Cosilting complexes and AIR-cotilting modules*

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Abstract

We introduce and study the new concepts of cosilting complexes, cosilting modules and AIR-cotilting modules. We prove that the three concepts AIR-cotilting modules, cosilting modules and quasi-cotilting modules coincide with each other, in contrast with the dual fact that AIR-tilting modules, silting modules and quasi-tilting modules are different. Further, we show that there are bijections between the following four classes (1) equivalent classes of AIR-cotilting (resp., cosilting, quasi-cotilting) modules, (2) equivalent classes of 2-term cosilting complexes, (3) torsion-free cover classes and (4) torsion-free special precover classes. We also extend a classical result of Auslander and Reiten on the correspondence between certain contravariantly finite subcategories and cotilting modules to the case of cosilting complexes.

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1 Introduction

The tilting theory is well known, and plays an important role in the representation theory of Artin algebra. The classical notion of tilting and cotilting modules was first considered in the case of finite dimensional algebras by Brenner and Butler [10] and by Happel and Ringel [17]. Cotilting theory (for arbitrary modules over arbitrary unital rings) extends Morita duality in analogy to the way tilting theory extends Morita equivalence. In particular, cotilting modules generalize injective cogenerators similarly as tilting modules generalize progenerators. Later, many scholars have done a lot of research on the tilting theory and cotilting theory, for instance [3, 5, 6, 7, 13, 14, 22, 26, 29] and so on.

The silting theory seems to be the tilting theory in the level of derived categories (while the tilting complexes play the role of progenerators). Silting complexes were first introduced by Keller and Vossieck [19] to study t-structures in the bounded derived category of representations of Dynkin quivers. Beginning with [2], such objects were recently shown to have various nice properties [20, 21]. The results in [27] show that silting complexes (i.e., semi-tilting complexes in [27]) have similar properties as that tilting modules have in the module categories. The recent paper by Buan and Zhou [12] also shows that it is reasonable to see the silting theory as the tilting theory in the level of derived categories.

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The τ-tilting theory recently introduced by Adachi, Iyama and Reiten [1] is an important generalization of the classical tilting theory. In particular, it was shown that support τ-tilting modules have close relations with 2-term silting complexes and cluster-tilting objects [1]. In [4], the authors introduce silting modules as a generalization of support τ-tilting modules over arbitrary rings and modules. We note that there is also another little different generalization of support τ-tilting modules over arbitrary rings, called large support τ-tilting modules, which was introduced by the second author [28].

In this paper, we concentre on the dual case, i.e., the correspondent cotilting parts to the above achievements. In the level of finitely generated modules over artin algebras, such dual cases proceed very well. So we only consider the dual over arbitrary rings and modules. As one will see, there are many interesting properties in such case.

Let us briefly introduce the contents and main results of this paper in the following.

After the introduction in Section 1, Section 2 is devoted to studying cosilting complexes. Namely, a complex T over a ring R is cosilting if it satisfies the following three conditions:

(1) $T \in \mathcal{K}^b(\mathrm{Inj}R)$,

(2) T is prod-semi-selforthogonal and,

(3) $\mathcal{K}^b(\mathrm{Inj}R)$ is just the smallest triangulated subcategory containing $\langle \mathrm{Adp}_{\mathcal{D}}T \rangle$, where $\mathrm{Adp}_{\mathcal{D}}T$ denotes the class of complexes isomorphic in the derived category $\mathcal{D}(\mathrm{Mod}R)$ to a direct summand of some direct products of copies of T.

It is clear that a cotilting complex [11] is cosilting. We show that an R-module is a cotilting module if and only if it is isomorphic in the derived category to a cosilting complex. Some characterizations of cosilting complexes are obtained. In particular, we extend a simple characterization of cotilting modules [7] to cosilting complexes (Theorem 2.14). In [6], Auslander and Reiten showed that, over an artin algebra, there is a one-one correspondence between certain contravariantly finite subcategories and basic cotilting modules. The result was extended to the derived category of artin algebras by Buan [11], where the author proved that there is a one-one correspondence between basic cotilting complexes and certain contravariantly finite subcategories of the derived category. Here, we further extend the result to cosilting complexes and to arbitrary rings (Theorem 2.17).

