mathsf{k}^r\mathsf{x}^m\otimes\mathsf{k}^s\mathsf{y}^n)(\mathsf{h}\otimes 1+1\otimes\mathsf{h}) + 2(m-n)(\mathsf{k}^r\mathsf{x}^m\otimes\mathsf{k}^s\mathsf{y}^n)$$

so the condition implies that $m = n$. Let $a_{k,r,s,n}$ denote $a_{k,r,s,n,n}$. Then

$$\mathsf{R} = \sum_{k,n,r,s} a_{k,r,s,n}\, e^{\frac{h}{4}k(\mathsf{h}\otimes\mathsf{h})}(\mathsf{k}^r\mathsf{x}^n\otimes\mathsf{k}^s\mathsf{y}^n). \tag{24}$$

For the generator x: For this generator, Condition (i) of Definition 5.2 asserts that $(\tau\circ\Delta)(\mathsf{x})\cdot\mathsf{R} = \mathsf{R}\cdot\Delta(\mathsf{x})$. By (ii) of Lemma 2.5,

$$(\tau\circ\Delta)(\mathsf{x})\cdot\mathsf{R} = \sum_{k,n,r,s} a_{k,r,s,n} e^{\frac{h}{4}k(\mathsf{h}\otimes\mathsf{h})}(\overline{\mathsf{k}}^{2k-1-r}\mathsf{x}^n\otimes\mathsf{x}\mathsf{k}^s\mathsf{y}^n + \mathsf{x}\mathsf{k}^r\mathsf{x}^n\otimes\overline{\mathsf{k}}^{2k+1-s}\mathsf{y}^n),$$

$$\mathsf{R}\cdot\Delta(\mathsf{x}) = \sum_{k,n,r,s} a_{k,r,s,n}\, e^{\frac{h}{4}k(\mathsf{h}\otimes\mathsf{h})}(\mathsf{k}^r\mathsf{x}^{n+1}\otimes\mathsf{k}^s\mathsf{y}^n\mathsf{k} + \mathsf{k}^r\mathsf{x}^n\overline{\mathsf{k}}\otimes\mathsf{k}^s\mathsf{y}^n\mathsf{x}).$$

The condition $(\tau\circ\Delta)(\mathsf{x})\cdot\mathsf{R} = \mathsf{R}\cdot\Delta(\mathsf{x})$ may be rewritten so that terms containing an x in the second tensor factor are gathered on the left in a series A, and the remaining are gathered on the right in a series B. Then condition is equivalent to

$$A = B \tag{25}$$

where

$$A := \sum_{k,n,r,s} a_{k,r,s,n}\, e^{\frac{h}{4}k(\mathsf{h}\otimes\mathsf{h})}\left(\left(\overline{\mathsf{k}}^{2k-1-r}\mathsf{x}^n\otimes\mathsf{x}\mathsf{k}^s\mathsf{y}^n\right) - \left(\mathsf{k}^r\mathsf{x}^n\overline{\mathsf{k}}\otimes\mathsf{k}^s\mathsf{y}^n\mathsf{x}\right)\right),$$

$$B := \sum_{k,n,r,s} a_{k,r,s,n}\, e^{\frac{h}{4}k(\mathsf{h}\otimes\mathsf{h})}\left(\left(\mathsf{k}^r\mathsf{x}^{n+1}\otimes\mathsf{k}^s\mathsf{y}^n\mathsf{k}\right) - \left(\mathsf{x}\mathsf{k}^r\mathsf{x}^n\otimes\overline{\mathsf{k}}^{2k+1-s}\mathsf{y}^n\right)\right).$$

To simplify A, note that $\mathsf{x}\,\mathsf{k}^s = \overline{q}^s\mathsf{k}^s\mathsf{x}$ and $\mathsf{x}^n\overline{\mathsf{k}} = q^n\overline{\mathsf{k}}\mathsf{x}^n$ from (iv) of Lemma (2.5), so

$$A = \sum_{k,n,r,s} a_{k,r,s,n}\, e^{\frac{h}{4}k(\mathsf{h}\otimes\mathsf{h})}\left(\overline{q}^s\left(\overline{\mathsf{k}}^{2k-1-r}\mathsf{x}^n\otimes\mathsf{k}^s\mathsf{x}\mathsf{y}^n\right) - q^n\left(\mathsf{k}^{r-1}\mathsf{x}^n\otimes\mathsf{k}^s\mathsf{y}^n\mathsf{x}\right)\right).$$

The commutator $[\mathsf{x},\mathsf{y}^n]$ appears from the second tensor factors of this expression, if we set $s=-n$ and $k=1$. Replacing $a_{1,r,-n,n}(q)$ by $a_{r,n}(q)$, then using (v) of Lemma (2.5) to calculate the commutator, and replacing n by $n+1$ gives

$$A = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} \sum_{n,r} a_{r,n+1}[n+1]\, q^{n+1}\mathsf{k}^{r-1}\mathsf{x}^{n+1} \otimes [\mathsf{h}+n]\overline{\mathsf{k}}^{n+1}\mathsf{y}^n. \tag{26}$$

To simplify B, we use the values of s and k determined above and the observations that $\mathsf{y}^n\mathsf{k} = q^n\mathsf{k}\mathsf{y}^n$ and $\mathsf{x}\mathsf{k}^r = \overline{q}^r\mathsf{k}^r\mathsf{x}$ from (iv) of Lemma (2.5) to obtain

$$B = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} \sum_{n,r} a_{r,n}(q)\mathsf{k}^r\mathsf{x}^{n+1} \otimes \left(q^n\overline{\mathsf{k}}^{n-1}\mathsf{y}^n - \overline{q}^r\overline{\mathsf{k}}^{n+3}\mathsf{y}^n\right).$$

which, on setting $r=n$ and denoting $a_{n,n}$ by a_n, gives

$$B = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} \sum_{n\geq 0} a_n\,(q-\overline{q})\mathsf{k}^n\mathsf{x}^{n+1} \otimes [\mathsf{h}+n]\,\overline{\mathsf{k}}^{n+1}\mathsf{y}^n.$$

With this setting of r, the expression for A given in (26) becomes

$$A = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} \sum_{n\geq 0} a_{n+1}[n+1]\, q^{n+1}\mathsf{k}^n\mathsf{x}^{n+1} \otimes [\mathsf{h}+n]\overline{\mathsf{k}}^{n+1}\mathsf{y}^n.$$

Since $A=B$ from (25), it follows that $a_n(q)$ must satisfy the two-term recurrence equation $a_{n+1}[n+1]q^{n+1} = a_n\,(q-\overline{q})$ with initial condition $a_0(q)=1$, whence $a_n = \frac{(q-\overline{q})^n}{[n]!}\overline{q}^{\frac{1}{2}n(n+1)}$. It follows from (24) that

$$\mathsf{R} = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} \sum_{n\geq 0} \frac{(q-\overline{q})^n}{[n]!}\overline{q}^{\frac{1}{2}n(n+1)}\,(\mathsf{k}^n\mathsf{x}^n \otimes \overline{\mathsf{k}}^n\mathsf{y}^n).$$

But $\mathsf{k}^n\mathsf{x}^n = q^{\frac{1}{2}n(n-1)}(\mathsf{k}\mathsf{x})^n$ and $\overline{\mathsf{k}}^n\mathsf{y}^n = q^{\frac{1}{2}n(n-1)}(\overline{\mathsf{k}}\mathsf{y})^n$ from (iv) of Lemma 2.5, so

$$\mathsf{R} = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}} \sum_{n\geq 0} \frac{(q-\overline{q})^n}{[n]!}q^{\frac{1}{2}n(n-3)}\,(\mathsf{k}\mathsf{x})^n \otimes (\overline{\mathsf{k}}\mathsf{y})^n. \tag{27}$$

For the generator y: It is readily shown that this condition is satisfied by the expression for R given in (27).

5.2.3. *Condition* (ii) *of Definition 5.2.* It is immediate from the definition of the quantum exponential function in (20) and the expression for R given in (27) that

$$\mathsf{R} = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}}\,\mathrm{Exp}_q\left(\lambda_q\mathsf{k}\mathsf{x}\otimes\overline{\mathsf{k}}\mathsf{y}\right) \qquad \text{where } \lambda_q := \overline{q}(q-\overline{q}). \tag{28}$$

Condition (ii) of Definition 5.2 requires that $(\Delta\otimes\mathrm{id}_{\mathcal{A}})(\mathsf{R}) = \mathsf{R}_{13}\cdot\mathsf{R}_{23}$. It is readily seen that

$$(\Delta\otimes\mathrm{id}_{\mathcal{A}})(\mathsf{R}) = e^{\frac{h}{4}((\mathsf{h}\otimes 1+1\otimes\mathsf{h})\otimes\mathsf{h})}\,\mathrm{Exp}_q\left(\lambda_q(\mathsf{k}\mathsf{x}\otimes\mathsf{k}^2 + 1\otimes\mathsf{k}\mathsf{x})\otimes\overline{\mathsf{k}}\mathsf{y}\right) \tag{29}$$

since Δ is an algebra morphism, and $\Delta(\mathsf{k}) = \mathsf{k}\otimes\mathsf{k}$ from (21).

On the other hand, setting $\mathsf{h}_1 := e^{\frac{h}{4}\,1\otimes\mathsf{h}\otimes\mathsf{h}}$ and $\mathsf{h}_2 := e^{\frac{h}{4}\,\mathsf{h}\otimes 1\otimes\mathsf{h}}$ in Definition 5.2, gives

$$\mathsf{R}_{13} = \mathsf{h}_2\,\mathrm{Exp}_q\left(\lambda_q(\mathsf{k}\mathsf{x}\otimes 1\otimes\overline{\mathsf{k}}\mathsf{y})\right) \quad\text{and}\quad \mathsf{R}_{23} = \mathsf{h}_1\,\mathrm{Exp}_q\left(\lambda_q(1\otimes\mathsf{k}\mathsf{x}\otimes\overline{\mathsf{k}}\mathsf{y})\right)$$

so

$$\mathsf{R}_{13}\cdot\mathsf{R}_{23} = \mathsf{h}_2\,\mathrm{Exp}_q\left(\lambda_q(\mathsf{k}\mathsf{x}\otimes 1\otimes\overline{\mathsf{k}}\mathsf{y})\right)\mathsf{h}_1\,\mathrm{Exp}_q\left(\lambda_q(1\otimes\mathsf{k}\mathsf{x}\otimes\overline{\mathsf{k}}\mathsf{y})\right).$$

Now, by Lemma 2.6, $\mathrm{Exp}_q\left(\lambda_q(\mathsf{kx}\otimes 1\otimes \overline{\mathsf{k}}\mathsf{y})\right)\mathsf{h}_1 = \mathsf{h}_1\mathrm{Exp}_q\left(\lambda_q(\mathsf{kx}\otimes \mathsf{k}^2\otimes \overline{\mathsf{k}}\mathsf{y})\right)$ and since h_1 and h_2 commute, it therefore follows that

$$\mathsf{R}_{13}\cdot\mathsf{R}_{23} = e^{\frac{h}{4}(\mathsf{h}\otimes 1\otimes \mathsf{h}+1\otimes\mathsf{h}\otimes\mathsf{h})}\cdot \mathrm{Exp}_q\left(\lambda_q(\mathsf{kx}\otimes \mathsf{k}^2\otimes \overline{\mathsf{k}}\mathsf{y})\right)\cdot \mathrm{Exp}_q\left(\lambda_q(1\otimes\mathsf{kx}\otimes \overline{\mathsf{k}}\mathsf{y})\right).$$

Let $C := \mathsf{kx}\otimes\mathsf{k}^2\otimes\overline{\mathsf{k}}\mathsf{y}$ and $D := 1\otimes\mathsf{kx}\otimes\overline{\mathsf{k}}\mathsf{y}$. Then, from (iv) of Lemma 2.5, we have $CD = \left(\mathsf{kx}\otimes\mathsf{k}^3\mathsf{x}\otimes(\overline{\mathsf{k}}\mathsf{y})^2\right)$ and $DC = \overline{q}^2\left(\mathsf{kx}\otimes\mathsf{k}^3\mathsf{x}\otimes(\overline{\mathsf{k}}\mathsf{y})^2\right)$ whence $CD = q^2DC$. Then, from Lemma 4.6,

$$\mathsf{R}_{13}\cdot\mathsf{R}_{23} = e^{\frac{h}{4}(\mathsf{h}\otimes 1\otimes \mathsf{h}+1\otimes\mathsf{h}\otimes\mathsf{h})}\cdot \mathrm{Exp}_q\left(\lambda_q(\mathsf{kx}\otimes \mathsf{k}^2\otimes \overline{\mathsf{k}}\mathsf{y}+1\otimes\mathsf{kx}\otimes \overline{\mathsf{k}}\mathsf{y})\right).$$

It follows from (29) that this is equal to $(\Delta\otimes\mathrm{id}_{\mathcal{A}})(\mathsf{R})$, so the Condition is satisfied.

5.2.4. *Condition* (iii) *of Definition 5.2.* It may be shown similarly that this condition is also satisfied.

We have therefore proved the following:

Theorem 5.3 (An R-morphism)**.** *A universal* R*-morphism for* $\mathcal{U}_h(\mathfrak{sl}_2)$ *is*

$$\mathsf{R} = e^{\frac{h}{4}\mathsf{h}\otimes\mathsf{h}}\sum_{n\geq 0}\frac{(q-\overline{q})^n}{[n]!}q^{\frac{1}{2}n(n-3)}\,(\mathsf{kx})^n\otimes(\overline{\mathsf{k}}\mathsf{y})^n.$$

Remark 5.4. We make three comments on the derivation of R.

(A) Condition (i) of Definition 5.2 was required only for the generators h and x to fully define the unknown series $a_{k,r,s,m,n}(q)$, in Ansatz (23) and thence R. It was then confirmed that this Condition also held for the other generator y, and that Conditions (ii) and (iii) also held. This suggests that Ansatz (23) is a particularly restrictive one, in that Conditions (ii) and (iii) were forced.

(B) The salient aspects of the derivation of R are **(i)** the appearance of the commutator $[\mathsf{x},\mathsf{y}^n]$ in (26), **(ii)** the fact that R may be expressed succinctly in terms of the quantum exponential function in (28). The factorization property given in Lemma 4.6 for the quantum exponental is crucial in showing that Conditions (ii) and (iii) of Definition 5.2 hold.

(C) The above argument may be used to show that if the form for R proposed in (22) is an R-morphism then it is the one given in Theorem 5.3.

6. Ribbon Hopf algebra structure on $\mathcal{U}_h(\mathfrak{sl}_2)$

Ribbon Hopf algebras were introduced by [RT90] in order to construct a polynomial invariant for any framed link. Briefly, ribbon Hopf algebras are quasi-triangular Hopf algebras in which a square root of a special element exists and satisfies some desirable properties. In the context of constructing link invariants, the existence of this square root allows one to untangle an even number of trivial twists.

6.1. **Ribbon Hopf Algebras.** Not only does the existence of a universal R-morphism in a quasi-triangular Hopf algebra $\mathfrak{A}$ yield solutions to the Yang-Baxter equation, but it also endows $\mathfrak{A}$ with rich structure. Indeed, one important consequence of having an R-morphism is that the action of S^2 the square of the antipode is completely determined by an element, conventionally denoted by $\mathbf{u}$, related to R.

Definition 6.1 (The element $\mathbf{u}$)**.** *The element* $\mathbf{u}$ *is defined by*

$$\mathbf{u} := \sum_i \mathsf{S}(\beta_i) \cdot \alpha_i$$

where the $\alpha_i \otimes \beta_i$ *the summands in* R*, as in* (iv) *of Definition 5.2.*

The following three properties were first discovered and proved by Drinfel'd in [Dri89]. Detailed proofs of these can be found in [Kas95, *pp.*180–184].

