Extending coherent state transforms to Clifford analysis

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Abstract

We introduce two extensions of the Segal-Bargmann coherent state transform from $L^2(\mathbb{R}, dx)$ to Hilbert spaces of slice monogenic and axial monogenic functions and study their properties. These two transforms are related by the dual Radon transform. Representation theoretic and quantum mechanical aspects of the new representations are studied.

Contents

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1 Introduction

Clifford analysis (see [BDS, DSS]) has been developed to extend the complex analysis of holomorphic functions to functions of Clifford algebra variables, satisfying properties generalizing the Cauchy–Riemann conditions.

On the other hand, in quantum physics, Clifford algebra or spinor representation valued functions describe some systems with internal degrees of freedom, such as particles with spin.

Recall that the Segal-Bargmann transform [Ba, Se1, Se2], for a particle on $\mathbb{R}$, establishes the unitary equivalence of the Schrödinger representation with Hilbert space $L^2(\mathbb{R}, dx)$, with (Fock space-like) representations with Hilbert spaces, $\mathcal{H}L^2(\mathbb{C}, d\nu)$, of holomorphic functions on the phase space, $\mathbb{R}^2 \cong \mathbb{C}$ which are L^2 with respect to a measure ν. Recall that in the Schrödinger representation the position operator $\hat{x}_{Sch}$ acts diagonally while the momentum operator $\hat{p}_{Sch} = i\frac{d}{dx}$. On the other hand, on the Hilbert space $\mathcal{H}L^2(\mathbb{C}, d\nu)$ it is the operator $\hat{x}_{SB} + i\hat{p}_{SB}$ that acts as multiplication by the holomorphic function $x + ip$.

In [Ha1], Hall has defined coherent state transforms (CSTs) for compact Lie groups G which are analogs of the Segal-Bargmann transform. These CSTs correspond to applying heat kernel evolution, $e^{\frac{\Delta}{2}}$, followed by analytic continuation to the complexification $G_{\mathbb{C}}$ of G [Ha2].

We use the fact that, after applying the heat kernel evolution, the resulting functions are in fact extendable to $\mathbb{R}^{m+1}$ in two natural ways motivated by Clifford analysis. These will lead to two generalizations of the CST, the slice monogenic CST, U_s, and the axial monogenic CST, U_a, which take values on spaces of $\mathbb{C}_m$–valued functions on $\mathbb{R}^{m+1}$, where $\mathbb{C}_m$ denotes the complex Clifford algebra with m generators. One, $\mathcal{H}_s = \operatorname{Im} U_s$, is a subspace of the recently introduced space of square integrable slice monogenic functions [CSS1], while the other, $\mathcal{H}_a = \operatorname{Im} U_a$, is a Hilbert space of, the more traditional in Clifford analysis, axial monogenic functions [BDS, DSS]. We show that the two coherent state transforms are related by the dual Radon transform $\check{R}$,

$$U_a = \check{R} \circ U_s.$$

A possibly interesting alternative way of defining a monogenic CST would be through Fueter's theorem [F, Q, KQS, PQS, Sc]. It would be very interesting to relate such a transform with the one studied in the present paper.

As in the case of the usual CST, the aim of these transforms is to describe the quantum states of a particle in $\mathbb{R}$ with internal degrees of freedom parametrized by a Clifford algebra, through slice/axial monogenic functions, thus making available, the powerful analytic machinery of Clifford analysis. In Section 5, we show that the operator $\hat{x}_0 + i\hat{p}_0$ has a simple action in both the slice and axial monogenic representations.

2 Preliminaries

2.1 Coherent state transforms (CST)

Let G be a compact Lie group with complexification $G_{\mathbb{C}}$. In 1994, Hall [Ha1] introduced a class of unitary integral transforms on $L^2(G, dx)$, where dx is Haar measure, to spaces of holomorphic functions on $G_{\mathbb{C}}$ which are L^2 with respect to an appropriate measure. These are known as coherent state transforms (CSTs) or generalized Segal–Bargmann transforms.

These transforms were extended to groups of compact type, which include the case of $G = \mathbb{R}^n$ considered in the present paper, by Driver in [Dr]. General Lie groups of compact type are products of compact Lie groups and $\mathbb{R}^n$, see Corollary 2.2 of [Dr]. For $G = \mathbb{R}^n$ the Hall transform coincides with the classical Segal–Bargmann transform [Ba, Se1, Se2].