In Section 3, we study quasi-cotilting modules and cosilting modules. In the tilting case, it is known that a silting module is always a finendo quasi-tilting module but the converse is not true in general [25]. However, in the dual case, we see that quasi-cotilting modules are always cofinendo and that they are also pure-injective [30]. We show that cosilting modules are always quasi-cotilting modules, and consequently, cosilting modules are pure injective and cofinendo. Interesting properties and characterizations of cosilting modules are also given in this section.

In Section 4, we introduce and study AIR-cotilting modules. We call an R-module M AIR-tilting if it is large support τ-tilting in sense of [28], i.e., it satisfies the following two conditions:

(1) there is an exact sequence $P_1 \to^f P_0 \to M \to 0$ with P_1, P_0 projective such that $\mathrm{Hom}(f, M^{(X)})$ is surjective for any set X and,

(2) there is an exact sequence $R \to^g M_0 \to M_1 \to 0$ with $M_0, M_1 \in \mathrm{Add}M$ such that $\mathrm{Hom}(g, M^{(X)})$ is surjective for any set X.

Dually, we call an R-module M AIR-cotilting if it satisfies the following two conditions:

(1) there is an exact sequence $0 \to T \to I_0 \to^f I_1$ with I_0, I_1 injective such that $\mathrm{Hom}(T^X, f)$ is surjective for any set X and,

(2) there is an exact sequence $0 \to T_1 \to T_0 \to^g Q$ with $T_0, T_1 \in \mathrm{Adp}T$ and Q an injective cogenerator such that $\mathrm{Hom}(T^X, g)$ is surjective for any set X.

Clearly, 1-tilting modules are AIR-tilting modules and 1-cotilting modules are AIR-cotilting modules. In the tilting case, a silting module is always AIR-tilting and an AIR-tilting module can be completed to a silting module [4, 28]. But it is a question if these two notions are the same in general (they are the same in the scope of finitely generated modules over artin algebras). It is also known that AIR-tilting modules are finendo quasi-tilting. But the converse is not true in general [25]. However, in the dual case, we prove that AIR-cotilting modules coincide with cosilting modules, as well as quasi-cotilting modules (Theorem 4.18). Moreover, we also show that there is a 1-1 correspondence between equivalent classes of AIR-cotilting modules and 2-term cosilting complexes (Theorem 4.12).

Summarized, we obtain the following main results.

Theorem 1.1 *There is a one-one correspondence, given by* $u : T \mapsto {}^{\perp_{i>0}}T$, *between equivalent classes of cosilting complexes in* $\mathcal{D}^{\geqslant}$ *and subcategories* $\mathcal{T} \subseteq \mathcal{D}^{\geqslant}$ *which is specially contravariantly finite in* $\mathcal{D}^{+}$, *resolving and closed under products such that* $\widehat{\mathcal{T}} = \mathcal{D}^{+}$.

Theorem 1.2 *Let* R *be a ring and* M *be an* R*-module. The following statements are equivalent:*

(1) M *is AIR-cotilting;*

(2) M *is quasi-cotilting.*

(3) M *is cosilting.*

Theorem 1.3 *There are bijections between*

(1) *equivalent classes of AIR-cotilting (resp., cosilting, quasi-cotilting) modules;*

(2) *equivalent classes of 2-term cosilting complexes;*

(3) *torsion-free cover classes and,*

(4) *torsion-free special precover classes.*

Throughout this paper, R will denote an associative ring with identity and we mainly work on the category of left R-modules which is denoted by Mod$R$. We denote by Inj$R$ (resp., Proj$R$) the class of all injective (resp., projective) R-modules. The notations $\mathcal{K}^b(\mathrm{Inj}R)$ (resp., $\mathcal{K}^b(\mathrm{Proj}R)$) denotes the homotopy category of bounded (always cochain) complexes of injective (resp., projective) modules. The unbounded derived category of Mod$R$ will be denoted by $\mathcal{D}(R)$, or simply $\mathcal{D}$, with [1] shift functor. We denoted by $\mathcal{D}^{\geqslant}$ the subcategory of complexes whose homologies are concentrated on non-negative terms. We use $\mathcal{D}^{+}$ to denote the subcategory of $\mathcal{D}$ consists of bounded-below complexes.

Note that $\mathcal{D}$ is a triangulated category and $\mathcal{K}^b(\mathrm{Inj}R)$, $\mathcal{K}^b(\mathrm{Proj}R)$, $\mathcal{D}^{+}$ are all full triangulated subcategories of $\mathcal{D}$. We refer to Happel's paper [16] for more details on derived categories and triangulated categories.