Lemma 6.2. *Let* $(\mathfrak{A}, \mathsf{R})$ *be a quasi-triangular Hopf algebra. Then the element* $\mathbf{u}$ *has the following properties:*

(i) *For all* $x \in \mathfrak{A}$*, the square of the antipode* $\mathsf{S}^2 : \mathfrak{A} \to \mathfrak{A}$ *acts by conjugation* $\mathsf{S}^2(x) = \mathbf{u}\, x\, \mathbf{u}^{-1}$*;*
(ii) $\Delta(\mathbf{u}) = (\tau(\mathsf{R}) \cdot \mathsf{R})^{-1}(\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \otimes \mathbf{u})(\tau(\mathsf{R}) \cdot \mathsf{R})^{-1}$.
(iii) $\mathbf{u}$ *is invertible, with*

$$\mathbf{u}^{-1} = \sum \mathsf{S}^{-1}(\overline{\beta}_i)\overline{\alpha}_i, \quad \textit{where } R^{-1} = \sum_i \overline{\alpha}_i \otimes \overline{\beta}_i.$$

Definition 6.3 (Ribbon Hopf Algebra)**.** *A* Ribbon Hopf Algebra *is a pair* $(\mathfrak{A}, \mathsf{R}, \mathbf{v})$ *where* $(\mathfrak{A}, \mathsf{R})$ *is a quasi-triangular Hopf algebra and* $\mathbf{v} \in \mathfrak{A}$ *satisfies:*

(i) $\mathbf{v}$ *is central and invertible.*
(ii) $\mathbf{v}^2 = \mathsf{S}(\mathbf{u}) \cdot \mathbf{u}$*,*
(iii) $\mathsf{S}(\mathbf{v}) = \mathbf{v}$*,*
(iv) $\varepsilon(\mathbf{v}) = 1_{\mathcal{A}}$*,*
(v) $\Delta(\mathbf{v}) = (\tau(\mathsf{R}) \cdot \mathsf{R})^{-1} \cdot (\mathbf{v} \otimes \mathbf{v})$.

Lemma 6.4. *If* $(\mathfrak{A}, \mathsf{R}, \mathbf{v})$ *is a ribbon Hopf algebra, then* $\mathbf{v}$ *is central and invertible.*

Proof. From Property (iii) of Lemma 6.2, $\mathbf{u}$ is invertible and, since $\mathcal{A}$ is a quasi-triangular Hopf algebra, S is invertible from Definition 5.2. Thus $\mathbf{v}^2$ is invertible by (ii) of Definition 6.3. Let $\mathbf{w} := (\mathbf{v}^2)^{-1}$. Then $\mathbf{v}(\mathbf{v}\mathbf{w}) = 1_{\mathcal{A}}$. But $\mathbf{v}$ is central by Part (i), so $(\mathbf{v}\mathbf{w})\mathbf{v} = 1_{\mathcal{A}}$. Thus $\mathbf{v}$ is invertible. □

6.2. **Constructing the element v.** We shall now construct an explicit element $\mathbf{v}$ so that $(\mathcal{U}_h(\mathfrak{sl}_2), \mathsf{R}, \mathbf{v})$, where R given by Theorem 5.3, is a Ribbon Hopf algebra. The element $\mathbf{u}$ is given in terms of R by the second and third expressions in (2). An explicit expression for $\mathbf{u}$ will be obtained first through these. The first expression in (2), which we shall take to be the defining property of $\mathbf{v}$, will then be used to obtain an expression for $\mathbf{v}$, and it will then be confirmed that this satisfies the appropriate conditions.

6.2.1. *The element* $\mathbf{u}$. The following is an explicit expression for $\mathbf{u}$.

Lemma 6.5. $\mathbf{u} = e^{-\frac{h}{4}\mathsf{h}^2} \sum_{n \geq 0} (-1)^n \dfrac{(q - \overline{q})^n}{[n]!} \, \overline{q}^{\frac{1}{2}n(n+3)} \, \overline{\mathsf{k}}^{2n} \mathsf{y}^n \mathsf{x}^n.$

Proof. From Theorem 5.3 and (iii) of Lemma 2.5,

$$\mathsf{R} = \sum_{m,n \geq 0} \frac{h^m}{4^m m!} \, c_n \, \overline{q}^{n(n-1)} \, (\mathsf{h}^m \mathsf{k}^n \mathsf{x}^n) \otimes \left(\mathsf{h}^m \overline{\mathsf{k}}^n \mathsf{y}^n\right)$$

where

$$c_n := \frac{(q-\overline{q})^n}{[n]!}\overline{q}^{\frac{1}{2}n(n-3)} \tag{30}$$

and so, from (2),

$$\mathbf{u} = \sum_{m,n\geq 0} \frac{h^m}{4^m m!}\, c_n\, \overline{q}^{n(n-1)} \left(\mathsf{S}(\mathsf{h}^m\overline{\mathsf{k}}^n\mathsf{y}^n)\right)\cdot(\mathsf{h}^m\mathsf{k}^n\mathsf{x}^n).$$

To straighten the summand, we note that $\mathsf{S}(\overline{\mathsf{k}}) = \mathsf{k}$ from (21), and that h and k commute. Using the action of S defined in Theorem 5.1, we have

$$\begin{aligned}\left(\mathsf{S}(\mathsf{h}^m\overline{\mathsf{k}}^n\mathsf{y}^n)\right)\cdot(\mathsf{h}^m\mathsf{k}^n\mathsf{x}^n) &= (-1)^{m+n}\overline{q}^n\mathsf{y}^n\mathsf{h}^{2m}\mathsf{k}^{2n}\mathsf{x}^n\\ &= (-1)^{m+n}\overline{q}^n q^{2n^2}(\mathsf{h}+2n)^{2m}\mathsf{k}^{2n}\mathsf{y}^n\mathsf{x}^n\end{aligned}$$

from (vi) and (vii) of Lemma (2.5), so

$$\mathbf{u} = \sum_{n\geq 0}(-1)^n c_n\, q^{n^2} e^{-\frac{h}{4}(\mathsf{h}+2n)^2}\mathsf{k}^{2n}\mathsf{y}^n\mathsf{x}^n.$$

The result then follows by observing that $e^{-\frac{h}{4}(\mathsf{h}+2n)^2} = \overline{q}^{2n^2}e^{-\frac{h}{4}\mathsf{h}^2}\cdot\overline{\mathsf{k}}^{4n}$. □

6.2.2. *The antipode acting on* $\mathbf{u}$. To construct an expression for $\mathbf{v}$ from (2), we will need an explicit expression for $\mathsf{S}(\mathbf{u})$. Happily, S acts quite simply on $\mathbf{u}$ in $\mathcal{U}_h(\mathfrak{sl}_2)$.

Lemma 6.6. $\mathsf{S}(\mathbf{u}) = \overline{\mathsf{k}}^4\mathbf{u}$.

Proof. Let

$$d_n := (-1)^n\frac{(q-\overline{q})^n}{[n]!}\,\overline{q}^{\frac{1}{2}n(n+3)}.$$

Then

$$\overline{\mathsf{k}}^4\mathbf{u} = e^{-\frac{h}{4}\mathsf{h}^2}\sum_{n\geq 0} d_n\,\overline{\mathsf{k}}^{2(n+2)}\mathsf{y}^n\mathsf{x}^n$$

since, from Lemma 6.5, $\mathbf{u} = e^{-\frac{h}{4}\mathsf{h}^2}\sum_{n\geq 0} d_n\,\overline{\mathsf{k}}^{2n}\mathsf{y}^n\mathsf{x}^n$. From the definition of S in Theorem 5.1 and from its action on k given in (21),

$$\mathsf{S}(\mathbf{u}) = \sum_{n\geq 0} d_n\,\mathsf{x}^n\mathsf{y}^n\mathsf{k}^{2n}e^{-\frac{h}{4}\mathsf{h}^2}.$$

This may be straightened by noting that k^{2n} and $e^{-\frac{h}{4}\mathsf{h}^2}$ commute and that, by (ii) of Lemma 2.5, both of these commute with $\mathsf{x}^n\mathsf{y}^n$, to obtain

$$\mathsf{S}(\mathbf{u}) = e^{-\frac{h}{4}\mathsf{h}^2}\sum_{n\geq 0} d_n\,\mathsf{k}^{2n}\mathsf{x}^n\mathsf{y}^n.$$

Using Theorem 2.3, we can establish the equality of $\overline{\mathsf{k}}^4\mathbf{u}$ and $\mathsf{S}(\mathbf{u})$ by comparing coefficients. First we compare the coefficients of $\mathsf{y}^n\mathsf{x}^n$ with the aid of Lemma 3.2. Doing so, the assertion of the Lemma is therefore equivalent to the identity

$$d_n\,\overline{\mathsf{k}}^{2(n+2)} = \sum_{m\geq n} d_m\,\frac{[m]!^2}{[n]!^2}\,\mathsf{k}^{2m}\begin{bmatrix}\mathsf{h}\\ m-n\end{bmatrix}$$

and so, by changing the index of summation to $s := m - n$, to showing that

$$\overline{\mathsf{k}}^{4(n+1)} = A_n \qquad \text{where} \qquad A_n := \sum_{s\geq 0} \frac{d_{n+s}}{d_n}\, \frac{[n+s]!^2}{[n]!^2}\, \mathsf{k}^{2s} \begin{bmatrix} \mathsf{h} \\ s \end{bmatrix}.$$

It is in this form that we shall prove the lemma.

From (5) and (8),

$$\begin{bmatrix} \mathsf{h} \\ s \end{bmatrix} = \frac{1}{[s]!} \prod_{r=0}^{s-1} \frac{\overline{q}^r \mathsf{k}^2 - q^r \overline{\mathsf{k}}^2}{q - \overline{q}} = \frac{1}{[s]!}\, \frac{\overline{q}^{\frac{1}{2}s(s-1)}}{(q-\overline{q})^s}\, \overline{\mathsf{k}}^{2s} \prod_{r=0}^{s-1} \left(\mathsf{k}^4 - q^{2r}\right)$$

so

$$A_n = \sum_{s\geq 0} (-1)^s\, \overline{q}^{s(n+s+1)} \begin{bmatrix} n+s \\ s \end{bmatrix} \prod_{r=0}^{s-1} \left(\mathsf{k}^4 - q^{2r}\right).$$

We shall now transform the quantum binomial coefficient into a q-binomial coefficient through (19) to obtain

$$A_n = \sum_{s\geq 0} (-1)^s \overline{q}^{s(2n+s+1)} \binom{n+s}{s}_{q^2} \prod_{r=0}^{s-1} \left(\mathsf{k}^4 - q^{2r}\right).$$

The q-binomial coefficient may be expressed as a negative q-binomial coefficient through

$$\binom{n+s}{s}_{q^2} = (-1)^s \binom{-(n+1)}{s}_{q^2} q^{s(2n+s+1)}.$$

Then

$$A_n = \sum_{s\geq 0} \binom{-(n+1)}{s}_{q^2} \prod_{r=0}^{s-1} \left(\mathsf{k}^4 - q^{2r}\right).$$

Using the expressions $q^2 = e^h$ and $\mathsf{k}^4 = e^{h\mathsf{h}}$ for these elements in $\mathcal{U}_h(\mathfrak{sl}_2)$, we have

$$A_n = e^{-(n+1)h\mathsf{h}} = \overline{\mathsf{k}}^{4(n+1)}$$

via Lemma 4.5. This completes the proof. □

6.2.3. *Construction of* $\mathbf{v}$. We begin by constructing a putative expression for $\mathbf{v}$ from (2). Lemma 6.6, we know that $\mathbf{v}^2 = \overline{\mathsf{k}}^4 \mathbf{u}^2$. But, by Lemma 6.2, $\mathbf{u}$ is almost-central so, in particular, $\mathbf{u}\overline{\mathsf{k}} = \mathsf{S}^2(\overline{\mathsf{k}})\mathbf{u}$. But, by (21), $\mathsf{S}(\overline{\mathsf{k}}) = \mathsf{k}$ and $\mathsf{S}(\mathsf{k}) = \overline{\mathsf{k}}$ so $\mathbf{u}\overline{\mathsf{k}} = \overline{\mathsf{k}}\mathbf{u}$. Thus

(i) $\mathbf{u}$ and $\overline{\mathsf{k}}$ commute; (ii) $\mathbf{v} = \overline{\mathsf{k}}^2 \mathbf{u}$.

Therefore, from Lemma 6.5, we have the putative expression

$$e^{-\frac{h}{4}\mathsf{h}^2} \cdot \sum_{n\geq 0} \frac{(\overline{q}-q)^n}{[n]!}\, \overline{q}^{\frac{1}{2}n(n+3)} \overline{\mathsf{k}}^{2(n+1)} \mathsf{y}^n \mathsf{x}^n$$

for $\mathbf{v}$. We show next that this expression satisfies the appropriate conditions.

Theorem 6.7. $(\mathcal{U}_h(\mathfrak{sl}_2), \mathsf{R}, \mathbf{v})$ *is a Ribbon Hopf algebra, where*

$$\mathbf{v} := e^{-\frac{h}{4}\mathsf{h}^2} \cdot \sum_{n\geq 0} \frac{(\overline{q}-q)^n}{[n]!}\, \overline{q}^{\frac{1}{2}n(n+3)} \overline{\mathsf{k}}^{2(n+1)} \mathsf{y}^n \mathsf{x}^n,$$

$$\mathsf{R} := e^{\frac{h}{4}\,\mathsf{h}\otimes\mathsf{h}} \cdot \sum_{n\geq 0} \frac{(q-\overline{q})^n}{[n]!} q^{\frac{1}{2}n(n-3)}\, (\mathsf{k}\mathsf{x})^n \otimes (\overline{\mathsf{k}}\mathsf{y})^n.$$

Proof. We have already shown that $(\mathcal{U}_h(\mathfrak{sl}_2), \mathsf{R})$ is a quasi-triangular Hopf algebra. To show that $(\mathcal{U}_h(\mathfrak{sl}_2), \mathsf{R}, \mathbf{v})$ is a Ribbon Hopf algebra, it remains only to check that each of the following conditions from Definition 6.3 are satisfied:

$$\begin{cases} 1) \quad \mathbf{v} \text{ is central and invertible,} & 2) \quad \mathbf{v}^2 = \mathsf{S}(\mathbf{u}) \cdot \mathbf{u}, \\ 3) \quad \mathsf{S}(\mathbf{v}) = \mathbf{v}, & 4) \quad \varepsilon(\mathbf{v}) = 1_{\mathcal{A}}, \\ 5) \quad \Delta(\mathbf{v}) = (\tau(\mathsf{R}) \cdot \mathsf{R})^{-1} \cdot (\mathbf{v} \otimes \mathbf{v}). & \end{cases}$$

For Condition 1: That $\mathbf{v}$ is invertible was established in Lemma 6.4. To show centrality of $\mathbf{v}$, it is sufficient to show $\mathbf{v}$ commutes with the generators h, x and y.

The generator h: From (ii) of Lemma 2.5, it is clear that h and $\mathsf{y}^n\mathsf{x}^n$ commute (and k commutes with h), so h commutes with $\mathbf{v}$.