We will briefly recall now the case $G = \mathbb{R}$, which we will extend to the context of Clifford analysis in the present paper. The case of arbitrary groups of compact type is very interesting and will be studied in forthcoming work. Let $\rho_t(x)$ denote the fundamental solution of the heat equation.

$$\frac{\partial}{\partial t}\rho_t = \frac{1}{2}\Delta\, \rho_t,$$

i.e.

$$\rho_t(x) = \frac{1}{(2\pi t)^{1/2}}\, e^{-\frac{x^2}{2t}},$$

where Δ is the Laplacian for the Euclidian metric. The Segal–Bargman or coherent state transform

$$U \ : L^2(\mathbb{R}, dx) \longrightarrow \mathcal{H}(\mathbb{C})$$

is defined by

$$\begin{aligned} U(f)(z) &= \int_{\mathbb{R}} \rho_{t=1}(z-x) f(x)\, dx = \\ &= \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{-\frac{(z-x)^2}{2}}\, f(x)\, dx\, . \end{aligned} \tag{2.1}$$

where ρ_1 has been analytically continued to $\mathbb{C}$. We see that the transform U in (2.1) factorizes according to the following diagram

$$\begin{tikzcd} & \mathcal{H}(\mathbb{C}) \\ L^2(\mathbb{R}, dx) \arrow[ur, hook, "U"] \arrow[r, hook, "e^{\frac{\Delta}{2}}"'] & \mathcal{A}(\mathbb{R}) \arrow[u, "\mathcal{C}"'] \end{tikzcd} \tag{2.2}$$

where $\mathcal{C}$ denotes the analytic continuation from $\mathbb{R}$ to $\mathbb{C}$ and $e^{\frac{\Delta}{2}}(f)$ is the (real analytic) heat kernel evolution of the function $f \in L^2(\mathbb{R}, dx)$ at time $t = 1$, that is the solution of

$$\left\{ \begin{array}{lcl} \frac{\partial}{\partial t} h_t &=& \frac{1}{2}\Delta\, h_t \\ h_0 &=& f \end{array} \right. , \tag{2.3}$$

evaluated at time $t = 1$,

$$e^{\frac{\Delta}{2}}(f) = h_1.$$

$\mathcal{A}(\mathbb{R})$ in (2.2) is the space of (complex valued) real analytic functions on $\mathbb{R}$ with unique analytic continuation to entire functions on $\mathbb{C}$. For $f \in \mathcal{A}(\mathbb{R})$ the solution of (2.3) is also given as the sum of the (uniformly convergent on compact subsets of $\mathbb{R}$) exponential series

$$h_t = \sum_{k=0}^{\infty} \frac{1}{k!} \frac{t^k}{2^k}\, \Delta^k(f).$$

Let $\widetilde{\mathcal{A}}(\mathbb{R}) \subset \mathcal{A}(\mathbb{R})$ denote the image of $L^2(\mathbb{R}, dx)$ by the operator $e^{\frac{\Delta}{2}}$.

Theorem 2.1 (Segal–Bargmann) *The transform (2.1)*

$$\begin{tikzcd} & \mathcal{H}L^2(\mathbb{C}, \nu\, dxdy) \\ L^2(\mathbb{R}, dx) \arrow[ur, "U"] \arrow[r, "e^{\frac{\Delta}{2}}"'] & \widetilde{\mathcal{A}}(\mathbb{R}) \arrow[u, "\mathcal{C}"'] \end{tikzcd} \tag{2.4}$$

is a unitary isomorphism, where $z = x + iy \in \mathbb{C}, x, y \in \mathbb{R}$ *and* $\nu(y) = e^{-y^2}$.

2.2 Clifford analysis

Clifford analysis has been developed to extend the complex analysis of holomorphic functions to functions of Clifford algebra variables, satisfying properties generalizing the Cauchy–Riemann conditions [BDS, DSS]. Let us briefly recall from [CSS1] and [DS], some definitions and results from Clifford analysis. Let $\mathbb{R}_m$ denote the real Clifford algebra with m generators, $e_j, j = 1, \ldots, m$, identified with the canonical basis of $\mathbb{R}^m \subset \mathbb{R}_m$ and satisfying the relations $e_ie_j + e_je_i = -2\delta_{ij}$. We have that $\mathbb{R}_m = \oplus_{k=0}^m \mathbb{R}_m^k$, where $\mathbb{R}_m^k$ denotes the space of k-vectors, defined by $\mathbb{R}_m^0 = \mathbb{R}$ and $\mathbb{R}_m^k = \mathrm{span}_{\mathbb{R}}\{e_A \; : \; A \subset \{1, \ldots, m\}, |A| = k\}$. We see that, in particular, $\mathbb{R}^m$ is identified with the space of 1-vectors, $\mathbb{R}^m = \mathbb{R}_m^1$, $\underline{x} = \sum_{j=1}^m x_je_j$ and $\mathbb{R}^{m+1}$ is identified with the space, $\mathbb{R}_m^{\leq 1}$, of vectors of the form,