2 Subcategories of the derived category and cosilting complexes

In this section we study the dual of silting complexes and give various characterizations of cosilting complexes. In particular, we extend a result of Bazzoni [7] and establish a one-one correspondence between certain subcategories of the derived category $\mathcal{D}$ and the equivalent classes of cotilting complexes stemming from Auslander and Reiten [6].

We begin with some basic notations and some useful facts in general triangulated categories.

Let $\mathcal{C}$ be a triangulated category with [1] the shift functor. Assume that $\mathcal{B}$ is a full subcategory of $\mathcal{C}$. Recall that $\mathcal{B}$ is closed under extension if for any triangle $X \to Y \to Z \to$ in $\mathcal{C}$ with $X, Z \in \mathcal{C}$, we have $V \in \mathcal{C}$. The subcategory $\mathcal{B}$ is resolving (resp., coresolving) if it is closed under extension and under the functor $[-1]$ (resp., [1]). It is easy to prove that $\mathcal{B}$ is resolving (resp., coresolving) if and only if for any triangle $X \to Y \to Z \to$ (resp., $Z \to Y \to X \to$) in $\mathcal{B}$ with $Z \in \mathcal{B}$, one has that ' $X \in \mathcal{B} \Leftrightarrow Y \in \mathcal{B}$ '.

We say that an object $M \in \mathcal{C}$ has a $\mathcal{B}$-resolution (resp., $\mathcal{B}$-coresolution) with the length at most m ($m \geqslant 0$), if there are triangles $M_{i+1} \to X_i \to M_i \to$ (resp., $M_i \to X_i \to M_{i+1} \to$) with $0 \leqslant i \leqslant m$ such that $M_0 = M$, $M_{m+1} = 0$ and each $X_i \in \mathcal{B}$. In the case, we denoted by $\mathcal{B}$-res.dim$(L) \leqslant m$ (resp., $\mathcal{B}$-cores.dim$(L) \leqslant m$). One may compare such notions with the usual finite resolutions and coresolutions respectively in the module category.

Associated with a subcategory $\mathcal{B}$, we have the following notations which are widely used in the tilting theory (see for instance [6]), where $n \geqslant 0$ and m is an integer.

$(\widehat{\mathcal{B}})_n = \{L \in \mathcal{C} \mid \mathcal{B}\text{-res.dim}(L) \leqslant n\}$.
$(\check{\mathcal{B}})_n = \{L \in \mathcal{C} \mid \mathcal{B}\text{-cores.dim}(L) \leqslant n\}$.
$\widehat{\mathcal{B}} \quad = \{L \in \mathcal{C} \mid L \in (\widehat{\mathcal{B}})_n \text{ for some } n\}$.
$\check{\mathcal{B}} \quad = \{L \in \mathcal{C} \mid L \in (\check{\mathcal{B}})_n \text{ for some } n\}$.
$\mathcal{B}^{\perp_{i \neq 0}} = \{N \in \mathcal{C} \mid \mathrm{Hom}(M, N[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i \neq 0\}$.
${}^{\perp_{i \neq 0}}\mathcal{B} = \{N \in \mathcal{C} \mid \mathrm{Hom}(N, M[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i \neq 0\}$.
$\mathcal{B}^{\perp_{i > m}} = \{N \in \mathcal{C} \mid \mathrm{Hom}(M, N[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i > m\}$.
${}^{\perp_{i > m}}\mathcal{B} = \{N \in \mathcal{C} \mid \mathrm{Hom}(N, M[i]) = 0 \text{ for all } M \in \mathcal{B} \text{ and all } i > m\}$.
$\mathcal{B}^{\perp_{i \gg 0}} = \{N \in \mathcal{C} \mid N \in \mathcal{B}^{\perp_{i > m}} \text{ for some } m\}$.

Note that $\mathcal{B}^{\perp_{i > m}}$ (resp., ${}^{\perp_{i > m}}\mathcal{B}$) is coresolving (resp., resolving) and closed under direct summands and that $\mathcal{B}^{\perp_{i \gg 0}}$ is a triangulated subcategory of $\mathcal{C}$.