The generator x: Let T_n denote the term $e^{-\frac{h}{4}\mathsf{h}^2}\overline{\mathsf{k}}^{2(n+1)}\mathsf{y}^n\mathsf{x}^n$ that appears as the contribution of the generators to the general term of $\mathbf{v}$. We consider $\mathsf{x}T_n$. From (ii) of Lemma 2.5, $\mathsf{x}e^{-\frac{h}{4}\mathsf{h}^2} = \overline{q}^2 e^{-\frac{h}{4}\mathsf{h}^2}\mathsf{k}^4\mathsf{x}$. Thus,

$$\begin{aligned} \mathsf{x}T_n &= e^{-\frac{h}{4}\mathsf{h}^2} q^{2n}\overline{\mathsf{k}}^{2(n-1)}(\mathsf{x}\,\mathsf{y}^n)\mathsf{x}^n && \text{((iii) of Lemma 2.5)} \\ &= e^{-\frac{h}{4}\mathsf{h}^2} q^{2n}\overline{\mathsf{k}}^{2(n-1)}\left(\mathsf{y}^n\mathsf{x}^{n+1} + [n]\,[\mathsf{h}+n-1]\,\mathsf{y}^{n-1}\mathsf{x}^n\right) && \text{((v) of Lemma 2.5)} \end{aligned}$$

and then, using the notation from (30),

$$\begin{aligned} \mathsf{x}\mathbf{v} &= e^{-\frac{h}{4}\mathsf{h}^2}\sum_{n\geq 0} c_n\, q^{2n}\overline{\mathsf{k}}^{2(n-1)}\left(\mathsf{y}^n\mathsf{x}^{n+1} + [n]\,[\mathsf{h}+n-1]\,\mathsf{y}^{n-1}\mathsf{x}^n\right) \\ &= e^{-\frac{h}{4}\mathsf{h}^2}\sum_{n\geq 1} q^{2n-2}\overline{\mathsf{k}}^{2(n-2)}\Big(c_{n-1} + c_n\, q^2\overline{\mathsf{k}}^2[n]\,[\mathsf{h}+n-1]\Big)\,\mathsf{y}^{n-1}\mathsf{x}^n \end{aligned}$$

by shifting the summation index for the term containing $\mathsf{y}^n\mathsf{x}^{n+1}$ by one. The bracketed term is equal to $\frac{(\overline{q}-q)^{n-1}}{[n-1]!}\,\overline{q}^{n-1}\overline{\mathsf{k}}^4\overline{q}^{\frac{1}{2}(n^2+3n-4)}$. Therefore

$$\mathsf{x}\mathbf{v} = e^{-\frac{h}{4}\mathsf{h}^2}\sum_{n\geq 1}\overline{q}^{\frac{1}{2}(n^2+n-2)}\frac{(\overline{q}-q)^{n-1}}{[n-1]!}\,\overline{\mathsf{k}}^{2n}\mathsf{y}^{n-1}\mathsf{x}^n.$$

The right hand side is readily seen to be equal to $\mathbf{v}\mathsf{x}$ by shifting the summation index for the expression for $\mathbf{v}$ to start at 0.

The generator y: This is proved similarly.

For Condition 2: This was used to construct the expression for $\mathbf{v}$, and therefore holds.

For Conditions 3 and 4: These are immediate from the actions of S and ε stated Theorem 5.1.

For Condition 5: Using the expression for $\mathbf{v}$ in terms of $\mathbf{u}$ and multiplying both sides by $\tau(\mathsf{R}) \cdot \mathsf{R}$, the required identity is equivalent to

$$(\tau(\mathsf{R}) \cdot \mathsf{R}) \cdot (\overline{\mathsf{k}}^2 \otimes \overline{\mathsf{k}}^2) \cdot \Delta(\mathbf{u}) = (\overline{\mathsf{k}}^2 \otimes \overline{\mathsf{k}}^2) \cdot (\mathbf{u} \otimes \mathbf{u}).$$

Since each term of R (and thus $\tau(\mathsf{R})$) commutes with $\overline{\mathsf{k}}^2 \otimes \overline{\mathsf{k}}^2$, it is sufficient to show

$$(\tau(\mathsf{R}) \cdot \mathsf{R}) \cdot \Delta(\mathbf{u}) = \mathbf{u} \otimes \mathbf{u}.$$

This is (ii) of Lemma 6.2. □

6.3. **Knot Invariants and the q-analogue $\mathcal{U}_q(\mathfrak{sl}_2)$ of $\mathcal{U}_h(\mathfrak{sl}_2)$.** Given a Ribbon Hopf algebra $\mathfrak{A}$, an isotopy $\mathfrak{A}$-valued invariant of (framed) links can be constructed. Recall that *links* are embeddings of a disjoint union of circles $\mathbb{S}^1$ into $\mathbb{R}^3$. Such invariant can be transformed into a more familiar object, say a scalar quantity or a polynomial, by taking a representation of $\mathfrak{A}$ and taking a trace of the resulting operator in the representation. For example, the Jones polynomial of a knot may be recovered by taking the 2-dimensional irreducible representation for $\mathcal{U}_h(\mathfrak{sl}_2)$. Taking higher dimensional representations will yield the HOMFLY polynomial. For more, see [CP94, Oht02].

As representations of our algebra give rise to invariants of links, it is desirable that our algebras have a rich representation theory. Unfortunately, the ribbon Hopf algebra $\mathcal{U}_h(\mathfrak{sl}_2)$, though more complicated overall in comparison to $\mathcal{U}(\mathfrak{sl}_2)$, has an algebra structure essentially equivalent to that of the usual universal enveloping algebra. More precisely, recall that $\mathcal{U}_h(\mathfrak{sl}_2) \cong \mathcal{U}(\mathfrak{sl}_2)[[h]]$. Thus, as algebras, $\mathcal{U}_h(\mathfrak{sl}_2)$ can be obtained from $\mathcal{U}(\mathfrak{sl}_2)$ by extending scalars from $\mathbb{C}$ to $\mathbb{C}[[h]]$. Consequently, the representation theory of $\mathcal{U}_h(\mathfrak{sl}_2)$ is not as useful for our purposes as it could be.

Representations of $\mathcal{U}_h(\mathfrak{sl}_2)$ would be more subtle if h could be specialized to particular complex numbers. Being a formal parameter, however, h cannot generally be set to anything other than 0. The issue is that $\mathcal{U}_h(\mathfrak{sl}_2)$ is too "big": the algebra contains many elements which would yield divergent sums if h were set to a nonzero quantity. Thus a "smaller" algebra, $\mathcal{U}_q(\mathfrak{sl}_2)$, in which the parameter q is an element of the underlying field of definition, is often considered.

Intuitively, $\mathcal{U}_h(\mathfrak{sl}_2)$ is the "subalgebra" of $\mathcal{U}_h(\mathfrak{sl}_2)$ generated by the elements x, y, k and $\overline{\mathsf{k}}$, over the ring of convergent power series in h, where k is to be regarded as an entirely new symbol. That is, in this algebra k is *not* determined by h, for the latter is not in this algebra. Since the inverse of k will be needed, it has to be added as a generator. More precisely, we have the following.

Definition 6.8. *The q-*analogue algebra $\mathcal{U}_q(\mathfrak{sl}_2)$ *of $\mathcal{U}(\mathfrak{sl}_2)$ is the algebra over the field $\mathbb{C}(q)$ of rational functions in q generated by elements x, y, k and $\overline{\mathsf{k}}$, subject to the relations:*

$$\mathsf{k}\overline{\mathsf{k}} = 1 = \overline{\mathsf{k}}\mathsf{k}, \quad \mathsf{k}\mathsf{x} = q\mathsf{x}\mathsf{k}, \quad \mathsf{k}\mathsf{y} = \overline{q}\mathsf{y}\mathsf{k}, \quad [\mathsf{x},\mathsf{y}] := \frac{\mathsf{k}^2 - \overline{\mathsf{k}}^2}{q - \overline{q}}. \tag{31}$$

Remark 6.9. Note that we have not introduced any topology on $\mathcal{U}_q(\mathfrak{sl}_2)$. Indeed, we take $\mathcal{U}_q(\mathfrak{sl}_2)$ to be a usual algebra over $\mathbb{C}(q)$ and thus we only consider *finite* sums of elements in $\mathcal{U}_q(\mathfrak{sl}_2)$. In particular, the expressions in $\mathcal{U}_h(\mathfrak{sl}_2)$ for R in Theorem 5.3, for $\mathbf{u}$ in Lemma 6.5 and for $\mathbf{v}$ Lemma 6.7 do not *a priori* make sense in $\mathcal{U}_q(\mathfrak{sl}_2)$. With some care, one can reconcile each structure with $\mathcal{U}_q(\mathfrak{sl}_2)$, thereby equipping this algebra with a Ribbon Hopf algebra structure similar to that of $\mathcal{U}_h(\mathfrak{sl}_2)$. As we shall see, however, a different Ribbon Hopf algebra structure, which makes sense within $\mathcal{U}_q(\mathfrak{sl}_2)$, can be constructed when q is taken to be a root of unity,

6.3.1. *Relating $\mathcal{U}_h(\mathfrak{sl}_2)$ and $\mathcal{U}_q(\mathfrak{sl}_2)$.* Let us first make precise what we mean by the statement that $\mathcal{U}_q(\mathfrak{sl}_2)$ can be viewed as a subalgebra of $\mathcal{U}_h(\mathfrak{sl}_2)$. For further discussion, see, for example, [Kas95, p.413].

Lemma 6.10. *There exists an embedding $\iota : \mathcal{U}_q(\mathfrak{sl}_2) \to \mathcal{U}_h(\mathfrak{sl}_2)$ of algebras over $\mathbb{C}$ such that*

$$\iota(\mathsf{x}) = \mathsf{x}, \quad \iota(\mathsf{y}) = \mathsf{y}, \quad \iota(q) = e^{\frac{h}{2}}, \quad \iota(\mathsf{k}) = e^{\frac{h}{4}\mathsf{h}}.$$

The only thing that needs to be checked for the existence statement is whether the relation $\mathsf{kx} = q\mathsf{xk}$ (and the analogous relation for y) holds in $\mathcal{U}_h(\mathfrak{sl}_2)$. But these hold from (iii) of Lemma 2.5, thereby yielding ι.

6.3.2. *Hopf Structure on* $\mathcal{U}_q(\mathfrak{sl}_2)$. A Hopf structure on $\mathcal{U}_q(\mathfrak{sl}_2)$ can be deduced from the structure on $\mathcal{U}_h(\mathfrak{sl}_2)$ given in Theorem 5.1, together with the embedding from Lemma 6.10.

Theorem 6.11. *The q-analogue $\mathcal{U}_q(\mathfrak{sl}_2)$ of the algebra $\mathcal{U}(\mathfrak{sl}_2)$ is a Hopf algebra with unit $\eta(1) := 1$, co-multiplication Δ, antipode S, co-unit ε and unit defined on generators by*

$$\begin{aligned} \Delta &: \mathsf{x} \mapsto \mathsf{x} \otimes \mathsf{k} + \overline{\mathsf{k}} \otimes \mathsf{x}, & \mathsf{S} &: \mathsf{x} \mapsto -q\mathsf{x}, & \varepsilon &: \mathsf{x} \mapsto 0, \\ & \mathsf{y} \mapsto \mathsf{y} \otimes \mathsf{k} + \overline{\mathsf{k}} \otimes \mathsf{y}, & & \mathsf{y} \mapsto -\overline{q}\mathsf{y}, & & \mathsf{y} \mapsto 0, \\ & \mathsf{k} \mapsto \mathsf{k} \otimes \mathsf{k}, & & \mathsf{k} \mapsto \overline{\mathsf{k}}, & & \mathsf{k} \mapsto 1, \\ & \overline{\mathsf{k}} \mapsto \overline{\mathsf{k}} \otimes \overline{\mathsf{k}}, & & \overline{\mathsf{k}} \mapsto \mathsf{k}, & & \overline{\mathsf{k}} \mapsto 1, \end{aligned}$$

where Δ and ε are extended as algebra morphisms, and m and η are extend as co-algebra morphisms.

6.3.3. *Ribbon Structure of $\mathcal{U}_q(\mathfrak{sl}_2)$ For q a Root of Unity.* Suppose now that the parameter q in $\mathcal{U}_q(\mathfrak{sl}_2)$ is taken to be a root of unity, say $q := \exp(\frac{1}{\ell}\pi i)$. Then, because of a factorization property of the q-binomial coefficient, $\mathcal{U}_q(\mathfrak{sl}_2)$ essentially factorizes into the product of a finite-dimensional quotient $\mathcal{U}_q$ and a subalgebra. The quotient $\mathcal{U}_q$ is obtained from $\mathcal{U}_q(\mathfrak{sl}_2)$ by imposing the additional relations

$$\mathsf{k}^{4\ell} = 1, \quad \text{and} \quad \mathsf{x}^\ell = \mathsf{y}^\ell = 0.$$

A quasi-triangular structure on $\mathcal{U}_q$ is induced from that of $\mathcal{U}_q(\mathfrak{sl}_2)$. Explicitly, the R-morphism can be written

$$\mathsf{R} := \frac{1}{4\ell} \sum_{p=0}^{4\ell-1} \sum_{n=0}^{\ell-1} \frac{(q-\overline{q})^n}{[n]!} q^{\frac{1}{2}n(n-3)+\frac{1}{2}p} \left(\prod_{\substack{r=0 \\ r\neq p}}^{4\ell-1} (\mathsf{k} - q^{\frac{1}{2}r}) \right) (\mathsf{kx})^n \otimes \mathsf{k}^p (\overline{\mathsf{k}}\mathsf{y})^n.$$

With this R-morphism, the element $\mathbf{u}$ can be written as

$$\mathbf{u} := \frac{1}{4\ell} \sum_{p=0}^{4\ell-1} \sum_{n=0}^{\ell-1} (-1)^n \frac{(q-\overline{q})^n}{[n]!} \overline{q}^{\frac{1}{2}n(n+3)+\frac{1}{2}p(p-1)} \overline{\mathsf{k}}^{2n} \left(\prod_{\substack{r=0 \\ r\neq p}}^{4\ell-1} (q^n \mathsf{k} - q^{\frac{1}{2}r}) \right) \mathsf{y}^n \mathsf{x}^n.$$

A ribbon structure is now induced on $\mathcal{U}_q$ by taking $\mathbf{v} = \overline{\mathsf{k}}^2 \mathbf{u}$, very much as in Theorem 6.7.

Not only is the algebra $\mathcal{U}_q$ finite-dimensional, it has a very rich representation theory. By considering a special family of representations of $\mathcal{U}_q$, Reshetikhin and Turaev were able to obtain in [RT91] a new class of invariants for 3-manifolds. By associating to a link its complement, these can be viewed as invariants of links.

7. Extension to $\mathcal{U}_h(\mathfrak{sl}_3)$

The definition of quantized universal enveloping algebras can be applied in great generality. Indeed, the definition of $\mathcal{U}_h(\mathfrak{sl}_2)$ presented in Section 2.3 can be modified to define a quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{g})$ for any complex semisimple Lie algebra $\mathfrak{g}$. For general definitions, see, for example [Ram98].

In the next two sections, we will discuss how the techniques developed in the $\mathcal{U}_h(\mathfrak{sl}_2)$ setting may be applied to derive an R-morphism for the quantized universal enveloping algebras of $\mathfrak{sl}_{n+1}$, $n \geq 2$. Here, we shall first consider the case of $\mathcal{U}_h(\mathfrak{sl}_3)$ in detail, as it serves as a model for all higher dimensional cases. In the following section, we will carry out the construction of R for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ for general n.

The results in this section are not new. To our knowledge, the first explicit description of an R-morphism for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ was given by [Ros89]. See also [Bur90]. An explicit R-morphism for $\mathcal{U}_h(\mathfrak{g})$, $\mathfrak{g}$ a general complex semisimple Lie algebra, was given in [LS91, KR90]. Our discussion here differs from these past works in the methods and techniques used, and in the perspective with which we view the objects in hand. Indeed, our methods are much more direct in comparison to previous accounts.

7.1. **Quantized Universal Enveloping Algebra $\mathcal{U}_h(\mathfrak{sl}_3)$.**

The definition of $\mathcal{U}_h(\mathfrak{sl}_3)$ is very similar to that of $\mathcal{U}_h(\mathfrak{sl}_2)$ in Section 2.3, except that crucially, a new type of relation needs is now present.

Definition 7.1. *The* quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{sl}_3)$ *for* $\mathfrak{sl}_3$ *is the associative* $\mathbb{C}[[h]]$*-algebra generated by two sets of* $\mathfrak{sl}_2$ *triples,* $(\mathsf{x}_i, \mathsf{h}_i, \mathsf{y}_i)$*,* $i = 1, 2$*, subject to the commutator relations* $[\mathsf{h}_i, \mathsf{h}_j] = 0$*,*

$$[\mathsf{h}_i, \mathsf{x}_j] = \begin{cases} 2\mathsf{x}_i & i = j, \\ -\mathsf{x}_j & i \neq j, \end{cases} \qquad [\mathsf{h}_i, \mathsf{y}_j] = \begin{cases} -2\mathsf{y}_i & i = j, \\ \mathsf{y}_j & i \neq j, \end{cases} \qquad [\mathsf{x}_i, \mathsf{y}_j] = \delta_{ij} \frac{\mathsf{k}_i - \overline{\mathsf{k}}_i}{q - \overline{q}},$$

where $\mathsf{k}_i := e^{\frac{h}{4}\mathsf{h}_i}$*, and the* q-Serre relations*,*

$$\mathsf{x}_1^2\mathsf{x}_2 - (q + \overline{q})\mathsf{x}_1\mathsf{x}_2\mathsf{x}_1 + \mathsf{x}_2\mathsf{x}_1^2 = 0, \qquad \mathsf{x}_2^2\mathsf{x}_1 - (q + \overline{q})\mathsf{x}_2\mathsf{x}_1\mathsf{x}_2 + \mathsf{x}_1\mathsf{x}_2^2 = 0, \tag{32}$$

with corresponding equations for the y_i*.*

A Hopf algebra structure on $\mathcal{U}_h(\mathfrak{sl}_3)$ can be defined on the generators $(\mathsf{x}_i, \mathsf{h}_i, \mathsf{y}_i)$ by taking the structure maps as given in Theorem 5.1 for $\mathcal{U}_h(\mathfrak{sl}_2)$ for each triple.