$$\vec{x} = x_0 + \underline{x} = x_0 + \sum_{j=1}^m x_je_j.$$

Notice also that $\mathbb{R}_1 \cong \mathbb{C}$ and $\mathbb{R}_2 \cong \mathbb{H}$. The inner product in $\mathbb{R}_m$ is defined by

$$< u, v >=< \sum_A u_Ae_A, \sum_B v_Be_B >= \sum_A u_Av_A,$$

and therefore, $\underline{x}^2 = -|\underline{x}|^2 = - < x, x >$. The Dirac operator is defined as

$$\partial_{\underline{x}} = \sum_{j=1}^m \partial_{x_j}e_j,$$

and the Cauchy-Riemann operator as

$$\partial_{\vec{x}} = \partial_{x_0} + \partial_{\underline{x}}.$$

We have that $\partial_{\underline{x}}^2 = -\Delta_m$ and $\partial_{\vec{x}}\overline{\partial_{\vec{x}}} = \Delta_{m+1}$.

Recall that a continuously differentiable function f on an open domain $U \subset \mathbb{R}^{m+1}$, with values on $\mathbb{R}_m$ or $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C}$, is called (left) monogenic on U if (see, for example, [BDS, DSS])

$$\partial_{\vec{x}}f(x_0, \underline{x}) = (\partial_{x_0} + \partial_{\underline{x}})f(x_0, \underline{x}) = 0.$$

For $m = 1$, monogenic functions on $\mathbb{R}^2$ correspond to holomorphic functions of the complex variable $x_0 + e_1x_1$.

3 Monogenic extensions of analytic functions

3.1 Slice monogenic extension

Recall from [CSS1, CSS3] that a function $f : U \subseteq \mathbb{R}^{m+1} \to \mathbb{R}_m$ is slice monogenic if, for any unit 1–vector $\underline{\omega} \in S^{m-1} \subset \mathbb{R}^1_m$, the restrictions $f_{\underline{\omega}}$ of f to the complex planes

$$H_{\underline{\omega}} = \{u + v\,\underline{\omega},\ u, v \in \mathbb{R}\},$$

are holomorphic,

$$(\partial_u + \underline{\omega}\,\partial_v) f_{\underline{\omega}}(u, v) = 0,\ \forall \underline{\omega} \in S^{m-1}. \tag{3.1}$$

Let $\mathcal{SM}(\mathbb{R}^{m+1})$ denote the space of slice monogenic functions on $\mathbb{R}^{m+1}$. As a consequence of the Proposition 2.7 of [CSS2] one obtains the following result.

Theorem 3.1 (Colombo-Sabadini-Struppa) *The slice-monogenic extension map,*

$$\begin{aligned} M_s \ :\ \mathcal{A}(\mathbb{R}) \otimes \mathbb{R}_m &\longrightarrow \mathcal{SM}(\mathbb{R}^{m+1}) \\ M_s(h)(x_0, \underline{x}) &= M_s(\sum_A h_A\, e_A)(x_0, \underline{x}) = \\ &= \sum_A h_A(x_0 + \underline{x})\, e_A := \sum_A e^{\underline{x}\,\frac{d}{dx_0}}\, h_A(x_0)\, e_A = \\ &= \sum_A \sum_{k=0}^{\infty} \frac{\underline{x}^k}{k!}\, \frac{d^k h_A}{dx_0^k}(x_0)\, e_A, \end{aligned} \tag{3.2}$$

is well defined and satisfies $M_s(h)(x_0, 0) = h(x_0), \forall x_0 \in \mathbb{R}$.

3.2 Axial monogenic extension and dual Radon transform

A monogenic function $f(x_0, \underline{x})$ is called axial monogenic of degree zero (see [DS], p. 322, for the definition of axial monogenic functions of degree k) if it is of the form

$$\begin{aligned} f(x_0, \underline{x}) &= \sum_A f_A(x_0, \underline{x})\, e_A \\ f_A(x_0, \underline{x}) &= B_A(x_0, |\underline{x}|) + \frac{\underline{x}}{|\underline{x}|}\, C_A(x_0, |\underline{x}|)\,, \end{aligned} \tag{3.3}$$

where B_A, C_A are scalar functions and the functions f_A are monogenic. The monogeneicity condition, $\partial_{\vec{x}} f_A = \partial_{x_0} f_A + \partial_{\underline{x}} f_A = 0$, then leads to the Vekua–type system for B_A, C_A, generalising the Cauchy-Riemann conditions,

$$\partial_{x_0} B_A - \partial_r C_A = \frac{m-1}{r}\, C_A\,, \quad \partial_{x_0} C_A + \partial_r B_A = 0,\ r = |\underline{x}|.$$

Let $\mathcal{AM}_0(\mathbb{R}^{m+1})$ denote the space of degree zero axial monogenic functions on $\mathbb{R}^{m+1}$.