The subcategory $\mathcal{B}$ is said to be semi-selforthogonal (resp., selforthogonal) if $\mathcal{B} \subseteq \mathcal{B}^{\perp_{i > m}}$ (resp., $\mathcal{B} \subseteq \mathcal{B}^{\perp_{i \neq 0}}$). For instance, both subcategories Proj$R$ and Inj$R$ are selforthogonal.

In the following results of this section, we always assume that $\mathcal{B}$ is semi-selforthogonal and that $\mathcal{B}$ is additively closed (i.e., $\mathcal{B} = \mathrm{add}_{\mathcal{C}}\mathcal{B}$ where $\mathrm{add}_{\mathcal{C}}\mathcal{B}$ denotes the subcategory of all objects in $\mathcal{C}$ which are isomorphic to a direct summand of finite direct sums of copies of objects in $\mathcal{B}$).

Associated with the subcategory $\mathcal{B}$, we also have the following two useful subcategories which are again widely used in the tilting theory (see for instance [6]).

$\mathcal{X}_{\mathcal{B}} \quad = \{N \in {}^{\perp_{i>0}}\mathcal{B} \mid$ there are triangles $N_i \to B_i \to N_{i+1} \to$ such that $N_0 = N$, $N_i \in {}^{\perp_{i>0}}\mathcal{B}$ and $B_i \in \mathcal{B}$ for all $i \geqslant 0\}$.

${}_{\mathcal{B}}\mathcal{X} \quad = \{N \in \mathcal{B}^{\perp_{i>0}} \mid$ there are triangles $N_{i+1} \to B_i \to N_i \to$ such that $N_0 = N$, $N_i \in \mathcal{B}^{\perp_{i>0}}$ and $B_i \in \mathcal{B}$ for all $i \geqslant 0\}$.

We summarize some results on subcategories associated with $\mathcal{B}$ in the following, where $\langle \mathcal{B} \rangle$ denotes the smallest triangulated subcategory containing $\mathcal{B}$. We refer to [27] for their proofs.

Proposition 2.1 *Let $\mathcal{B}$ be a semi-selforthogonal subcategory of a triangulated category $\mathcal{C}$ such that $\mathcal{B}$ is additively closed. Then*

(1) *The three subcategories $\check{\mathcal{B}} \subseteq \mathcal{X}_{\mathcal{B}} \subseteq {}^{\perp_{i>0}}\mathcal{B}$ is resolving and closed under direct summands.*
(2) *The three subcategories $\widehat{\mathcal{B}} \subseteq {}_{\mathcal{B}}\mathcal{X} \subseteq \mathcal{B}^{\perp_{i>0}}$ is coresolving and closed under direct summands.*
(3) $\mathcal{B} = \check{\mathcal{B}} \bigcap \mathcal{B}^{\perp_{i>0}} = \widehat{\mathcal{B}} \bigcap {}^{\perp_{i>0}}\mathcal{B}$.

(4) $(\check{\mathcal{B}})_n = \mathcal{X}_{\mathcal{B}} \bigcap (\mathcal{X}_{\mathcal{B}})^{\perp_{i>n}} = \mathcal{X}_{\mathcal{B}} \bigcap ({}^{\perp_{i>0}}\mathcal{B})^{\perp_{i>n}}$. *In particular, it is closed under extensions and direct summands.*

(5) $(\widehat{\mathcal{B}})_n = {}_{\mathcal{B}}\mathcal{X} \bigcap {}^{\perp_{i>n}}({}_{\mathcal{B}}\mathcal{X}) = {}_{\mathcal{B}}\mathcal{X} \bigcap {}^{\perp_{i>n}}(\mathcal{B}^{\perp_{i>0}})$. *In particular, it is closed under extensions and direct summands.*

(6) *The following three subcategories coincide with each other.*

  (*i*) $\langle \mathcal{B} \rangle$*: the smallest triangulated subcategory containing* $\mathcal{B}$*;*

  (*ii*) $(\widehat{\mathcal{B}})_- = \{X \in \mathcal{C} |$ *there exists some* $Y \in \widehat{\mathcal{B}}$ *and some* $i \leqslant 0$ *such that* $X = Y[i]\}$*;*

  (*iii*) $(\check{\mathcal{B}})_+ = \{X \in \mathcal{C} |$ *there exists some* $Y \in \check{\mathcal{B}}$ *and some* $i \geqslant 0$ *such that* $X = Y[i]\}$*.*

(7) $\widehat{\mathcal{B}} = \mathcal{B}^{\perp_{i>0}} \bigcap \langle \mathcal{B} \rangle$.