7.1.1. *Simplifying the q-Serre Relations.* When attempting to apply straightening techniques, the q-Serre relations, which are cubic relations amongst generators, are difficult to handle. Indeed, the straightening formalism relies on being able to swap pairs of generators *via* a quadratic relation. From this perspective, it is natural to hope that (32) may be replaced by some set commutation relations. Rearranging the first equation in (32), we have that

$$\begin{aligned} 0 &= \mathsf{x}_1^2\mathsf{x}_2 - (q + \overline{q})\mathsf{x}_1\mathsf{x}_2\mathsf{x}_1 + \mathsf{x}_2\mathsf{x}_1^2 \\ &= \mathsf{x}_1(\mathsf{x}_1\mathsf{x}_2 - \overline{q}\mathsf{x}_2\mathsf{x}_1) - (q\mathsf{x}_1\mathsf{x}_2 - \mathsf{x}_2\mathsf{x}_1)\mathsf{x}_1 \\ &= \overline{q}^{\frac{1}{2}}\mathsf{x}_1(q^{\frac{1}{2}}\mathsf{x}_1\mathsf{x}_2 - \overline{q}^{\frac{1}{2}}\mathsf{x}_2\mathsf{x}_1) - q^{\frac{1}{2}}(q^{\frac{1}{2}}\mathsf{x}_1\mathsf{x}_2 - \overline{q}^{\frac{1}{2}}\mathsf{x}_2\mathsf{x}_1)\mathsf{x}_1. \end{aligned} \tag{33}$$

Similarly, manipulating the second equation, we find

$$\overline{q}^{\frac{1}{2}}(q^{\frac{1}{2}}\mathsf{x}_1\mathsf{x}_2 - \overline{q}^{\frac{1}{2}}\mathsf{x}_2\mathsf{x}_1)\mathsf{x}_2 - q^{\frac{1}{2}}\mathsf{x}_2(q^{\frac{1}{2}}\mathsf{x}_1\mathsf{x}_2 - \overline{q}^{\frac{1}{2}}\mathsf{x}_2\mathsf{x}_1) = 0. \tag{34}$$

Comparing (33) and (34) suggests that we should define a new generator

$$\mathsf{x}_{12} := q^{\frac{1}{2}}\mathsf{x}_1\mathsf{x}_2 - \overline{q}^{\frac{1}{2}}\mathsf{x}_2\mathsf{x}_1. \tag{35}$$

In terms of x_{12}, the q-Serre relations (32) may be re-expressed through (33) and (34) as q-commutation relations

$$\overline{q}^{\frac{1}{2}}\mathsf{x}_1\mathsf{x}_{12} - q^{\frac{1}{2}}\mathsf{x}_{12}\mathsf{x}_1 = 0, \qquad \overline{q}^{\frac{1}{2}}\mathsf{x}_{12}\mathsf{x}_2 - q^{\frac{1}{2}}\mathsf{x}_2\mathsf{x}_{12} = 0. \tag{36}$$

Thus we see that, by introducing an additional algebra generator, $\mathcal{U}_h(\mathfrak{sl}_3)$ can be presented completely in terms of commutation and q-commutation relations. To be explicit, we record the commutation relations derived from (35) and (36) with the precedence order $\mathsf{x}_1 \succ \mathsf{x}_{12} \succ \mathsf{x}_2$:

$$\mathsf{x}_2\mathsf{x}_1 = q\,\mathsf{x}_1\mathsf{x}_2 - q^{\frac{1}{2}}\,\mathsf{x}_{12}, \qquad \mathsf{x}_{12}\mathsf{x}_1 = \overline{q}\mathsf{x}_1\mathsf{x}_{12}, \qquad \mathsf{x}_2\mathsf{x}_{12} = \overline{q}\mathsf{x}_{12}\mathsf{x}_2. \tag{37}$$

The analogous construction for the y generators give relations among the y_i of the same form.

7.1.2. *A PBW Basis.* As before, using the commutation relations, like (35) and (36), in $\mathcal{U}_h(\mathfrak{sl}_3)$, arbitrary monomials in the generators h_i, x_i and y_i can be straightened to be expressed as a sum of terms in the set

$$\mathcal{B} := \left\{\mathsf{h}_1^{r_1}\mathsf{h}_2^{r_2}\mathsf{y}_1^{s_1}\mathsf{y}_{12}^{s_{12}}\mathsf{y}_2^{s_2}\mathsf{x}_1^{t_1}\mathsf{x}_{12}^{t_{12}}\mathsf{x}_2^{t_2} : r_i, s_i, t_i \in \mathbb{Z}_{\geq 0}\right\}.$$

It is clear that $\mathcal{B}$ linearly spans $\mathcal{U}_h(\mathfrak{sl}_3)$; linear independence is more tedious to see and can be established either directly through methods in in [Ros89] or *via* the general theory in [Lus93, Chapter 40]. The relevant fact for us is that $\mathcal{B}$ is an analogue of a PBW basis for $\mathcal{U}_h(\mathfrak{sl}_3)$, with respect to the ordering

$$\mathsf{h}_1 \succ \mathsf{h}_2 \succ \mathsf{y}_1 \succ \mathsf{y}_{12} \succ \mathsf{y}_2 \succ \mathsf{x}_1 \succ \mathsf{x}_{12} \succ \mathsf{x}_2.$$

7.1.3. *Notation.* As is evident with the statement of the basis above, notation becomes quite cumbersome in higher dimensional $\mathfrak{sl}_n$. To simplify calculations, we introduce the following shorthands:

$$\mathsf{X}(n_1, n_{12}, n_2) := \mathsf{x}_1^{n_1}\mathsf{x}_{12}^{n_{12}}\mathsf{x}_2^{n_2}, \qquad \mathsf{K}(m_1, m_2) := \mathsf{k}_1^{m_1}\mathsf{k}_2^{m_2},$$
$$\mathsf{Y}(n_1, n_{12}, n_2) := \mathsf{y}_1^{n_1}\mathsf{y}_{12}^{n_{12}}\mathsf{y}_2^{n_2}, \qquad \overline{\mathsf{K}}(m_1, m_2) := \overline{\mathsf{k}}_1^{m_1}\overline{\mathsf{k}}_2^{m_2}.$$

Moreover, set $\mathbf{n} := (n_1, n_{12}, n_2)$, $\mathbf{m} := (m_1, m_2)$ and write $\mathsf{X}(\mathbf{n}) := \mathsf{X}(n_1, n_{12}, n_2)$, $\mathsf{K}(\mathbf{m}) := \mathsf{K}(m_1, m_2)$ and so on. Finally, we shall write things like

$$\mathsf{X}(n_1 - 1; \mathbf{n}) := \mathsf{X}(n_1 - 1, n_{12}, n_2)$$

to indicate changes in the exponent of one of the monomials. If no confusion can arise, we will sometimes omit the reference to $\mathbf{n}$ and simply indicate the exponents that have changed.

As in the $\mathcal{U}_h(\mathfrak{sl}_2)$ case, we will also need to handle an exponential of the h_i. For each pair of i, j, write

$$\epsilon_{ij} := e^{\frac{h}{4}\mathsf{h}_i \otimes \mathsf{h}_j}.$$

Also, set

$$\begin{aligned}\epsilon^K &:= \epsilon_{11}^{\kappa_{11}}\epsilon_{12}^{\kappa_{12}}\epsilon_{21}^{\kappa_{21}}\epsilon_{22}^{\kappa_{22}} \\ &= \exp\left(\frac{h}{4}\left(\kappa_{11}\,\mathsf{h}_1 \otimes \mathsf{h}_1 + \kappa_{12}\,\mathsf{h}_1 \otimes \mathsf{h}_2 + \kappa_{21}\,\mathsf{h}_2 \otimes \mathsf{h}_1 + \kappa_{22}\,\mathsf{h}_2 \otimes \mathsf{h}_2\right)\right).\end{aligned}$$

where

$$K := \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix},$$

is a matrix conveniently packaging the exponents κ_{ij}.

7.1.4. *Straightening in* $\mathcal{U}_h(\mathfrak{sl}_3)$. We are almost in position to calculate an R-morphism for $\mathcal{U}_h(\mathfrak{sl}_3)$. It remains to develop a armamentarium of straightening laws, extending the $\mathcal{U}_h(\mathfrak{sl}_2)$ relations developed in Section 2.4. Since commutation relations among individual $\mathfrak{sl}_2$ triples in $\mathcal{U}_h(\mathfrak{sl}_3)$—*i.e.* generators with the same subscript—are the same as those in (4), the calculations done in $\mathcal{U}_h(\mathfrak{sl}_2)$ may be applied when straightening expressions involving generators with the same lower index. Thus we need only derive new straightening laws for expressions involving generators with different subscripts. The following result collects a few of these "mixed" straightening laws. Proofs of these are straightforward calculations.

Lemma 7.2. *Let a be a nonnegative integer, b any integer, $i, j \in \{1, 2\}$ distinct, $l \in \{1, 2\}$, and $f(x)$ a formal power series in x. Then the following identities hold in $\mathcal{U}_h(\mathfrak{sl}_3)$:*

(i) $\mathsf{x}_i f(\mathsf{h}_j) = f(\mathsf{h}_j + 1)\mathsf{x}_i, \qquad \mathsf{y}_i f(\mathsf{h}_j) = f(\mathsf{h}_j - 1)\mathsf{y}_i,$

(ii) $\mathsf{x}_{12} f(\mathsf{h}_j) = f(\mathsf{h}_j - 1)\mathsf{x}_{12}, \qquad \mathsf{y}_{12} f(\mathsf{h}_j) = f(\mathsf{h}_j + 1)\mathsf{y}_{12},$

(iii) $\mathsf{x}_i^a \mathsf{k}_j^b = q^{\frac{1}{2}ab}\mathsf{k}_j \mathsf{x}_i, \qquad \mathsf{y}_i^a \mathsf{k}_j^b = \overline{q}^{\frac{1}{2}ab}\mathsf{k}_j^b \mathsf{y}_i^a,$

(iv) $\mathsf{x}_{12}^a \mathsf{k}_j^b = \overline{q}^{\frac{1}{2}ab}\mathsf{k}_j^b \mathsf{x}_{12}, \qquad \mathsf{y}_{12}^a \mathsf{k}_j^b = q^{\frac{1}{2}ab}\mathsf{k}_j^b \mathsf{y}_{12},$

(v) $\mathsf{x}_{12}^a \mathsf{x}_1^b = \overline{q}^{ab}\mathsf{x}_1^b \mathsf{x}_{12}^a, \qquad \mathsf{x}_2^b \mathsf{x}_{12}^a = \overline{q}^{ab}\mathsf{x}_{12}^a \mathsf{x}_2^b.$

From the defining relations, x_i and y_j commute for $i \neq j$. However, as y_{12} contains both y_1 and y_2 in its definition, this monomial commutes with neither x_1 nor x_2. Rather, what we shall call a *shortening* process takes place: when commuting, say, x_1 past a power of y_{12}, the x_1 pairs with one the y_1 in a y_{12}, resulting in a left over y_2. More precisely, we have the following calculation.

Lemma 7.3 (Shortening)**.** *Let n_{12} be a positive integer. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_3)$.*

(i) $$\mathsf{y}_{12}^{n_{12}}\mathsf{x}_1 = \mathsf{x}_1\mathsf{y}_{12}^{n_{12}} - \overline{q}^{\frac{1}{2}}[n_{12}]\mathsf{k}_1^2\mathsf{y}_{12}^{n_{12}-1}\mathsf{y}_2,$$

(ii) $$\mathsf{x}_2\mathsf{y}_{12}^{n_{12}} = \mathsf{y}_{12}^{n_{12}}\mathsf{x}_2 - q^{\frac{1}{2}}\overline{q}^{n_{12}-1}[n_{12}]\overline{\mathsf{k}}_2^2\mathsf{y}_1\mathsf{y}_{12}^{n_{12}-1}.$$

Proof. We calculate the relation (i) when $n_{12} = 1$, from which the general case can be established using the methods of Lemma 2.5. Using the definition of y_{12} from (35), the straightening relations between x_1 and y_1 from (4), and (i) of Lemma 7.2,

$$\begin{aligned}
\mathsf{y}_{12}\mathsf{x}_1 &= (q^{\frac{1}{2}}\mathsf{y}_1\mathsf{y}_2 - \overline{q}^{\frac{1}{2}}\mathsf{y}_2\mathsf{y}_1)\mathsf{x}_1 \\
&= q^{\frac{1}{2}}(\mathsf{x}_1\mathsf{y}_1 - [\mathsf{h}_1])\mathsf{y}_2 - \overline{q}^{\frac{1}{2}}\mathsf{y}_2(\mathsf{x}_1\mathsf{y}_1 - [\mathsf{h}_1]) \\
&= \mathsf{x}_1\mathsf{y}_{12} - (q^{\frac{1}{2}}[\mathsf{h}_1] - \overline{q}^{\frac{1}{2}}[\mathsf{h}_1 - 1])\mathsf{y}_2 \\
&= \mathsf{x}_1\mathsf{y}_{12} - \frac{\left(q^{\frac{1}{2}}\mathsf{k}_1^2 - q^{\frac{1}{2}}\overline{\mathsf{k}}_1^2\right) - \left(\overline{q}^{\frac{3}{2}}\mathsf{k}_1^2 + q^{\frac{1}{2}}\overline{\mathsf{k}}_1^2\right)}{q - \overline{q}}\mathsf{y}_2 \\
&= \mathsf{x}_1\mathsf{y}_{12} - \overline{q}^{\frac{1}{2}}\mathsf{k}_1^2\mathsf{y}_2.
\end{aligned}$$

The calculations for (ii) are analogous. □

Inverse to shortening, a *lengthening* takes place when commuting x_1 past x_2. The proofs involve the same techniques as the Shortening relations.

Lemma 7.4 (Lengthening)**.** *Let n_1 and n_2 be positive integers. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_3)$.*

(i) $$\mathsf{x}_2^{n_2}\mathsf{x}_1 = q^{n_2}\mathsf{x}_1\mathsf{x}_2^{n_2} - q^{\frac{1}{2}}[n_2]\mathsf{x}_{12}\mathsf{x}_2^{n_2-1},$$

(ii) $$\mathsf{x}_2\mathsf{x}_1^{n_1} = q^{n_1}\mathsf{x}_1^{n_1}\mathsf{x}_2 - q^{\frac{1}{2}}[n_1]\mathsf{x}_1^{n_1-1}\mathsf{x}_{12}.$$

Finally, we have the following analogue of Lemma 2.6 for moving generators past the exponential factor in the R-morphism.