Axial monogenic functions of degree zero are determined by their restriction to the real axis, $f(x_0, 0)$. The inverse map of extending (when such an extension exists) a real analytic function h on $\mathbb{R}$ to a degree zero axial monogenic function on $\mathbb{R}^{m+1}$ is called generalized axial Cauchy-Kowalewski extension and has been studied by many authors (see, for example, [DS]).

Using the dual Radon transform to map slice monogenic functions to monogenic functions as proposed in [CLSS], we will factorize the axial monogenic extension into the slice monogenic extension followed by the dual Radon transform. Let us first recall the definition of the dual Radon transform. (See, for example, [He].)

Definition 3.2 *The dual Radon transform of a smooth function f on $\mathbb{R}^{m+1}$ is*

$$\check{R}(f)(x_0,\underline{x}) = \int_{S^{m-1}} f(x_0, <\underline{x},\underline{t}> \underline{t})\, d\underline{t}. \tag{3.4}$$

It is known from [CLSS] that $\check{R}$ maps entire slice monogenic functions to entire monogenic functions.

Let us denote a function $f \in \mathcal{A}(\mathbb{R})$ and its analytic continuation to the complex plane $H_{\underline{t}}$ by the same symbol, f. The following is a small modification of the Theorem 4.2 in [DS].

Theorem 3.3 *The axial monogenic or axial Cauchy-Kovalewski extension map*

$$\begin{aligned} M_a \ :\ \mathcal{A}(\mathbb{R})\otimes\mathbb{R}_m &\longrightarrow \mathcal{AM}_0(\mathbb{R}^{m+1}) \\ M_a(h)(x_0,\underline{x}) &= M_a(\sum_A h_A e_A)(x_0,\underline{x}) = \\ &= \sum_A \int_{S^{m-1}} h_A(x_0 + <\underline{x},\underline{t}> \underline{t})\, d\underline{t}\, e_A, \end{aligned} \tag{3.5}$$

where $d\underline{t}$ denotes the invariant normalized (probability) measure on S^{m-1}, is well defined and satisfies $M_a(h)(x_0,0) = h(x_0), \forall x_0 \in \mathbb{R} = \mathbb{R}^0_m$.

Proof. From (3.2) and (3.4) we see that the map M_a in (3.5) factorizes to

$$M_a = \check{R} \circ M_s\ . \tag{3.6}$$

The fact that the image of this map is a subspace of the space of entire monogenic functions on $\mathbb{R}^{m+1}$ is a consequence of the theorem A of [CLSS]. We still need to show that the functions $M_a(h)$ are axial monogenic for all $h \in \mathcal{A}(\mathbb{R})\otimes\mathbb{R}_m$. Notice that the Taylor series of h, with center at any point of $\mathbb{R}$ has infinite radius of convergence. Using (3.2), Theorem 3.1, and the fact that for $\underline{\omega} \in S^{m-1}$ one has $\underline{\omega}^{2k} = (-1)^k$, we obtain

$$M_a(h)(x_0,\underline{x}) = M_a(\sum_A h_A\, e_A)(x_0,\underline{x}) =$$

$$= \sum_A \check{R}\circ M_s(h_A)(x_0,\underline{x})\, e_A = \sum_A \int_{S^{m-1}} \sum_{k=0}^{\infty} \frac{(\langle \underline{x},\underline{\omega}\rangle \underline{\omega})^k}{k!} h_A^{(k)}(x_0) d\underline{\omega}\, e_A$$

$$= \sum_A \left(\sum_{j=0}^{\infty} \int_{S^{m-1}} \frac{(-1)^j}{(2j)!} h_A^{(2j)}(x_0) \langle \underline{x},\underline{\omega}\rangle^{2j} + \underline{\omega}\, \frac{(-1)^j}{(2j+1)!}\, h_A^{(2j+1)}(x_0)\langle \underline{x},\underline{\omega}\rangle^{2j+1} d\underline{\omega} \right) e_A.$$

and therefore,

$$M_a(h)(x_0,\underline{x}) = \sum_A \left(\sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} h_A^{(2j)}(x_0) C_{m,2j} |\underline{x}|^{2j} + \underline{x}\, \frac{(-1)^j}{(2j+1)!} h_A^{(2j+1)}(x_0) C_{m,2j+2} |\underline{x}|^{2j} \right) e_A,$$

where

$$C_{m,2j} = \int_0^{\pi} \sin^{m-1}(\theta) \cos^{2j}(\theta)\, d\theta.$$

This is of the form (3.3) which completes the proof. ■

We therefore get the following commutative diagram.