(8) $\check{\mathcal{B}} = {}^{\perp_{i>0}}\mathcal{B} \bigcap \langle \mathcal{B} \rangle$.

We also need the following results.

Lemma 2.2 *Suppose that* $n \geqslant 1$ *and there are triangles* $L_i \to M_i \to L_{i+1} \to$ *with each* $M_i \in \mathcal{X}_{\mathcal{B}}$*, where* $0 \leqslant i \leqslant n-1$*. Then there exist* $X_n, Y_n \in \mathcal{C}$ *such that*

(1) $Y_n \in \mathcal{X}_{\mathcal{B}}$*,*

(2) *there is a triangle* $L_n \to X_n \to Y_n \to$*, and*

(3) *there are triangles* $X_{i-1} \to B_{i-1} \to X_i \to$ *with each* $B_{i-1} \in \mathcal{B}$*, for all* $1 \leqslant i \leqslant n$*, where* $X_0 = L_0$*.*

Proof. We use induction on n to prove this conclusion.

For $n = 1$, there is a triangle $M_0 \to B_0 \to Y_1 \to$ with $B_0 \in \mathcal{B}$ and $Y_1 \in \mathcal{X}_{\mathcal{B}}$ since $M_0 \in \mathcal{X}_{\mathcal{B}}$. Then we can get the following triangle commutative diagram:

$$\begin{array}{cccccc}
L_0 & \to & M_0 & \to & L_1 & \to \\
\downarrow & & \downarrow & & \downarrow & \\
L_0 & \to & B_0 & \to & X_1 & \to \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & Y_1 & \to & Y_1 & \to \\
\downarrow & & \downarrow & & \downarrow &
\end{array}$$

Obviously, X_1 and Y_1 in diagram above are just the objects we look for.

We suppose that the result holds for $n-1$. Next, we will verify that the result holds for n. According to the known condition, we have the triangle $L_{n-1} \to M_{n-1} \to L_n \to$ with $M_{n-1} \in \mathcal{X}_{\mathcal{B}}$. Using the induction on L_{n-1}, one can obtain some triangles $L_{n-1} \to X_{n-1} \to Y_{n-1} \to$ with $Y_{n-1} \in \mathcal{X}_{\mathcal{B}}$ and $X_{i-1} \to B_{i-1} \to X_i \to$ with $B_i \in \mathcal{B}$, for all $1 \leqslant i \leqslant n-1$, where $X_0 = L_0$. Hence we have the following triangle commutative diagram:

$$\begin{array}{cccccc}
L_{n-1} & \to & M_{n-1} & \to & L_n & \to \\
\downarrow & & \downarrow & & \downarrow & \\
X_{n-1} & \to & H & \to & L_n & \to \\
\downarrow & & \downarrow & & \downarrow & \\
Y_{n-1} & \to & Y_{n-1} & \to & 0 & \to \\
\downarrow & & \downarrow & & \downarrow &
\end{array}$$

From the second column in diagram above, we can obtain that $H \in \mathcal{X}_\mathcal{B}$ by Proposition 2.1. So one have a triangle $H \to B_{n-1} \to X_n \to$ with $B_{n-1} \in \mathcal{B}$, $X_n \in \mathcal{X}_\mathcal{B}$. Consequently, one have the following triangle commutative diagram:

$$\begin{array}{ccccccc}
X_{n-1} & \to & H & \to & L_n & \to \\
\downarrow & & \downarrow & & \downarrow & \\
X_{n-1} & \to & B_{n-1} & \to & X_n & \to \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & Y_n & \to & Y_n & \to \\
\downarrow & & \downarrow & & \downarrow &
\end{array}$$

It is easy to see that X_n and Y_n from diagram above are just the objects we want. $\square$

Corollary 2.3 *For any $L \in (\widehat{\mathcal{X}_\mathcal{B}})_n$, then there are two triangles $L \to X \to Y \to$ with $X \in (\widehat{\mathcal{B}})_n$, $Y \in \mathcal{X}_\mathcal{B}$ and $U \to V \to L \to$ with $U \in (\widehat{\mathcal{B}})_{n-1}$ and $V \in {}^{\perp_{i>0}}\mathcal{B}$.*