Lemma 7.5. *Let $f(x)$ be a formal power series in x, n a positive integer, κ any integer and $i,j,l \in \{1,2\}$ with $i \neq j$. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_3)$:*

(i) $$(1\otimes \mathsf{x}_i^n)\epsilon_{li}^{\kappa} = \epsilon_{li}^{\kappa}(\overline{\mathsf{k}}_l^{2n\kappa}\otimes \mathsf{x}_i^n), \qquad (1\otimes \mathsf{y}_i^n)\epsilon_{li}^{\kappa} = \epsilon_{li}^{\kappa}(\mathsf{k}_l^{2n\kappa}\otimes \mathsf{y}_i^n),$$

(ii) $$(1\otimes \mathsf{x}_i^n)\epsilon_{lj}^{\kappa} = \epsilon_{lj}^{\kappa}(\mathsf{k}_l^{n\kappa}\otimes \mathsf{x}_i^n), \qquad (1\otimes \mathsf{y}_i^n)\epsilon_{lj}^{\kappa} = \epsilon_{lj}^{\kappa}(\overline{\mathsf{k}}_l^{n\kappa}\otimes \mathsf{y}_i^n).$$

7.2. **Constructing an R-morphism for $\mathcal{U}_h(\mathfrak{sl}_3)$.** We now carry out the program of Section 5 to construct an R-morphism for $\mathcal{U}_h(\mathfrak{sl}_3)$. An *ansatz* for R can be constructed from first principles through general reasoning about straightening relations, as was done for $\mathcal{U}_h(\mathfrak{sl}_2)$. Alternatively, and more efficiently, we use the general principle that objects in semisimple Lie algebras can be built by appropriately combining ingredients from the constituent $\mathfrak{sl}_2$ to obtain an *ansatz* for R here. This principle, together with the form of R given in Theorem 5.3, leads us to propose the following form for R in $\mathcal{U}_h(\mathfrak{sl}_3)$:

(38) $$\mathsf{R} := \epsilon^K \sum_{\mathbf{m},\mathbf{n}} \alpha(\mathbf{m},\mathbf{n})\,\mathsf{K}(\mathbf{m})\mathsf{X}(\mathbf{n})\otimes \overline{\mathsf{K}}(\mathbf{m})\mathsf{Y}(\mathbf{n})$$

for some coefficients $\alpha(\mathbf{m},\mathbf{n})$, which are rational functions in q. We have used the notation introduced in Subsection 7.1.3. Denote by $T_{\mathbf{m},\mathbf{n}}$ the summand in R indexed by $\mathbf{m}$ and $\mathbf{n}$.

As with Section 5.2, the coefficients and the exponents in (38) will be determined by imposing Conditions (i), (ii) and (iii) of Definition 5.2.

7.2.1. *Condition* (i) *of Definition 5.2.* As with $\mathcal{U}_h(\mathfrak{sl}_2)$, it suffices to impose Condition (i) for the algebra generators x_i, h_i and y_i, $i = 1,2$, in $\mathcal{U}_h(\mathfrak{sl}_3)$; in particular, it is unnecessary to carry out these calculations for x_{12} and y_{12}.

For the generators h_i: Since the h_i commute with k_1 and k_2, we need only consider the x_j and y_j components of each general term in (38). Disregard, for the moment, our knowledge of R in $\mathcal{U}_h(\mathfrak{sl}_2)$ and suppose that we write the x_j and y_j dependence of general term in R as $\mathsf{x}_1^{n_1}\mathsf{x}_{12}^{n_{12}}\mathsf{x}_2^{n_2}\otimes \mathsf{y}_1^{n_1'}\mathsf{y}_{12}^{n_{12}'}\mathsf{y}_2^{n_2'}$. Using (i) and (ii) of Lemma 7.2, Condition (i) for h_1 and h_2 is equivalent to the pair of equations

$$2(n_1 - n_1') + (n_{12} - n_{12}') - (n_2 - n_2') = 0,$$
$$-(n_1 - n_1') + (n_{12} - n_{12}') + 2(n_2 - n_2') = 0.$$

These conditions provide further support to the assumption that $n_i' = n_i$ in (38).

For the generators x_i: Recall that $\Delta(\mathsf{x}_i) = \mathsf{x}_i \otimes \mathsf{k}_i + \overline{\mathsf{k}}_i \otimes \mathsf{x}_i$. Thus this condition requires that

$$(\mathsf{k}_i \otimes \mathsf{x}_i + \mathsf{x}_i \otimes \overline{\mathsf{k}}_i) \cdot \mathsf{R} = \mathsf{R} \cdot (\mathsf{x}_i \otimes \mathsf{k}_i + \overline{\mathsf{k}}_i \otimes \mathsf{x}_i). \tag{39}$$

For $i = 1, 2$ and for each $\mathbf{n} = (n_1, n_{12}, n_2)$, $\mathbf{m} = (m_1, m_2)$, set

$$\begin{aligned} A_i^+(\mathbf{m},\mathbf{n}) &:= (\mathsf{k}_i \otimes \mathsf{x}_i) \cdot \epsilon^K\, \mathsf{K}(\mathbf{m})\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{m})\mathsf{Y}(\mathbf{n}),\\ B_i^-(\mathbf{m},\mathbf{n}) &:= (\mathsf{x}_i \otimes \overline{\mathsf{k}}_i) \cdot \epsilon^K\, \mathsf{K}(\mathbf{m})\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{m})\mathsf{Y}(\mathbf{n}),\\ B_i^+(\mathbf{m},\mathbf{n}) &:= \epsilon^K\, \mathsf{K}(\mathbf{m})\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{m})\mathsf{Y}(\mathbf{n})(\mathsf{x}_i \otimes \mathsf{k}_i),\\ A_i^-(\mathbf{m},\mathbf{n}) &:= \epsilon^K\, \mathsf{K}(\mathbf{m})\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{m})\mathsf{Y}(\mathbf{n})(\overline{\mathsf{k}}_i \otimes \mathsf{x}_i). \end{aligned} \tag{40}$$

Using relations (iii) and (iv) of Lemma 7.2 together with (i) and (ii) of Lemma 7.5, we can straighten the individual terms in (39). First, for the left hand side when $i = 1$,

$$\begin{aligned} A_1^+(\mathbf{m},\mathbf{n}) &= q^{m_1 - \frac{1}{2}m_2}\, \epsilon^K\, \mathsf{K}(\mathbf{m}_{A_1^+})\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{m})\, (\mathsf{x}_1 \mathsf{y}_1^{n_1})\, \mathsf{y}_{12}^{n_{12}} \mathsf{y}_2^{n_2}\\ B_1^-(\mathbf{m},\mathbf{n}) &= \overline{q}^{m_1 - \frac{1}{2}m_2}\, \epsilon^K\, \mathsf{K}(\mathbf{m})\mathsf{X}(n_1+1) \otimes \overline{\mathsf{K}}(\mathbf{m}_{B_1^-})\mathsf{Y}(\mathbf{n}). \end{aligned}$$

where

$$\begin{aligned} \mathbf{m}_{A_1^+} &:= (m_1 + 1 - 2\kappa_{11} + \kappa_{22}, m_2 - 2\kappa_{21} + \kappa_{12})\\ \mathbf{m}_{B_1^-} &:= (m_1 - 1 - 2\kappa_{11} + \kappa_{21}, m_2 - 2\kappa_{12} + \kappa_{22}) \end{aligned}$$

The right hand side of (39) needs to be straightened before we can compare it with the left. First, using the Lengthening Lemma and (v) of Lemma 7.2,

$$\begin{aligned} B_1^+(\mathbf{n},\mathbf{m}) = {}& q^{n_1 - \frac{1}{2}n_{12} + \frac{1}{2}n_2}\, \epsilon^K\, \mathsf{K}(\mathbf{m})\mathsf{X}(n_1+1) \otimes \overline{\mathsf{K}}(m_1-1)\mathsf{Y}(\mathbf{n})\\ &- q^{n_1 + \frac{1}{2}n_{12} - \frac{1}{2}n_2 + \frac{1}{2}}[n_2]\, \epsilon^K\, \mathsf{K}(\mathbf{n})\mathsf{X}(n_{12}+1, n_2-1) \otimes \overline{\mathsf{K}}(m_1-1)\mathsf{Y}(\mathbf{n}) \end{aligned}$$

Next, using the Shortening Lemma,

$$\begin{aligned} A_1^-(\mathbf{n},\mathbf{m}) = {}& q^{n_1 + \frac{1}{2}n_{12} - \frac{1}{2}n_2}\, \epsilon^K\, \mathsf{K}(m_1-1)\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{m})\, (\mathsf{y}_1^{n_1}\mathsf{x}_1)\, \mathsf{y}_{12}^{n_{12}}\mathsf{y}_2^{n_2}\\ &- q^{3n_1 + \frac{1}{2}n_{12} - \frac{1}{2}n_2 - \frac{1}{2}}[n_{12}]\, \epsilon^K\, \mathsf{K}(m_1-1)\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(m_1-2)\mathsf{Y}(n_{12}-1, n_2+1). \end{aligned}$$

As observed in (B) of Remark 5.4, the commutator of x and y played an essential role in deriving R in $\mathcal{U}_h(\mathfrak{sl}_2)$. Thus we are led to rearrange (39) to yield an equation $A_1 = B_1$, where

$$A_1 := \sum_{\mathbf{m},\mathbf{n}} A_1^+(\mathbf{m},\mathbf{n}) - A_1^-(\mathbf{m},\mathbf{n}), \qquad B_1 := \sum_{\mathbf{m},\mathbf{n}} B_1^+(\mathbf{m},\mathbf{n}) - B_1^-(\mathbf{m},\mathbf{n}).$$

In order to combine the terms in A_1 to yield a commutator $[\mathsf{x}_1, \mathsf{y}_1^{n_1}]$, we need to combine the term in A_1^+ with the first term of A_1^-. For that, we must set $\mathbf{m}_{A_1^+} = (m_1 - 1, m_2)$ and also set the exponent of the prefactor q to be equal. This yields the systems of equations

$$\begin{aligned} m_1 - \tfrac{1}{2}m_2 &= n_1 + \tfrac{1}{2}n_{12} - \tfrac{1}{2}n_2,\\ m_1 + 1 - 2\kappa_{11} + \kappa_{12} &= m_1 - 1,\\ m_2 - 2\kappa_{21} + \kappa_{22} &= m_2. \end{aligned}$$

Performing the same calculations for $i = 2$, we obtain another system of equations

$$\begin{aligned} -\tfrac{1}{2}m_1 + m_2 &= -\tfrac{1}{2}n_1 + \tfrac{1}{2}n_{12} + n_2, \\ m_1 + \kappa_{11} - 2\kappa_{12} &= m_1, \\ m_2 + 1 + \kappa_{21} - 2\kappa_{22} &= m_2 - 1. \end{aligned}$$

These equations together imply that

$$K := \begin{pmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and that

$$m_1 + m_2 = n_1 + 2n_{12} + n_2, \qquad m_1 - m_2 = n_1 - n_2.$$

Solving for m_1 and m_2 gives

$$m_1 = n_1 + n_{12}, \qquad m_2 = n_2 + n_{12} \tag{41}$$

In fact, setting $\mathsf{k}_{12} := \mathsf{k}_1\mathsf{k}_2$, this suggests that we write the general term of R in the form

$$\mathsf{k}_1^{n_1}\mathsf{k}_{12}^{n_{12}}\mathsf{k}_2^{n_2}\mathsf{x}_1^{n_1}\mathsf{x}_{12}^{n_{12}}\mathsf{x}_2^{n_2} \otimes \overline{\mathsf{k}}_1^{n_1}\overline{\mathsf{k}}_{12}^{n_{12}}\overline{\mathsf{k}}_2^{n_2}\mathsf{y}_1^{n_1}\mathsf{y}_{12}^{n_{12}}\mathsf{y}_2^{n_2}.$$

Substituting these findings into the expressions for A_1 and computing the commutator $[\mathsf{x}_1, \mathsf{y}_1^{n_1}]$ with (v) of Lemma 2.5, we have

$$\begin{aligned} A_1 = \epsilon^K \sum_{\mathbf{n}} \alpha(\mathbf{n})\Big(& q^{n_1+\frac{1}{2}n_{12}-\frac{1}{2}n_2}[n_1]\, \mathsf{K}(n_1-1)\mathsf{X}(\mathbf{n}) \otimes [\mathsf{h}_1 + n_1 - 1]\overline{\mathsf{K}}(\mathbf{n})\mathsf{Y}(n_1-1) \\ & + q^{3n_1+\frac{1}{2}n_{12}-\frac{1}{2}n_2-\frac{1}{2}}[n_{12}]\, \mathsf{K}(n_1-1)\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(n_1-2)\mathsf{Y}(n_{12}-1, n_2+1)\Big). \end{aligned}$$

Similarly, for B_1, we have

$$\begin{aligned} B_1 &= \epsilon^K \sum_{\mathbf{n}} \alpha(\mathbf{n})\Big(q^{\frac{1}{2}n_2-\frac{1}{2}n_{12}}\mathsf{K}(\mathbf{n})\mathsf{X}(n_1+1) \otimes \Big(q^{n_1}\overline{\mathsf{k}}_1^{n_1-1} - \overline{q}^{n_1}\overline{\mathsf{k}}_1^{n_1+3}\Big)\overline{\mathsf{k}}_{12}^{n_{12}}\overline{\mathsf{k}}_2^{n_2}\mathsf{Y}(\mathbf{n}) \\ &\qquad - q^{n_1+\frac{1}{2}n_{12}-\frac{1}{2}n_2+\frac{1}{2}}[n_2]\, \mathsf{K}(\mathbf{n})\mathsf{X}(n_{12}+1, n_2-1) \otimes \overline{\mathsf{K}}(n_1-1)\mathsf{Y}(\mathbf{n})\Big) \\ &= \epsilon^K \sum_{\mathbf{n}} \alpha(\mathbf{n})\Big(q^{\frac{1}{2}n_2-\frac{1}{2}n_{12}}(q-\overline{q})\mathsf{K}(\mathbf{n})\mathsf{X}(n_1+1) \otimes [\mathsf{h}_1 + n_1]\, \overline{\mathsf{K}}(n_1+1)\mathsf{Y}(\mathbf{n}) \\ &\qquad - q^{n_1+\frac{1}{2}n_{12}-\frac{1}{2}n_2+\frac{1}{2}}[n_2]\, \mathsf{K}(\mathbf{n})\mathsf{X}(n_{12}+1, n_2-1) \otimes \overline{\mathsf{K}}(n_1-1)\mathsf{Y}(\mathbf{n})\Big). \end{aligned}$$

Comparing the first term of A_1 the first term of B_1—after making an index shift $n_1 \mapsto n_1 - 1$—we obtain a recursion relation for $\alpha(\mathbf{n})$ with respect to n_1:

$$\alpha(\mathbf{n}) = \alpha(n_1 - 1; \mathbf{n})\frac{(q-\overline{q})}{[n_1]}\overline{q}^{n_1+n_{12}-n_2}. \tag{42}$$

Comparing the second term of A_1 with the second term of B_1, after making index shifts $n_2 \mapsto n_2 - 1$ and $n_{12} \mapsto n_{12} - 1$ respectively, we obtain a relation for $\alpha(\mathbf{n})$ with respect to n_{12} and n_2:

$$\alpha(n_{12} - 1; \mathbf{n}) = -\alpha(n_2 - 1; \mathbf{n})\frac{[n_{12}]}{[n_2]}q^{2n_1}. \tag{43}$$

Performing the analogous manipulations for A_2 and B_2 yields recurrences for $\alpha(\mathbf{n})$ with respect to n_2:

$$\alpha(\mathbf{n}) = \alpha(n_2 - 1; \mathbf{n})\frac{(q-\overline{q})}{[n_2]}\overline{q}^{n_2+n_{12}-n_1}, \tag{44}$$

and with respect to n_{12} and n_1:

$$\alpha(n_{12}-1;\mathbf{n}) = -\alpha(n_1-1;\mathbf{n})\frac{[n_{12}]}{[n_1]}q^{2n_2}. \tag{45}$$

Now, by combining either (43) with (44) or (45) with (42), a recurrence for $\alpha_{\mathbf{n}}$ with respect to n_{12} is found:

$$\alpha(\mathbf{n}) = -\alpha(n_{12}-1;\mathbf{n})\frac{(q-\overline{q})}{[n_{12}]}\overline{q}^{n_1+n_{12}+n_2}. \tag{46}$$

The recurrences for $\alpha(\mathbf{n})$ for each individual index can be solved individually, all assuming the initial condition $\alpha(\mathbf{0}) = 1$, and combined together to yield

$$\alpha(\mathbf{n}) = (-1)^{n_{12}}\frac{(q-\overline{q})^{|\mathbf{n}|}}{[\mathbf{n}]!}\overline{q}^{\frac{1}{2}n_1(n_1+1)+\frac{1}{2}n_{12}(n_{12}+1)+\frac{1}{2}n_2(n_2+1)+2n_1n_{12}-2n_1n_2+2n_{12}n_2},$$

where we have used the notation $|\mathbf{n}| := n_1+n_{12}+n_2$ and $[\mathbf{n}]! := [n_1]![n_{12}]![n_2]!$.