$$\begin{tikzcd} & \mathcal{AM}_0(\mathbb{R}^{m+1}) \\ \mathcal{A}(\mathbb{R}) \otimes \mathbb{R}_m \arrow[ur, "M_a"] \arrow[dr, "M_s"'] & \\ & \mathcal{SM}_0(\mathbb{R}^{m+1}) \arrow[uu, "\check{R}"'] \end{tikzcd} \tag{3.7}$$

As an illustration let us consider the axial monogenic extension of plane waves. From Example 2.2.1 and Remark 2.1 of [DS] we obtain the following.

Proposition 3.4 *The (degree zero) axial monogenic plane waves are given by*

$$M_a(\varphi_p)(x_0,\underline{x}) = M_a(e^{ipx_0}) = \Gamma(\frac{m}{2}) \left(\frac{2i}{p|\underline{x}|} \right)^{m/2-1} \left(I_{m/2-1}(p|\underline{x}|) + i \frac{\underline{x}}{|\underline{x}|}\, I_{m/2}(p|\underline{x}|) \right) e^{ipx_0}\,, \tag{3.8}$$

where I_α are the hyperbolic Bessel functions.

Proof. By representing, as in example 2.2.1 of [DS], $M_a(\varphi_p)(x_0)$ in the form

$$M_a(\varphi_p)(x_0,\underline{x}) = \sum_{j=0}^{\infty} c_j \underline{x}^j B_j e^{ipx_0},$$

and expressing the monogeneicity of the transform

$$(\partial_{x_0} + \partial_{\underline{x}}) \sum_{j=0}^{\infty} c_j \underline{x}^j B_j e^{ipx_0} = 0,$$

we obtain the following recurrence relation for the functions $B_j(x_0)$,

$$B_{j+1}(x_0) = -ipB_j(x_0) - B_j'(x_0), \quad B_0(x_0) = 1.$$

The solution is $B_j(x_0) = (-ip)^j$. Then we see that the transform is obtained by replacing $\underline{x}$ by $ip\underline{x}$ in the expressions of example 2.2.1 of [DS]. ■

Remark 3.5 From Theorem A of [CLSS], $\check{R} : \mathcal{SM}_0(\mathbb{R}^{m+1}) \to \mathcal{AM}_0(\mathbb{R}^{m+1})$ is an injective map. In fact, from Corollary 4.4 of [CLSS], we see that (non-zero) degree 0 slice monogenic functions do not belong to $\mathrm{Ker}\,\check{R}$. ◊

Remark 3.6 Note that, as in [DS], considering $h \in \mathcal{A}(\mathbb{R}) \otimes \mathbb{C}_m$, one also has,

$$M_a(h)(x_0, \underline{x}) = \sum_A \int_{S^{m-1}} h_A(x_0 + i\langle \underline{x}, \underline{t}\rangle)(1 - i\underline{t}))\, d\underline{t}\, e_A, \tag{3.9}$$

which is equivalent to (3.5) and can be readily verified by expansion in power series. $\Diamond$

4 Clifford extensions of the CST

The two extensions (3.2) and (3.5) give two natural paths to define coherent state transforms by replacing the vertical arrow of analytic continuation in the diagram (2.4).

We refer the reader interested in the representation theoretic and the quantum mechanical meaning of the Hilbert spaces introduced in the present section to section 5.

4.1 Slice monogenic coherent state transform (SCST)

The slice monogenic CST is naturally defined by substituting the vertical arrow in the diagram (2.4) by the slice monogenic extension (3.2) leading to

$$\begin{tikzcd} & \mathcal{SM}(\mathbb{R}^{m+1}) \otimes \mathbb{C} \\ L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m \arrow[ur, hook, "U_s"] \arrow[r, hook, "e^{\frac{\Delta_0}{2}}"'] & \widetilde{\mathcal{A}}(\mathbb{R}) \otimes \mathbb{C}_m \arrow[u, "M_s"'] \end{tikzcd} \tag{4.1}$$

where $\Delta_0 = \frac{d^2}{dx_0^2}$. Notice that even though the plane waves, $\varphi_p(x_0) = e^{ipx_0}$, are not in the Hilbert space $L^2(\mathbb{R}, dx_0)$, they are generalized eigenfunctions of Δ_0 with eigenvalue $-p^2$, and therefore

$$e^{\frac{\Delta_0}{2}}(\varphi_p)(x_0) = e^{\frac{\Delta_0}{2}}\, e^{ipx_0} = e^{-\frac{p^2}{2}}\, e^{ipx_0} = e^{-\frac{p^2}{2}}\varphi_p(x_0). \tag{4.2}$$

On the other hand since the plane waves $\varphi_p \in \mathcal{A}(\mathbb{R})$ we can use (3.2) to obtain the following very simple result.