Proof. For any $L \in (\widehat{\mathcal{X}_\mathcal{B}})_n$, we have triangles $L_i \to M_i \to L_{i+1} \to$ with $M_i \in \mathcal{X}_\mathcal{B}$, where $0 \leqslant i \leqslant n$, $L_0 = 0, L_{n+1} = L$. By Lemma 2.2, we obtain the triangle $L \to X \to Y \to$ with $X \in (\widehat{\mathcal{B}})_n$, $Y \in \mathcal{X}_\mathcal{B}$. Since $X \in (\widehat{\mathcal{B}})_n$, one can get a triangle $U \to B_0 \to X \to$ with $T_0 \in \mathcal{B}$ and $U \in (\widehat{\mathcal{B}})_{n-1}$. Then we have the following triangles commutative diagram:

$$\begin{array}{ccccccc}
U & \to & U & \to & 0 & \to \\
\downarrow & & \downarrow & & \downarrow & \\
V & \to & B_0 & \to & Y & \to \\
\downarrow & & \downarrow & & \downarrow & \\
L & \to & X & \to & Y & \to \\
\downarrow & & \downarrow & & \downarrow &
\end{array}$$

From the second row in diagram above, one can easy see that $V \in {}^{\perp_{i>0}}\mathcal{B}$ since $Y, T_0 \in {}^{\perp_{i>0}}\mathcal{B}$. Hence the triangle $U \to V \to L \to$ is just what we want. $\square$

Now let R be a ring and T be a complex. Recall that $\mathrm{Adp}_\mathcal{D}T$ denotes the class of complexes isomorphic in the derived category $\mathcal{D}$ to a direct summand of some direct products of T. We say that T is *prod-semi-selforthogonal* if $\mathrm{Adp}_\mathcal{D}T$ is semi-selforthogonal. It is easy to see that $\mathrm{Adp}_\mathcal{D}T$ is additively closed in this case. So the results above applies when we set $\mathcal{B} = \mathrm{Adp}_\mathcal{D}T$.

We introduce the following definition.

Definition 2.4 *A complex T is said to cosilting if it satisfies the following conditions:*

(1) $T \in \mathcal{K}^b(\mathrm{Inj}R)$,

(2) T *is prod-semi-selforthogonal, and*

(3) $\mathcal{K}^b(\mathrm{Inj}R) = \langle \mathrm{Adp}_\mathcal{D}T \rangle$, *i.e.,* $\mathcal{K}^b(\mathrm{Inj}R)$ *coincides with the smallest triangulated subcategory containing* $\mathrm{Adp}_\mathcal{D}T$.

Now let Q be an injective cogenerator for $\mathrm{Mod}R$. Recall that $\mathcal{D}^+$ is the triangulated subcategory of the derived category $\mathcal{D}$ consists of bounded-below complexes, $\mathcal{K}^b(\mathrm{Inj}R)$ is homotopy category of bounded complexes of injective modules. Also, recall that $\mathcal{D}^{\geqslant}$ is the subcategory of the complexes whose homologies are concentrated on non-negative terms. It is not difficult to verify that $\mathcal{D}^{\geqslant} = {}^{\perp_{i>0}}Q$ and $\mathcal{D}^{+} = {}^{\perp_{i\gg 0}}Q$.

The following result gives a characterization of cosilting complexes.

Theorem 2.5 *Let T be a complex and Q be an injective cogenerator of* Mod$R$. *Up to shifts, we may assume that $T \in \mathcal{D}^{\geqslant}$. Then T is cosilting if and only if it satisfies the following three conditions:*

(i) $T \in \widetilde{\mathrm{Adp}_{\mathcal{D}}Q}$,

(ii) T *is prod-semi-selforthogonal, and*

(iii) $Q \in \widehat{\mathrm{Adp}_{\mathcal{D}}T}$.