Finally, like in the $\mathcal{U}_h(\mathfrak{sl}_2)$ case, the general term of R can be rearranged to the form

$$(\mathsf{k}_1\mathsf{x}_1)^{n_1}(\mathsf{k}_{12}\mathsf{x}_{12})^{n_{12}}(\mathsf{k}_2\mathsf{x}_2)^{n_2} \otimes (\overline{\mathsf{k}}_1\mathsf{y}_1)^{n_1}(\overline{\mathsf{k}}_{12}\mathsf{y}_{12})^{n_{12}}(\overline{\mathsf{k}}_2\mathsf{y}_2)^{n_2}.$$

Doing so would eliminate the cross terms in the exponent of $\overline{q}$ in $\alpha(\mathbf{n})$ and would also transform the terms $\overline{q}^{\frac{1}{2}n(n+1)}$ to $q^{\frac{1}{2}n(n-3)}$ as in (27). With this form, we see that the R-morphism factors into a product

$$\mathsf{R} = \epsilon^K \mathrm{Exp}_q(\lambda_q\mathsf{k}_1\mathsf{x}_1\otimes\overline{\mathsf{k}}_1\mathsf{y}_1)\mathrm{Exp}_q(-\lambda_q\mathsf{k}_{12}\mathsf{x}_{12}\otimes\overline{\mathsf{k}}_{12}\mathsf{y}_{12})\mathrm{Exp}_q(\lambda_q\mathsf{k}_2\mathsf{x}_2\otimes\overline{\mathsf{k}}_2\mathsf{y}_2) \tag{47}$$

where λ_q is as in (28).

For the generators y_i: It is readily checked that with the quantities found above, Condition (i) of Definition 5.2 is satisfied for the y_i. In fact, it is easy to see that the relevant calculations for the y_i completely mirror those done when considering the x_i, with only some differences in signs. The entire program above can therefore be carried out with the y_i in *lieu* of the x_i.

7.2.2. *Conditions* (ii) *and* (iii) *of Definition 5.2.* These conditions can be checked by using the techniques from the $\mathcal{U}_h(\mathfrak{sl}_2)$ case together the presentation (47) of R as a product of q-exponential functions.

8. Extensions to $\mathcal{U}_h(\mathfrak{sl}_{n+1})$

Generally, an R-morphism for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ can be constructed by comparing the coefficients of the equation $A = B$ from Condition (i) of Definition 5.2 corresponding terms. The method is essentially the same as in the $\mathcal{U}_h(\mathfrak{sl}_3)$ case, except more notationally cumbersome. This section will complete this construction for general $\mathcal{U}_h(\mathfrak{sl}_{n+1})$, $n \geq 2$.

Definition 8.1. *Let* $n \geq 1$. *The* quantized universal enveloping algebra $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ *is the* $\mathbb{C}[[h]]$ *algebra generated by* n $\mathfrak{sl}_2$ *triples* $(\mathsf{x}_i, \mathsf{h}_i, \mathsf{y}_i)$, $i = 1, \dots, n$, *subject to the relations* $[\mathsf{h}_i, \mathsf{h}_j] = 0$,

(48)

$$[\mathsf{h}_i,\mathsf{x}_j] = \begin{cases} 2\mathsf{x}_i & j=i, \\ -\mathsf{x}_j & j = i\pm 1, \\ 0 & \textit{otherwise}, \end{cases} \quad [\mathsf{h}_i,\mathsf{y}_j] = \begin{cases} -2\mathsf{y}_i & j=i, \\ \mathsf{y}_j & j = i\pm 1, \\ 0 & \textit{otherwise}, \end{cases} \quad [\mathsf{x}_i,\mathsf{y}_j] = \delta_{ij}\frac{\mathsf{k}_i-\overline{\mathsf{k}}_i}{q-\overline{q}},$$

where $\mathsf{k}_i := e^{\frac{h}{4}\mathsf{h}_i}$, *and for* $|i-j| = 1$,

$$\mathsf{x}_i^2\mathsf{x}_j - (q+\overline{q})\mathsf{x}_i\mathsf{x}_j\mathsf{x}_i + \mathsf{x}_j\mathsf{x}_i^2 = 0, \qquad \mathsf{y}_i^2\mathsf{y}_j - (q+\overline{q})\mathsf{y}_i\mathsf{y}_j\mathsf{y}_i + \mathsf{y}_j\mathsf{y}_i^2 = 0,$$

8.1. **q-Serre Relations and PBW Bases.** As in $\mathcal{U}_h(\mathfrak{sl}_3)$, the q-Serre relations can be simplified to commutation relations by introducing additional generators. However, because of the increase in number of $\mathfrak{sl}_2$ triples, more generators will be necessary. For instance, in $\mathcal{U}_h(\mathfrak{sl}_4)$, x_{12}, defined as in (35), is related to x_3 through a cubic relation since x_3 is related to x_2. In fact, reasoning like this shows that we must introduce a generator

$$\mathsf{x}_{i,j} := \mathsf{x}_{i,i+1,\dots,j}$$

for each pair $1 \le i \le j \le n$. In other words, we will need a generator for each interval of integers in $\{1,\dots,n\}$. As per the comments in Section 7.1.2, these generators together with a fixed ordering of the indices will give a PBW basis for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$. Throughout, we will fix the following ordering on the indices:

$$1 \succ 12 \succ \cdots \succ 12\dots n \succ 2 \succ 23 \succ \cdots \succ n-1, n \succ n. \tag{49}$$

After introducing these higher degree generators, we will establish some basic straightening relations for these generators and then calculate an R-morphism for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$. We shall carry through with this below.

Remark 8.2. Those knowledgeable about semisimple Lie algebras will immediately note that, upon identifying the $1,\dots,n$ with labels for a choice of simple roots for $\mathfrak{sl}_{n+1}$, the consecutive strings used to index the basis elements correspond to the set of positive roots for $\mathfrak{sl}_{n+1}$.

8.2. **Higher Degree Generators in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$.** To define $\mathsf{x}_{i,j}$, set $\mathsf{x}_{i,i} := \mathsf{x}_i$ and for $i < j$, inductively define

$$\mathsf{x}_{i,j} := q^{\frac{1}{2}}\mathsf{x}_i\mathsf{x}_{i+1,j} - \overline{q}^{\frac{1}{2}}\mathsf{x}_{i+1,j}\mathsf{x}_i. \tag{50}$$

In fact, there is a much more symmetric description of $\mathsf{x}_{i,j}$.

Lemma 8.3 (Splitting). *Let $1 \le i < j \le n$ and consider a splitting of the interval of integers $\{i, i+1,\dots,j\}$ into $\{i,\dots,s\}$ and $\{s+1,\dots,j\}$. Then*

$$\mathsf{x}_{i,j} = q^{\frac{1}{2}}\mathsf{x}_{i,s}\mathsf{x}_{s+1,j} - \overline{q}^{\frac{1}{2}}\mathsf{x}_{s+1,j}\mathsf{x}_{i,s}.$$

To establish this Lemma, we give a combinatorial description of $\mathsf{x}_{i,j}$ in terms of the degree one generators $\mathsf{x}_i,\dots,\mathsf{x}_j$. In short, $\mathsf{x}_{i,j}$ will be a sum of terms indexed by orientations on a path of length $j-i$.

Let P_n denote the Dynkin diagram for $\mathfrak{sl}_{n+1}$, *i.e.* a path consisting of n vertices labelled left to right by the integers 1 to n. For $1 \le i < j \le n$, let $P_{i,j}$ denote the induced subgraph of P obtained by taking the vertices labelled $i, i+1,\dots,j$. Let $\mathcal{D}_{i,j}$ denote the set of orientations on $P_{i,j}$. For each $D \in \mathcal{D}_{i,j}$, let

$$D_{\to} := \#\{\ell \to \ell+1 \in D\}, \qquad D_{\leftarrow} := \#\{\ell \leftarrow \ell+1 \in D\}$$

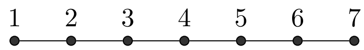
be the number of right- and left-pointing arrows in the orientation D. Set

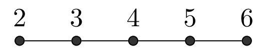
$$q^D := (-1)^{D_\leftarrow} q^{\frac{1}{2}(D_\to - D_\leftarrow)}.$$

For each $D \in \mathcal{D}_{i,j}$, we also construct a monomial x_D in $\mathsf{x}_i,\dots,\mathsf{x}_j$ as follows. Begin by writing x_j. Next, if the edge from $j-1$ to j is right-point, that is if $j-1 \to j \in D$, then place x_{j-1} to the *left* of x_j; otherwise, place it on the right. Next, if

$j-2 \to j+1 \in D$, then place x_{j-2} at the leftmost end; otherwise, place it on the rightmost. At each step, think of a right-pointing arrow $\ell-1 \to \ell$ as indicating that $\mathsf{x}_{\ell-1}$ and x_ℓ are to be positioned "in order", so that $\mathsf{x}_{\ell-1}$ appears before x_ℓ; similarly, a left-pointing arrow $\ell-1 \leftarrow \ell$ indicates that $\mathsf{x}_{\ell-1}$ and x_ℓ appear "out of order". Continue this process until all of x_j to x_i have been placed, resulting in x_D.

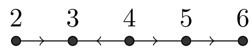
Example 8.4. *Suppose $n=8$, $i=2$ and $j=6$. Then*

$P_7 =$ 

, *and* $P_{2,6} =$ 

.

The set $\mathcal{D}_{2,6}$ has $2^4 = 16$ elements, one being

$D =$ 

.

Then $D_\to = 3$, $D_\leftarrow = 1$, $q^D = -q$. The construction of x_D proceeds through the following steps: x_6, $\mathsf{x}_5\mathsf{x}_6$, $\mathsf{x}_4\mathsf{x}_5\mathsf{x}_6$, $\mathsf{x}_4\mathsf{x}_5\mathsf{x}_6\mathsf{x}_3$, and finally $\mathsf{x}_D = \mathsf{x}_2\mathsf{x}_4\mathsf{x}_3\mathsf{x}_5\mathsf{x}_6$.

A combinatorial description of $\mathsf{x}_{i,j}$ can now be given.

Lemma 8.5. *Let $1 \le i < j \le n$. Then*

$$\mathsf{x}_{i,j} = \sum_{D \in \mathcal{D}_{i,j}} q^D \mathsf{x}_D.$$

Proof. We use induction on the difference $j-i$. When $j = i+1$, this formula reduces to the definition of $\mathsf{x}_{i,i+1}$ in (50). In general,

$$\begin{aligned}\mathsf{x}_{i,j} &= q^{\frac{1}{2}}\mathsf{x}_i\mathsf{x}_{i+1,j} - \overline{q}^{\frac{1}{2}}\mathsf{x}_{i+1,j}\mathsf{x}_i \\ &= \sum_{D\in\mathcal{D}_{i+1,j}} (-1)^{D_\leftarrow} q^{\frac{1}{2}((D_\to+1)-D_\leftarrow)}\mathsf{x}_i\mathsf{x}_D + (-1)^{D_\leftarrow+1} q^{\frac{1}{2}(D_\to-(D_\leftarrow+1))}\mathsf{x}_D\mathsf{x}_i.\end{aligned}$$

Since i is the smallest index and by the construction of $\mathsf{x}_{D'}$ with $D' \in \mathcal{D}_{i,j}$, the first terms in the sum correspond to orientations D' with right-pointing arrow $i \to i+1$. Similarly the second terms correspond to orientations with left-pointing $i \leftarrow i+1$. □

It is important to note that the x_D constructed are convenient representatives of commutation equivalence class of monomials in the x_i—recall that $\mathsf{x}_i\mathsf{x}_j = \mathsf{x}_j\mathsf{x}_i$ whenever $|i-j| > 1$. The important data encoded in the orientation $D \in \mathcal{D}_{i,j}$ is the relative position of adjacent generators. It is clear that any product of $\mathsf{x}_i, \dots, \mathsf{x}_j$ satisfying x_{i+1} is right of x_i if and only if $i \to i+1$ in D is equivalent to x_D.

Proof of Lemma 8.3. This follows immediately from Lemma 8.5 and the discussion above after the observation that an orientation on $P_{i,j}$ consists of orientations on $P_{i,s}$ and $P_{s+1,j}$ and an orientation on the edge $(s, s+1)$. The term $q^{\frac{1}{2}}\mathsf{x}_{i,s}\mathsf{x}_{s+1,j}$ accounts for all orientations with $s \to s+1$ where as $-\overline{q}^{\frac{1}{2}}\mathsf{x}_{s+1,j}\mathsf{x}_{i,s}$ accounts for orientations with $s \leftarrow s+1$. □

8.3. **Commutation Relations in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$.** Although the number of generators has increased in number, the commutation relations in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ are largely of the same type as those in $\mathcal{U}_h(\mathfrak{sl}_3)$. Indeed, rather than complicate matters, the increase in number of generators clarifies the rigid structure of $\mathcal{U}_h(\mathfrak{sl}_{n+1})$. As with the $\mathcal{U}_h(\mathfrak{sl}_3)$ case, Shortening and Lengthening phenomena take place and play an important role in the construction of the R-morphism. We collect the relevant relations in the next two Lemmas.

Lemma 8.6 (Lengthening)**.** *Let $1 \leq i < j \leq n$. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$.*

(i) $$\mathsf{x}_{i+1,j}^{n_{i+1,j}}\mathsf{x}_i = q^{n_{i+1,j}}\mathsf{x}_i\mathsf{x}_{i+1,j} - q^{\frac{1}{2}}[n_{i+1,j}]\,\mathsf{x}_{i,j}\mathsf{x}_{i+1,j}^{n_{i+1,j}-1},$$

(ii) $$\mathsf{x}_j\mathsf{x}_{i,j-1}^{n_{i,j-1}} = q^{n_{i,j-1}}\mathsf{x}_{i,j-1}^{n_{i,j-1}}\mathsf{x}_j - q^{\frac{1}{2}}[n_{i,j-1}]\,\mathsf{x}_{i,j-1}^{n_{i,j-1}-1}\mathsf{x}_{i,j}.$$

Lemma 8.7 (Shortening)**.** *Let $1 \leq i < j \leq n$. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$.*

(i) $$\mathsf{y}_{i,j}^{n_{i,j}}\mathsf{x}_i = \mathsf{x}_i\mathsf{y}_{i,j}^{n_{i,j}} - \overline{q}^{\frac{1}{2}}[n_{i,j}]\,\mathsf{k}_i^2\mathsf{y}_{i,j}^{n_{i,j}-1}\mathsf{y}_{i+1,j},$$

(ii) $$\mathsf{x}_j\mathsf{y}_{i,j}^{n_{i,j}} = \mathsf{y}_{i,j}^{n_{i,j}}\mathsf{x}_j - q^{\frac{1}{2}}\overline{q}^{n_{i,j}-1}[n_{i,j}]\,\overline{\mathsf{k}}_j^2\mathsf{y}_{i,j-1}\mathsf{y}_{i,j}^{n_{i,j}-1}.$$

The proofs of these Lemmas involve using the Splitting Lemma 8.3 to show the case when the exponent is 1, and then the techniques from Lemma 2.5 are used to establish the general case.

One last type of commutation we will need in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ is when the indexing interval of a some generator is completely contained in that of another. In these cases, all relations end up being commutation relations, possibly with some scalar factor. The proofs use the Splitting Lemma 8.3 to reduce the statement to a computation in $\mathcal{U}_h(\mathfrak{sl}_4)$.