Lemma 4.1 *The slice monogenic plane waves are given by*

$$M_s(\varphi_p)(x_0) = M_s(e^{ipx_0}) = e^{ip\vec{x}} = \left(\cosh(p|\underline{x}|) + i\frac{\underline{x}}{|\underline{x}|}\sinh(p|\underline{x}|)\right) e^{ipx_0}. \tag{4.3}$$

Proof. From (3.2) we obtain

$$M_s(\varphi_p)(x_0) = e^{ipx_0}\sum_{k=0}^{\infty}\frac{(ip\underline{x})^k}{k!} = \left(\cosh(p|\underline{x}|) + i\frac{\underline{x}}{|\underline{x}|}\sinh(p|\underline{x}|)\right) e^{ipx_0}.$$

■

Proposition 4.2 *Let $f \in L^2(\mathbb{R}, dx_0)$ and*

$$f(x_0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ipx_0} \, \tilde{f}(p) \, dp.$$

We have

$$\begin{aligned} U_s(f)(x_0, \underline{x}) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{p^2}{2}} \, e^{ip\vec{x}} \, \tilde{f}(p) \, dp = \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{p^2}{2}} \, e^{ipx_0} \, \cosh(p|\underline{x}|) \, \tilde{f}(p) \, dp + i \frac{\underline{x}}{|\underline{x}|} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{p^2}{2}} \, e^{ipx_0} \, \sinh(p|\underline{x}|) \, \tilde{f}(p) \, dp \end{aligned} \tag{4.4}$$

Proof. This result follows from Lemma (4.1), (3.2) and (4.2). ■

Consider the standard inner product on $\mathbb{C}_m$. Our main result in this section is the following.

Theorem 4.3 *The SCST, U_s in Diagram (4.1), is unitary onto its image for the measure $d\nu_m$ on $\mathbb{R}^{m+1}$ given by*

$$d\nu_m = \frac{2}{\sqrt{\pi}} \frac{1}{Vol(S^{m-1})} \frac{e^{-|\underline{x}|^2}}{|\underline{x}|^{m-1}} \, dx_0 d\underline{x},$$

where $Vol(S^{m-1})$ denotes the volume of the unit radius sphere in $\mathbb{R}^m$, i.e. the map U_s in the diagram

$$\begin{tikzcd} & \mathcal{H}_s \\ L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m \arrow[ur, "U_s"] \arrow[r, "e^{\frac{\Delta_0}{2}}"'] & \widetilde{\mathcal{A}}(\mathbb{R}) \otimes \mathbb{C}_m \arrow[u, "M_s"'] \end{tikzcd} \tag{4.5}$$

is a unitary isomorphism, where $\mathcal{H}_s = U_s(L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m) \subset \mathcal{SML}^2(\mathbb{R}^{m+1}, d\nu_m)$.

Proof. Let $\mathcal{S}(\mathbb{R})$ be the space of Schwarz functions on $\mathbb{R}$. For $f, h \in \mathcal{S}(\mathbb{R}) \otimes \mathbb{R}_m$, with $f = \sum_A f_A e_A$, $h = \sum_A h_A e_A$ we have

$$\begin{aligned} < U_s(f), U_s(h) > &= \frac{2}{\sqrt{\pi}} \frac{1}{Vol(S^{m-1})} \sum_A \int_{\mathbb{R}\times\mathbb{R}^m} \left[e^{2i\underline{x}p}\right]_0 \, e^{-p^2} \, \tilde{f}_A(p) \overline{\tilde{h}_A}(p) \, \frac{e^{-|\underline{x}|^2}}{|\underline{x}|^{m-1}} d^m x dp = \\ &= \frac{2}{\sqrt{\pi}} \frac{1}{Vol(S^{m-1})} \sum_A \int_{\mathbb{R}} e^{-p^2} \, \tilde{f}_A(p) \overline{\tilde{h}_A}(p) \, \left(\int_{\mathbb{R}^m} \cosh(2|\underline{x}|p) \frac{e^{-|\underline{x}|^2}}{|\underline{x}|^{m-1}} \, d^m x \right) dp = \\ &= \frac{2}{\sqrt{\pi}} \frac{1}{Vol(S^{m-1})} \sum_A \int_{\mathbb{R}} e^{-p^2} \, \tilde{f}_A(p) \overline{\tilde{h}_A}(p) \, \left(\int_0^\infty \cosh(2up) e^{-u^2} \, du \right) dp = \\ &= \sum_A \int_{\mathbb{R}} \tilde{f}_A(p) \overline{\tilde{h}_A}(p) \, dp =< f, h > . \end{aligned}$$