Proof. $\Leftarrow$ Since $T \in \widetilde{\mathrm{Adp}_{\mathcal{D}}Q}$, there are triangles $T_i \to Q_i \to T_{i+1} \to$ with $Q_i \in \mathrm{Adp}_{\mathcal{D}}Q$ for all $0 \leqslant i \leqslant n$, where $T_0 = T, T_{n+1} = 0$. It is easy to see that each $T_i \in \mathcal{K}^b(\mathrm{Inj}R)$, for $i = n, n-1, \cdots, 0$, since $\mathrm{Adp}_{\mathcal{D}}Q$ =Inj $R \subseteq \mathcal{K}^b(\mathrm{Inj}R)$. In particular, $T \in \mathcal{K}^b(\mathrm{Inj}R)$. Now we need only prove that $\mathcal{K}^b(\mathrm{Inj}R) = \langle \mathrm{Adp}_{\mathcal{D}}T\rangle$. Note that $\mathrm{Adp}_{\mathcal{D}}Q \subseteq \widehat{\mathrm{Adp}_{\mathcal{D}}T}$ by Proposition 2.1, since $Q \in \widehat{\mathrm{Adp}_{\mathcal{D}}T}$. and that $T \in \mathcal{K}^b(\mathrm{Inj}R)$, so we have

$$\mathcal{K}^b(\mathrm{Inj}R) = \langle \mathrm{Adp}_{\mathcal{D}}Q\rangle \subseteq \langle \widehat{\mathrm{Adp}_{\mathcal{D}}T}\rangle = \langle \mathrm{Adp}_{\mathcal{D}}T\rangle \subseteq \mathcal{K}^b(\mathrm{Inj}R).$$

Thus, T is cosilting.

$\Rightarrow$ Since $T \in \mathcal{D}^{\geqslant} = {}^{\perp_{i>0}}Q$ and $T \in \langle \mathrm{Adp}_{\mathcal{D}}T\rangle = \mathcal{K}^b(\mathrm{Inj}R) = \langle \mathrm{Adp}_{\mathcal{D}}Q\rangle$, one can get that $T \in \widetilde{\mathrm{Adp}_{\mathcal{D}}Q}$ by Proposition 2.1. Note that $\mathrm{Adp}_{\mathcal{D}}T \subseteq {}^{\perp_{i>0}}Q$, since $T \in \mathcal{D}^{\geqslant} = {}^{\perp_{i>0}}Q$, so $Q \in (\mathrm{Adp}_{\mathcal{D}}T)^{\perp_{i>0}}$. Combining with the fact that $Q \in \mathcal{K}^b(\mathrm{Inj}R) = \langle \mathrm{Adp}_{\mathcal{D}}T\rangle$, we have that $Q \in \widehat{\mathrm{Adp}_{\mathcal{D}}T}$ by Proposition 2.1. $\square$

Recall that an R-module T is (n-)cotilting (see for instance [7]) if it satisfies the following three conditions (1) id$T \leqslant n$, i.e., the injective dimension of T is finite, (2) $\mathrm{Ext}^i_R(T^X, T) = 0$ for any X and, (3) there is an exact sequence $0 \to T_n \to \cdots \to T_0 \to Q \to 0$, where $T_i \in \mathrm{Adp}_R T$ and Q is an injective cogenerator of Mod$R$.

Proposition 2.6 *Assume that T is an R-module. Then T is a cotilting module if and only if T is isomorphic in the derived category to a cosilting complex.*

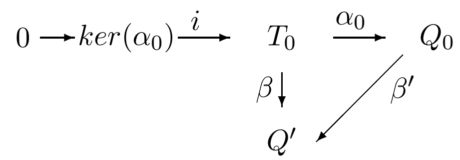
Proof. $\Rightarrow$ Since short exact sequences give triangles in the derive category, it is easy to see that every cotilting module is cosilting in the derived category by Theorem 2.5.

$\Leftarrow$ Note that there is a faithful embedding from Mod$R$ into $\mathcal{D}$. i.e., for any two modules $M, N \in$ Mod$R$, we have that $\mathrm{Hom}_{\mathcal{D}}(M, N) \cong \mathrm{Hom}_R(M, N)$. Moreover, we have $\mathrm{Hom}_{\mathcal{D}}(M, N[i]) \cong \mathrm{Ext}^i_R(M, N)$ for all $i > 0$ and for any two modules M, N. So the condition (2) in the definition of cotilting modules is satisfied.

As to the condition (1) in the definition of cotilting modules, since T is isomorphic in the derived category to a cosilting complex and $T \in \mathcal{D}^{\geqslant}$, by Theorem 2.5, we have that $T \in \widetilde{\mathrm{Adp}_{\mathcal{D}}Q}$. i.e., there are triangles $T_i \to^{\alpha_i} Q_i \to T_{i+1} \to$ with $Q_i \in \mathrm{Adp}_{\mathcal{D}}Q$ for all $0 \leqslant i \leqslant n$, where $T_0 = T$, $T_{n+1} = 0$. We will show that these triangles are in Mod$R$ and hence give short exact sequences in Mod$R$.