Lemma 8.8 (Passing)**.** *Let $1 \leq i < j \leq n$. Then the following hold in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$.*

(i) $$\mathsf{x}_{i,j}^{n_{i,j}}\mathsf{x}_i^{n_i} = \overline{q}^{n_i n_{i,j}}\mathsf{x}_i^{n_i}\mathsf{x}_{i,j}^{n_{i,j}}, \qquad \mathsf{x}_j^{n_j}\mathsf{x}_{i,j}^{n_{i,j}} = \overline{q}^{n_{i,j}n_j}\mathsf{x}_{i,j}^{n_{i,j}}\mathsf{x}_j^{n_j}.$$

Moreover, for any $i < s < j$,

(ii) $$\mathsf{x}_s^{n_s}\mathsf{x}_{i,j}^{n_{i,j}} = \mathsf{x}_{i,j}^{n_{i,j}}\mathsf{x}_s^{n_s}, \qquad \mathsf{x}_s^{n_s}\mathsf{y}_{i,j}^{n_{i,j}} = \mathsf{y}_{i,j}^{n_{i,j}}\mathsf{x}_s^{n_s}.$$

With our choice of ordering on the generators, we need not compute any other commutation relations.

8.4. **Exponential Prefactor of R.** We are now ready to derive an R-morphism for $\mathcal{U}_h(\mathfrak{sl}_{n+1})$. First, either from our experience in $\mathcal{U}_h(\mathfrak{sl}_2)$ and $\mathcal{U}_h(\mathfrak{sl}_3)$ or, again, by general reasoning about how straightening should contribute terms, we propose that a R-morphism takes the form

$$\mathsf{R} = \epsilon^K \sum_{\mathbf{n}} \alpha(\mathbf{n})\,\mathsf{K}(\mathbf{n})\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{n})\mathsf{Y}(\mathbf{n}),$$

where the coefficient $\alpha(\mathbf{n})$ is a rational function in q, K, $\overline{\mathsf{K}}$, X and Y are products of the generators in $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ ordered with respect to the fixed precedence order, and

$$\epsilon^K := \exp\left(\sum_{i,j=1}^{n} \kappa_{i,j}\mathsf{h}_i \otimes \mathsf{h}_j\right)$$

for some matrix of coefficients $K := (\kappa_{ij})$. Our first task will be to determine this matrix K. This is done by observing that, in the $\mathcal{U}_h(\mathfrak{sl}_3)$ case, the equations which we used to solve for K came from equating the exponents of powers of k and $\overline{\mathsf{k}}$ in the terms which can Condition (i) of Definition 5.2. These terms will come from

$$A_i^+(\mathbf{n}) := (\mathsf{k}_i \otimes \mathsf{x}_i) \cdot \left(\epsilon^K \, \mathsf{K}(\mathbf{n})\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{n})\mathsf{Y}(\mathbf{n})\right),$$
$$A_i^-(\mathbf{n}) := \left(\epsilon^K \, \mathsf{K}(\mathbf{n})\mathsf{X}(\mathbf{n}) \otimes \overline{\mathsf{K}}(\mathbf{n})\mathsf{Y}(\mathbf{n})\right) \cdot (\overline{\mathsf{k}}_i \otimes \mathsf{x}_i).$$

From the Shortening Lemma 8.7, we see that we will obtain a single term with $\cdots \mathsf{x}_i \mathsf{y}_i^{n_i} \cdots$ in A_i^+ and a single term with $\cdots \mathsf{y}_i^{n_i} \mathsf{x}_i \cdots$ in A_i^-. We would like to combine these terms. Using the defining relations together with slight generalizations of (iii) of Lemma 7.2 and Lemma 7.5, for each $j = 1, \dots, n$, we obtain n linear equations:

$$\kappa_{i,j-1} - 2\kappa_{ij} + \kappa_{i,j+1} = -2\delta_{ij}, \qquad j = 1, \dots, n,$$

where $\kappa_{i,-1} := 0$ and $\kappa_{i,n+1} := 0$. In matrix form,

$$KA_n := K \cdot \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & & & 0 \\ 0 & -1 & 2 & & & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & & & & 2 & -1 \\ 0 & \cdots & \cdots & \cdots & -1 & 2 \end{pmatrix} = 2I_n$$

where I_n is the $n \times n$ identity matrix. It is readily seen that A_n is invertible, so $K = 2A_n^{-1}$. The inverse can be explicitly calculated, yielding

$$\kappa_{ij} = \begin{cases} \frac{2}{n+1} i(n-j+1) & \text{if } j \le i, \\ \frac{2}{n+1} j(n-i+1) & \text{if } i \le j. \end{cases} \tag{51}$$

Remark 8.9. The matrix A_n is the Cartan matrix of the Lie algebra $\mathfrak{sl}_{n+1}$.

8.5. **Calculating the Coefficient.** We now proceed to calculate the coefficients $\alpha(\mathbf{n})$ of R. The technique is essentially the same as the $\mathcal{U}_h(\mathfrak{sl}_3)$ case, except more cumbersome notationally which makes the situation seem more technical than it is. The main steps are as follows: for each $i = 1, \dots, n$,

(1) Impose Condition (i) of Definition 5.2 for x_i;
(2) Rearrange the resulting equation to obtain series A_i and B_i;
(3) Compare coefficients of the like terms in A_i and B_i, resulting in recurrence relations for the coefficients $\alpha(\mathbf{n})$ with respect the various summation indices;
(4) Simultaneously solve the system of recurrences.

In between steps (2) and (3), the summands of A_i and B_i may need to straightened. Due to Shortening and Lengthening phenomena, each summand of A_i and B_i will itself be a sum of different terms and the crux of the entire procedure is to systematically track the coefficients of each summand. In the sections that follow, we will carry out the steps above.

8.5.1. *Individual Terms of $A_i^{\pm}$ and $B_i^{\pm}$.* For each tuple of exponents $\mathbf{n} := (n_1, n_{12}, \ldots, n_n)$, denote by $\mathsf{R}(\mathbf{n})$ the summand in R indexed by $\mathbf{n}$. Then, similar to (40), define for each $i = 1, \ldots, n$ and $\mathbf{n}$

$$A_i^+(\mathbf{n}) := (\mathsf{k}_i \otimes \mathsf{x}_i) \cdot \mathsf{R}(\mathbf{n}), \qquad B_i^+(\mathbf{n}) := \mathsf{R}(\mathbf{n}) \cdot (\mathsf{x}_i \otimes \mathsf{k}_i),$$
$$A_i^-(\mathbf{n}) := \mathsf{R}(\mathbf{n}) \cdot (\overline{\mathsf{k}}_i \otimes \mathsf{x}_i), \qquad B_i^-(\mathbf{n}) := (\mathsf{x}_i \otimes \overline{\mathsf{k}}_i) \cdot \mathsf{R}(\mathbf{n})$$

and then set

$$A_i := \sum_{\mathbf{n}} A_i^+(\mathbf{n}) - A_i^-(\mathbf{n}), \qquad B_i := \sum_{\mathbf{n}} B_i^+(\mathbf{n}) - B_i^-(\mathbf{n}). \tag{52}$$

As usual, Condition (i) of Definition 5.2 is equivalent to the equality $A_i = B_i$. This equality can be used to deduce properties of $\alpha(\mathbf{n})$ by understanding the terms appearing after straightening $A_i^{\pm}(\mathbf{n})$ and $B_i^{\pm}(\mathbf{n})$. To be clear, straightening the term $\mathsf{x}_i\mathsf{Y}$ means that we need to move x_i through the y until it is immediately left of y_i; similarly, straightening $\mathsf{Y}\mathsf{x}_i$ means that x_i needs to be moved until it is immediately right of y_i.

From the Shortening and Lengthening Lemmas, straightening x_i through Y or X will cause Shortening and Lengthening, respectively, to occur. As a consequence, the $A_i^{\pm}$ and $B_i^{\pm}$ are generally sums of various terms resulting from these transformations. We shall see that these terms are indexed by certain segments of $\{1, \ldots, n\}$.

Terms of $A_i^+ = (\mathsf{k}_i \otimes \mathsf{x}_i) \cdot \mathsf{R}$: When straightening A_i^+, x_i needs to be moved past the $\overline{\mathsf{K}}$ in front of the Y. From the defining relations (48), commuting x_i past $\overline{\mathsf{k}}_{a,b}$ will yield a nontrivial contribution to the coefficient of A_i^+ whenever the segment $\{a, a+1, \ldots, b\}$ contains either one or two of $\{i-1, i, i+1\}$. Precisely, the coefficient contribution when x_i is moved past a $\overline{\mathsf{k}}_{a,b}$ indexed by a segment

- $\{a, \ldots, i-1\}$ for $1 \le a < i$ is $\overline{q}^{\frac{1}{2}n_{a,i-1}}$;
- $\{a, \ldots, i\}$ for $1 \le a < i$ is $q^{\frac{1}{2}n_{a,i}}$;
- $\{i\}$ is q^{n_i};
- $\{i, \ldots, b\}$ for $i < b \le n$ is $q^{\frac{1}{2}n_{i,b}}$; and
- $\{i+1, \ldots, b\}$ for $i < b \le n$ is $\overline{q}^{\frac{1}{2}n_{i+1,b}}$.

This leads to an overall coefficient contribution of

$$\left(\prod_{a=1}^{i-1} q^{\frac{1}{2}n_{a,i} - \frac{1}{2}n_{a,i-1}}\right) q^{n_i} \left(\prod_{b=i+1}^{n} q^{\frac{1}{2}n_{i,b} - \frac{1}{2}n_{i+1,b}}\right) \tag{53}$$

to the terms of A_i^+.

Since x_i is being straightened starting from the left of A_i^+, Shortening occurs for each interval $\{s, \ldots, i\}$, where $s = 1, \ldots, i$; denote by $A_{s,i}^+$ the term where $\mathsf{y}_{s,i}^{n_{s,i}}$ is shortened by x_i. For $s < i$, (ii) of the Shortening Lemma 8.7 shows that the Y term in $A_{s,i}^+$ looks like

$$\left(\cdots \mathsf{y}_{s,i-1}^{n_{s,i-1}} \mathsf{y}_{s,i}^{n_{s,i}} \cdots\right) \mapsto \left(\cdots \mathsf{y}_{s,i-1}^{n_{s,i-1}} \left(-q^{\frac{1}{2}} \overline{q}^{n_{s,i}-1} [n_{s,i}] \,\overline{\mathsf{k}}_i^2 \mathsf{y}_{s,i-1} \mathsf{y}_{s,i}^{n_{s,i}}\right) \cdots\right).$$

The $\overline{\mathsf{k}}_i^2$ must now be moved left through the Y. Powers of q coming from this straightening are indexed by segments

- $\{a, \ldots, i-1\}$ for $1 \le a \le s$ with contribution $q^{n_{a,i-1}}$;
- $\{a, \ldots, i\}$ for $1 \le a < s$ with contribution $\overline{q}^{n_{a,i}}$.

In total, the coefficient of $A^+_{s,i}(\mathbf{n})$, $1 \le s < i$, is

$$(54)\quad -\alpha(\mathbf{n})[n_{s,i}]q^{\frac{3}{2}+n_i}\prod_{a=1}^{s}\overline{q}^{\frac{1}{2}n_{a,i}-\frac{1}{2}n_{a,i-1}}\prod_{a=s+1}^{i-1}q^{\frac{1}{2}n_{a,i}-\frac{1}{2}n_{a,i-1}}\prod_{b=i+1}^{n}q^{\frac{1}{2}n_{i,b}-\frac{1}{2}n_{i+1,b}}.$$

Moreover, the exponents of the generators in $A^+_{s,i}(\mathbf{n})$ change as

$$(55)\qquad \mathsf{X}(\mathbf{n})\otimes\mathsf{Y}(\mathbf{n})\mapsto\mathsf{X}(\mathbf{n})\otimes\mathsf{Y}(n_{s,i-1}+1,n_{s,i}-1;\mathbf{n}).$$

Finally, the term $A^+_{i,i}$ arises from commuting x_i through the Y until it is to the left of y_i, *i.e.* this comes from from repeatedly taking the first term in (ii) of Lemma 8.7. The coefficient of $A^+_{i,i}$ does not acquire any additional factors and thus is simply equal to (53).

<u>*Terms of* $A^-_i = \mathsf{R}\cdot(\overline{\mathsf{k}}_i\otimes\mathsf{x}_i)$</u>: The overall exponent in A^-_i comes from straightening a $\overline{\mathsf{k}}_i$ through X from the right. It is easy to see the total number of q factors arising from this straightening is again given by (53). When moving the x_i through the Y from the right, Shortening occurs for segments $\{i,\dots,t\}$, $i\le t\le n$; notate this term by $A^-_{i,t}$.

When $i<t$, (i) of the Shortening Lemma shows that the Y part of $A^-_{i,t}$ is now

$$\left(\cdots\mathsf{y}_{i,t}^{n_{i,t}}\mathsf{y}_{i,t+1}^{n_{i,t+1}}\cdots\right)\mapsto\left(\cdots\left(-\overline{q}^{\frac{1}{2}}[n_{i,t}]\,\mathsf{k}_i^2\mathsf{y}_{i,t}^{n_{i,t}}\mathsf{y}_{i+1,t}\right)\mathsf{y}_{i,t+1}^{n_{i,t+1}}\cdots\right).$$

The k_i^2 needs to be moved through the Y to the left and the $\mathsf{y}_{i+1,t}$ needs to be straightened to the right. From the Passing Lemma 8.8 and a calculation like the A^+_i case, we find the coefficient of $A^-_{i,t}(\mathbf{n})$, $i<t\le n$, to be

$$(56)\quad -\alpha(\mathbf{n})[n_{i,t}]q^{3n_i-\frac{1}{2}}\prod_{a=1}^{i-1}q^{\frac{3}{2}n_{a,i}-\frac{3}{2}n_{a,i-1}}\prod_{b=i+1}^{t-1}q^{\frac{3}{2}n_{i,b}-\frac{3}{2}n_{i+1,b}}\prod_{b=t}^{n}q^{\frac{1}{2}n_{i,b}-\frac{1}{2}n_{i+1,b}}$$

and that the exponents in $A^-_{i,t}(\mathbf{n})$ change as

$$(57)\qquad \mathsf{X}(\mathbf{n})\otimes\mathsf{Y}(\mathbf{n})\mapsto\mathsf{X}(\mathbf{n})\otimes\mathsf{Y}(n_{i,t}-1,n_{i+1,t}+1;\mathbf{n}).$$

As with before, when $t=i$, no additional powers of q are accrued while commuting x_i through the y until it is right of y_i. Thus $A^-_{i,i}$ has coefficient (53).