From the denseness of $\mathcal{S}(\mathbb{R}) \otimes \mathbb{C}$ in $L^2(\mathbb{R})$ we conclude that U_s is unitary onto its image. ■

Remark 4.4 For each complex plane $H_{\underline{\omega}} := \{u + v\underline{\omega} : u, v \in \mathbb{R}\}$ and for $f \in L^2(\mathbb{R}, dx) \otimes \mathbb{C}_m$, $f = \sum_A f_A \, e_A$, the map $f \mapsto U_s(f)\big|_{H_{\underline{\omega}}}$ coincides, for each component f_A of f, with the Segal–Bargmann transform, which is surjective to $\mathcal{H}L^2(H_{\underline{\omega}}, d\nu)$ and unitary for the measure $d\nu = e^{-v^2} du \, dv$ on $H_{\underline{\omega}}$. ◊

4.2 Axial monogenic coherent state transform (ACST)

The ACST is also naturally defined as the heat kernel evolution followed by axial Cauchy-Kowalewski extension

$$U_a = M_a \circ e^{\frac{\Delta_0}{2}}\,,$$

i.e. by substituting the vertical arrow in the diagram (2.2) by the axial monogenic extension (3.5)

$$\begin{tikzcd} & \mathcal{AM}(\mathbb{R}^{m+1}) \otimes \mathbb{C}_m \\ L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m \arrow[ur, hook, "U_a"] \arrow[r, hook, "e^{\frac{\Delta_0}{2}}"'] & \mathcal{A}(\mathbb{R}) \otimes \mathbb{C}_m \arrow[u, "M_a"'] \end{tikzcd} \tag{4.6}$$

Theorem 4.5 *Let $\mathcal{H}_a \subset \mathcal{AM}(\mathbb{R}^{m+1})$ denote the image of $L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m$ under U_a. The restriction of the dual Radon transform to $\mathcal{H}_s$ defines an isomorphism to $\mathcal{H}_a$.*

By defining the inner product, $\langle \cdot, \cdot \rangle_{\mathcal{H}_a}$,

$$\langle F, G\rangle_{\mathcal{H}_a} = \int_{\mathbb{R}^{m+1}} (\check{R})^{-1}(F)(\check{R})^{-1}(G) d\nu_m, \tag{4.7}$$

where $d\nu_m$ was defined in Theorem 4.3, U_a becomes a unitary map. The following diagram

$$\begin{tikzcd} & & \mathcal{H}_a \\ L^2(\mathbb{R}, dx_0) \otimes \mathbb{C}_m \arrow[urr, "U_a"] \arrow[r, "e^{\Delta_0/2}"] \arrow[drr, "U_s"'] & \widetilde{\mathcal{A}}(\mathbb{R}) \otimes \mathbb{C}_m \arrow[ur, "M_a"'] \arrow[dr, "M_s"] & \\ & & \mathcal{H}_s. \arrow[uu, "\check{R}"'] \end{tikzcd} \tag{4.8}$$

is commutative and its exterior arrows are unitary isomorphisms.

Proof. The injectivity of $\check{R}_{|\mathcal{H}_s}$ follows from the remark 3.5. From (3.6) we conclude that $U_a = \check{R} \circ U_s$ which implies the surjectivity of $\check{R}_{|\mathcal{H}_s} : \mathcal{H}_s \longrightarrow \mathcal{H}_a$. Then the inner product (4.7) is well defined, the diagram 4.8 is commutative and the exterior arrows are unitary isomorphisms. ■

Remark 4.6 As mentioned in the introduction, a possibly interesting alternative way of defining a monogenic CST would be by replacing the dual Radon transform in (3.7) and (4.8) by the Fueter mapping, $\Delta^{\frac{m-1}{2}}$ (see [F, Q, KQS, PQS, Sc]). Notice however that the map $\Delta^{\frac{m-1}{2}} \circ M_s$ does not correspond to a monogenic extension of entire functions of one variable as the restriction to the real line does not give back the original functions. It leads nevertheless to an interesting transform and it would be very interesting to relate it with U_a.
$\Diamond$