<u>*Terms of* $B^+_i = (\mathsf{R}\cdot(\mathsf{x}_i\otimes\mathsf{k}_i))$</u>: Straightening k_i through the Y from the right leads to an overall coefficient for the terms of B^+_i given, yet again, by (53). Terms in B^+_i come from Lengthening of monomials $\mathsf{x}_{i+1,t}$, $i\le t\le n$ denote the resulting term by $B^+_{i,t}$. For $i<t$, (i) of the Lengthening Lemma 8.6 shows $B^+_{i,t}$ looks like

$$\left(\cdots\mathsf{x}_{i+1,t-1}^{n_{i+1,t-1}}\mathsf{x}_{i+1,t}^{n_{i+1,t}}\cdots\right)\mapsto\left(\cdots\mathsf{x}_{i+1,t-1}^{n_{i+1,t-1}}\left(-q^{\frac{1}{2}}[n_{i+1,j}]\,\mathsf{x}_{i,t}\mathsf{x}_{i+1,t}^{n_{i+1,t}-1}\right)\cdots\right).$$

Straightening this form means commuting $\mathsf{x}_{i,t}$ to the left through some $\mathsf{x}_{i+1,t'}$, $t'<t$, and then through $\mathsf{x}_{i,t''}$, $t<t''$. By (ii) of the Passing Lemma 8.8, there are no additional factors of q when moving past the $\mathsf{x}_{i+1,b}$ terms; by (i) of the Passing Lemma, a factor of $\overline{q}^{n_{i,b}}$ is obtained when moving past $\mathsf{x}_{i,b}^{n_{i,b}}$. The coefficient of $B^+_{i,t}$ is thus

$$(58)\quad -\alpha(\mathbf{n})[n_{i,t}]q^{n_i+\frac{1}{2}}\prod_{a=1}^{i-1}q^{\frac{1}{2}n_{a,i}-\frac{1}{2}n_{a,i-1}}\prod_{b=i+1}^{t}q^{\frac{1}{2}n_{i,b}-\frac{1}{2}n_{i+1,b}}\prod_{b=t+1}^{n}\overline{q}^{\frac{1}{2}n_{i,b}+\frac{1}{2}n_{i+1,b}}$$

and the exponents in $B^+_{i,t}(\mathbf{n})$ will change as

$$\mathsf{X}(\mathbf{n}) \otimes \mathsf{Y}(\mathbf{n}) \mapsto \mathsf{X}(n_{i,t}+1, n_{i+1,t}-1; \mathbf{n}) \otimes \mathsf{Y}(\mathbf{n}). \tag{59}$$

When $t = i$, additional contributions of q will come from the first summand in the Lengthening Lemma 8.6 and from the relations (i) of the Passing Lemma 8.8. Thus the coefficient of $B^+_{i,i}(\mathbf{n})$ is

$$\alpha(\mathbf{n}) q^{n_i} \prod_{a=1}^{i-1} q^{\frac{1}{2}n_{a,i} - \frac{1}{2}n_{a,i-1}} \prod_{b=i+1}^{n} \overline{q}^{\frac{1}{2}n_{i,b} - \frac{1}{2}n_{i+1,b}}. \tag{60}$$

<u>*Terms of* $B^-_i = (\mathsf{x}_i \otimes \overline{\mathsf{k}}_i) \cdot \mathsf{R}$:</u> Commuting x_i through K from the left yields an overall coefficient of the inverse of (53) to all terms of B^-_i. When straightening x_i from the left through X, Lengthening occurs for every segment $\{s, \ldots, i-1\}$, $1 \le s \le i-1$; the resulting term will be denoted by $B^-_{s,i}$. Note that this does not include the term $B^-_{i,i}$ arising from moving the new x_i to the x_i term in X.

To determine the coefficient of $B^-_{s,i}$ when $1 \le s < i$, note that, by the Passing Lemma, moving x_i through the X to $\mathsf{x}_{s,i-1}$ picks up powers of q whenever x_i passes $\mathsf{x}_{a,i-1}^{n_{a,i-1}}$, contributing $q^{n_{a,i-1}}$, and $\mathsf{x}_{a,i}^{n_{a,i}}$, contributing $\overline{q}^{n_{a,i}}$. From part (ii) of the Lengthening Lemma 8.6, the change from Lengthening is

$$\left(\cdots \mathsf{x}_{s,i-1}^{n_{s,i-1}} \mathsf{x}_{s,i}^{n_{s,i}} \cdots\right) \mapsto \left(\cdots \left(-q^{\frac{1}{2}}[n_{s,i-1}]\, \mathsf{x}_{s,i-1}^{n_{s,i-1}-1} \mathsf{x}_{s,i}\right) \mathsf{x}_{s,i}^{n_{s,i}} \cdots\right).$$

No further straightening needs to take place. Thus the coefficient of $B^-_{s,i}$ is

$$-\alpha_{\mathbf{n}}[n_{s,i-1}]\overline{q}^{n_i - \frac{1}{2}} \prod_{i=1}^{s-1} \overline{q}^{\frac{3}{2}n_{a,i} - \frac{3}{2}n_{a,i-1}} \prod_{a=s}^{i-1} \overline{q}^{\frac{1}{2}n_{a,i} - \frac{1}{2}n_{a,i-1}} \prod_{b=i+1}^{n} \overline{q}^{\frac{1}{2}n_{i,b} - \frac{1}{2}n_{i+1,b}} \tag{61}$$

and the exponent in $B^-_{s,i}(\mathbf{n})$ changes as

$$\mathsf{X}(\mathbf{n}) \otimes \mathsf{Y}(\mathbf{n}) \mapsto \mathsf{X}(n_{s,i-1}-1, n_{s,i}+1; \mathbf{n}) \otimes \mathsf{Y}(\mathbf{n}). \tag{62}$$

For $s = i$, considerations similar to those for $B^+_{i,i}$ show that the coefficient of $B^-_{i,i}(\mathbf{n})$ is

$$\alpha(\mathbf{n}) \overline{q}^{n_i} \prod_{a=1}^{i-1} \overline{q}^{\frac{3}{2}n_{a,i} - \frac{3}{2}n_{a,i-1}} \prod_{b=i+1}^{n} \overline{q}^{\frac{1}{2}n_{i,b} - \frac{1}{2}n_{i+1,b}}. \tag{63}$$

8.6. **Combining Diagonal Terms.** The diagonal A terms $A^+_{i,i}(\mathbf{n})$ and $A^-_{i,i}(\mathbf{n})$ differ only in the Y component as

$$A^+_{i,i} = \mathsf{K}\mathsf{X} \otimes \overline{\mathsf{K}}(\cdots \mathsf{x}_i \mathsf{y}_i^{n_i} \cdots), \qquad A^-_{i,i} = \mathsf{K}\mathsf{X} \otimes \overline{\mathsf{K}}(\cdots \mathsf{y}_i^{n_i} \mathsf{x}_i \cdots).$$

These terms may be combined in A_i to obtain a commutator $[\mathsf{x}_i, \mathsf{y}_i^{n_i}]$, which can be simplified using (v) of Lemma 2.5; denote the resulting term by $A_{i,i}$. This term has coefficient

$$\alpha(\mathbf{n})[n_i]\, q^{n_i} \prod_{a=1}^{i-1} q^{\frac{1}{2}n_{a,i} - \frac{1}{2}n_{a,i-1}} \prod_{b=i+1}^{n} q^{\frac{1}{2}n_{i,b} - \frac{1}{2}n_{i+1,b}} \tag{64}$$

and exponent change of

$$\mathsf{X}(\mathbf{n}) \otimes \mathsf{Y}(\mathbf{n}) \mapsto \mathsf{X}(\mathbf{n}) \otimes \mathsf{Y}(n_i - 1; \mathbf{n}). \tag{65}$$

Similarly, $B^{+}_{i,i}(\mathbf{n})$ and $B^{-}_{i,i}(\mathbf{n})$ can be combined, in a manner analogous to how B_1 was simplified in the $\mathcal{U}_h(\mathfrak{sl}_3)$ case. Specifically, factor out

$$(q-\overline{q})\prod_{a=1}^{i-1}\overline{q}^{\frac{1}{2}n_{a,i}-\frac{1}{2}n_{a,i-1}}\prod_{b=i+1}^{n}\overline{q}^{\frac{1}{2}n_{i,b}-\frac{1}{2}n_{i+1,b}}$$

from each of $B^{\pm}_{i,i}(\mathbf{n})$. This allows us to combine the generators in $B^{\pm}_{i,i}(\mathbf{n})$, with the difference

$$\frac{1}{q-\overline{q}}\left(\left(\prod_{a=1}^{i-1}q^{n_{a,i}-n_{a,i-1}}\right)q^{n_i}\overline{\mathsf{k}}_i^{n_i-1}-\left(\prod_{a=1}^{i-1}\overline{q}^{n_{a,i}-n_{a,i-1}}\right)\overline{q}^{n_i}\overline{\mathsf{k}}_i^{n_i+1}\right)$$

in place of the $\overline{\mathsf{k}}_i$ in $\overline{\mathsf{K}}$. After factoring out $\overline{\mathsf{k}}_i^{n_i+1}$, this can be simplified to the quantum number

$$\left[\mathsf{h}_i+n_i+\sum_{a=1}^{i-1}n_{a,i}-n_{a,i-1}\right].$$

After straightening, this will contribute the same quantum number as obtain from simplifying the commutator in $A_{i,i}$. Overall, the coefficient of $B_{i,i}$ is

$$\alpha(\mathbf{n})(q-\overline{q})\prod_{a=1}^{i-1}\overline{q}^{\frac{1}{2}n_{a,i}-\frac{1}{2}n_{a,i-1}}\prod_{b=i+1}^{n}\overline{q}^{\frac{1}{2}n_{i,b}-\frac{1}{2}n_{i+1,b}} \tag{66}$$

and the change in exponent is due only to the additional x_i term on the right

$$\mathsf{X}(\mathbf{n})\otimes\mathsf{Y}(\mathbf{n})\mapsto\mathsf{X}(n_i+1;\mathbf{n})\otimes\mathsf{Y}(\mathbf{n}). \tag{67}$$

8.6.1. *Comparing Terms.* Finally, we can construct recurrence relations for the coefficient $\alpha(\mathbf{n})$ by comparing coefficients of like terms in the equation $A_i = B_i$. First, from (65) and (67), we see that the terms of $A_{i,i}$ agree in shape with those of $B_{i,i}$ and their coefficients can be compared after making the index shift $n_i \mapsto n_i - 1$ in $B_{i,i}$. From (64) and (66), we obtain a recursion relation for $\alpha(\mathbf{n})$ with respect to the index n_i:

$$\alpha(\mathbf{n})=\alpha(n_i-1;\mathbf{n})\frac{(q-\overline{q})}{[n_i]}\overline{q}^{n_i}\prod_{a=1}^{i-1}\overline{q}^{n_{a,i}-n_{a,i-1}}\prod_{b=i+1}^{n}\overline{q}^{n_{i,b}-n_{i+1,b}}. \tag{68}$$

From (55) and (62), we see that we can compare the coefficient of $A^{+}_{s,i}$ with that of $B^{-}_{s,i}$ after shifting the exponents $n_{s,i-1}\mapsto n_{s,i-1}$ for $A^{+}_{s,i}$ and $n_{s,i}\mapsto n_{s,i}-1$ for $B_{s,i}$. From (54) and (61), we obtain a recursion relation of the form

$$\begin{aligned}\alpha(n_{s,i}-1;\mathbf{n}) &= \\ &-\alpha(n_{s,i-1}-1;\mathbf{n})\frac{[n_{s,i}]}{[n_{s,i-1}]}q^{2n_i-n_{s,i}+n_{s,i-1}}\prod_{a=1}^{i-1}q^{n_{a,i}-n_{a,i-1}}\prod_{b=i+1}^{n}q^{n_{i,b}-n_{i+1,b}}.\end{aligned} \tag{69}$$

Comparing (57) and (59) shows that the coefficients of $A^{-}_{i,t}$ and $B^{+}_{i,t}$ can be compared, after making an appropriate exponent shift, and similar type of recursion relation can be constructed using (56) and (58). However, the two sets of relations (68) and (69) are sufficient to solve for $\alpha(\mathbf{n})$.

8.6.2. *Solving Recurrences.* The recursion (68) can be solved directly. For (69), notice that the relation expresses the changes in $\alpha(\mathbf{n})$ with respect to $n_{s,i}$ in terms of changes with respect to $n_{s,i-1}$, an exponent indexed by a shorter interval. Thus, by iterating (69), the relation can be transformed into on expressing the change with respect to $n_{s,i}$ in terms of the change with respect to n_s, at which point (68) can be used to set up a true recursion relation for $\alpha(\mathbf{n})$ with respect to $n_{s,i}$. Carrying out this calculation, marvellous telescoping occurs and the resulting recursion relation is

$$\alpha(\mathbf{n}) = (-1)^{i-s}\alpha(n_{s,i}-1;\mathbf{n})\frac{(q-\overline{q})}{[n_{s,i}]}q^{n_{s,i}}\prod_{a=1}^{i}\overline{q}^{n_{a,i}}\prod_{b=i+1}^{n}q^{n_{i+1,b}}\prod_{a=1}^{s-1}q^{n_{a,s-1}}\prod_{b=s}^{n}\overline{q}^{n_{s,b}}. \tag{70}$$

To obtain an expression for $\alpha(\mathbf{n})$, we solve the recursions (68) and (70) individually and put the resulting expressions together.

Before writing down an expression for R, observe that the cross terms

$$\left(\prod_{\substack{a=1\\a\neq s}}^{i}\overline{q}^{n_{s,i}n_{a,i}}\prod_{b=i+1}^{n}q^{n_{s,i}n_{i+1,b}}\right)\left(\prod_{a=1}^{s-1}q^{n_{a,s-1}n_{s,i}}\prod_{\substack{b=s\\b\neq i}}^{n}\overline{q}^{n_{s,b}n_{s,i}}\right)$$

in $\alpha(\mathbf{n})$ will be cancelled by factors of q arising from straightening

$$\mathsf{K}(\mathbf{n})\mathsf{X}(\mathbf{n})\otimes\overline{\mathsf{K}}(\mathbf{n})\mathsf{Y}(\mathbf{n})\mapsto((\mathsf{k}_1\mathsf{x}_1)^{n_1}(\mathsf{k}_{12}\mathsf{x}_{12})^{n_{12}}\cdots)\otimes\left((\overline{\mathsf{k}}_1\mathsf{y}_1)^{n_1}(\overline{\mathsf{k}}_{12}\mathsf{y}_{12})^{n_{12}}\cdots\right)$$

the monomials so that powers of $\mathsf{k}_{a,b}$, $\mathsf{x}_{a,b}$ and $\overline{\mathsf{k}}_{a,b}$, $\mathsf{y}_{a,b}$ occur together. The resulting coefficient $\alpha'(\mathbf{n})$ is then of the form

$$\alpha'(\mathbf{n}) = \prod_{a=1}^{n}\prod_{b=1}^{n}(-1)^{b-a}\frac{(q-\overline{q})^{n_{a,b}}}{[n_{a,b}]!}\,q^{\frac{1}{2}n_{a,b}(n_{a,b}-3)}.$$

From this, we deduce the following.

Theorem 8.10. *A universal* R*-morphism for* $\mathcal{U}_h(\mathfrak{sl}_{n+1})$*,* $n\geq 1$*, is*

$$\mathsf{R} = \exp\left(\frac{h}{4}\sum_{i,j=1}^{n}\kappa_{ij}\mathsf{h}_i\otimes\mathsf{h}_j\right)\prod_{1\leq a,b\leq n}^{\prec}\mathrm{Exp}_q\left((-1)^{b-a}\lambda_q\,\mathsf{k}_{a,b}\mathsf{x}_{a,b}\otimes\overline{\mathsf{k}}_{a,b}\mathsf{y}_{a,b}\right)$$

Here, the κ_{ij} are as in (51), λ_q is as in (28) and the $\prec$ on the product signifies that terms in the product in accordance to the precedence order (49).

8.6.3. *Comment on the Other Conditions.* The construction above ensures that R verifies Condition (i) of Definition 5.2. Using q-Exponential form of R, the two other Conditions can be established in the same way as the $\mathcal{U}_h(\mathfrak{sl}_2)$ case.

9. Concluding Remarks

In this final section, we collect some observations about our straightening method for constructing R-morphisms.

9.1. **Relation With the Quantum Double.** Although different, our method and the Quantum Double method of [Dri87] do appear to be somewhat related. Computing a R-morphism using the Quantum Double amounts to pairing the subalgebra of $\mathcal{U}_h(\mathfrak{sl}_{n+1})$ generated by the x_i and h_i with its dual, finding dual bases for these algebras and then identifying the dual with the subalgebra generated by the y_i and the h_i. The coefficients of R are then computed by pairing, in an appropriate sense, the x_i with the y_i.

An essential feature of our method is the appearance of the commutator $[\mathsf{x}_i, \mathsf{y}_i^{n_i}]$. From the straightening formulae involving commutators, we do see that $[\mathsf{x}_i, \cdot\,]$ behaves like a derivation on y_i and thus may be seen as a sort of pairing between x_i and y_i. Upon identifying this, one may perhaps be naturally led to the Quantum Double. Thus we have achieved one of the goals we set out in the beginning: we are able to see how certain remarkable algebraic results could have been obtained to begin with.

9.2. **Extensions to Other Lie Algebras.** Quantized universal enveloping algebras of other complex semisimple Lie algebras $\mathfrak{g}$ are defined in a way analogous to that of $\mathfrak{sl}_{n+1}$, except that the q-Serre relations change based on the structure of the Dynkin diagram of $\mathfrak{g}$. Nonetheless, constructions similar to the ones made can be used to simplify these higher order relations to commutation relations, at which point straightening can be used to calculate a R-morphism.

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