5 Representation theoretic and quantum mechanical interpretation

Recall that the Schrödinger representation in quantum mechanics is the representation for which the position operator $\hat{x}_0$ acts by multiplication on $L^2(\mathbb{R}, dx_0)$. The momentum operator is then given by

$$\hat{p}_0 = i\frac{d}{dx_0}.$$

The CST from Section 2.1 intertwines the Schrödinger representation with the Segal-Bargmann representation, on which the operator $\hat{x}_0+i\hat{p}_0$ acts as the operator of multiplication by the holomorphic function $x_0 + ip_0$ (see Theorem 6.3 of [Ha2])

$$\left(U \circ (\hat{x}_0 + i\hat{p}_0) \circ U^{-1}\right)(f)(x_0, p_0) = (x_0 + ip_0)f(x_0, p_0). \tag{5.1}$$

We will prove now the analogous result that the slice monogenic CST intertwines the Schrödinger representation with the representation on which $\hat{x}_0 + i\hat{p}_0$ acts as the operator of left multiplication by the slice monogenic function $x_0 + \underline{x}$.

Proposition 5.1 *The slice monogenic CST satisfies*

$$\left(U_s \circ (\hat{x}_0 + i\hat{p}_0) \circ U_s^{-1}\right)(f)(x_0, \underline{x}) = (x_0 + \underline{x})f(x_0, \underline{x}), \;\; f \in \mathcal{H}_s. \tag{5.2}$$

Proof. We have $U_s = M_s \circ e^{\frac{\Delta_0}{2}}$. From the injectivity of the slice monogenic extension map M_s, (5.2) is equivalent to

$$\left(e^{\frac{\Delta_0}{2}} \circ (x_0 - \frac{d}{dx_0}) \circ e^{-\frac{\Delta_0}{2}}\right)(f)(x_0) = x_0 f(x_0).$$

This follows from Theorem 6.3 of [Ha2]. ■

For the axial monogenic coherent state transform, on the other hand, we have a more complicated representation involving the Cauchy-Kowalesky extension of the polynomials $\underline{x}^j$, $j \in \mathbb{N}_0$.

Recall, from Theorem 2.2.1 of [DSS], that the Cauchy-Kowalesky extension of $\underline{x}^j$ is given by the polynomial $X_0^{(j)}(x_0, \underline{x})$, such that $X_0^{(j)}(0, \underline{x}) = \underline{x}^j$, where

$$X_0^{(j)}(x_0, \underline{x}) = CK(\underline{x}^j) = \mu_0^j |x|^j \left(C_j^{(m-1)/2}\left(\frac{x_0}{|x|}\right) + \frac{m-1}{m+j-1} C_{j-1}^{(m+1)/2}\left(\frac{x_0}{|x|}\right)\frac{\underline{x}}{|x|}\right),$$

with

$$\mu_0^{2j} = (-1)^j (C_{2j}^{(m-1)/2}(0))^{-1}, \;\; \mu_0^{2j+1} = (-1)^j \frac{m+2j}{m-1}(C_{2j}^{(m+1)/2}(0))^{-1}$$

and the Gegenbauer polynomials

$$C_j^{\nu}(y) = \sum_{i=0}^{[j/2]} \frac{(-1)^i (\nu)_{j-i}}{i!(j-2i)!}(2y)^{j-2i},$$

where $(\nu)_j = \nu(\nu+1)\cdots(\nu+j-1)$.

Proposition 5.2 *Let $f \in \mathcal{H}_a$ be given by*

$$f(x_0, \underline{x}) = \sum_{i=0}^{\infty} X_0^{(i)}(x_0, \underline{x})\, f_i. \tag{5.3}$$

The axial monogenic CST satisfies

$$\left(U_a \circ (\hat{x}_0 + i\hat{p}_0) \circ U_a^{-1}\right)(f)(x_0, \underline{x}) = \sum_{i=0}^{\infty} \left(\frac{2i+1}{2i+m} X_0^{(2i+1)}(x_0, \underline{x}) f_{2i} + X_0^{(2i+2)}(x_0, \underline{x}) f_{2i+1} \right). \tag{5.4}$$

Proof. From Theorem 3.4 of [CLSS], any entire monogenic function of degree zero has an expansion of the form (5.3). On the other hand, from equations (22) and (23) of [CLSS] we obtain

$$\begin{aligned} \left(\check{R} \circ (x_0 + \underline{x}) \circ \check{R}^{-1}\right)\left(X_0^{2j}\right) &= \frac{2j+1}{2j+m} X_0^{2j+1} \\ \left(\check{R} \circ (x_0 + \underline{x}) \circ \check{R}^{-1}\right)\left(X_0^{2j+1}\right) &= X_0^{2j+2}, \qquad j \in \mathbb{N}_0. \end{aligned}$$

These identities, together with the Proposition 5.1 and the fact that $U_a = \check{R} \circ U_s$ prove (5.4). ■

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