Consider firstly the triangle $T_0 \to^{\alpha_0} Q_0 \to T_1 \to$, where $T_0 = T$ is already an R-module. Then $\alpha_0 \in \mathrm{Hom}_{\mathcal{D}}(T_0, Q_0) \simeq \mathrm{Hom}_R(T_0, Q_0)$ shows that α_0 is homomorphism between modules. We claim that α_0 is injective. To see this, taking any momomorphism β: $T_0 \to Q'$ with Q' an injective module. Since $T \in \widetilde{\mathrm{Adp}_{\mathcal{D}}Q}$, it is easy to see that all $T_i \in {}^{\perp_{i>0}}Q$. Hence

$\mathrm{Hom}_R(\alpha_0, Q') \cong \mathrm{Hom}_{\mathcal{D}}(\alpha_0, Q')$ is surjective. Then we have the following commutative diagram in Mod$R$ for some homomorphisms β'.



From the diagram above, we have that $\beta\alpha_0 = \beta'\alpha_0 i = 0$. Note that β is injective, so we obtain that $i = 0$ and consequently, α_0 is injective. Then we have an exact sequence $0 \to T_0 \to Q_0 \to \mathrm{Coker}(\alpha_0) \to 0$ which induces a triangle $T_0 \to Q_0 \to \mathrm{Coker}(\alpha_0) \to$. It follows that $T_1 \cong \mathrm{Coker}(\alpha_0)$ is (quasi-isomorphic to) an R-module. Repeating discussion above for all i, we can get that each α_i is injective and each T_i is (quasi-isomorphic to) an R-module.

Note that $T_n = Q_n$. By discussion above, we can get a long exact sequence $0 \to T \to Q_0 \to Q_1 \to \cdots \to Q_n \to 0$. So id$T \leqslant n$, i.e., the condition (1) in the definition of cotilting modules is satisfied.

Finally, still by Theorem 2.5, we have that $Q \in \widehat{\mathrm{Adp}_{\mathcal{D}}T}$. Similarly to the above process, we can get a long exact sequence $0 \to T_n \to \cdots \to T_1 \to T_0 \to Q \to 0$ with $T_i \in \mathrm{Adp}T$, i.e., the condition (3) in the definition of cotilting modules is satisfied. □

Proposition 2.7 *Suppose that $T \in \mathcal{D}^{\geqslant}$ is a cosilting complex.*

(1) *If $S \bigoplus T$ is also a cosilting complex for some S, then $S \in \mathrm{Adp}_{\mathcal{D}}T$.*

(2) *If there are triangles $T_i \to Q_i \to T_{i+1} \to$ with $Q_i \in \mathrm{Adp}_{\mathcal{D}}Q$ for all $0 \leqslant i \leqslant n$, where $T_0 = T$, $T_{n+1} = 0$, then $\mathrm{Adp}_{\mathcal{D}}(\bigoplus_{i=0}^n Q_i) = \mathrm{Adp}_{\mathcal{D}}Q$.*

(3) *If there are triangles $Q_{i+1} \to T_i \to Q_i \to$ with $T_i \in \mathrm{Adp}_{\mathcal{D}}T$ for all $0 \leqslant i \leqslant m$, where $Q_0 = Q$, $Q_{m+1} = 0$, then $\bigoplus_{i=0}^m T_i$ is a cosilting complex. Moreover, $\mathrm{Adp}_{\mathcal{D}}(\bigoplus_{i=0}^m T_i) = \mathrm{Adp}_{\mathcal{D}}T$.*

Proof. (1) Since $S \bigoplus T$ is a cosilting complex, we have that $\langle \mathrm{Adp}_{\mathcal{D}}(S \bigoplus T)\rangle = \mathcal{K}^b(\mathrm{Inj}R) = \langle \mathrm{Adp}_{\mathcal{D}}T\rangle$. It is easy to verify that $S \in {}^{\perp_{i>0}}T$ and $S \in (\mathrm{Adp}_{\mathcal{D}}T)^{\perp_{i>0}}$ since $S \bigoplus T$ is prod-semi-selforthogonal. It follows from Proposition 2.1 that

$$S \in {}^{\perp_{i>0}}T \bigcap \langle \mathrm{Adp}_{\mathcal{D}}(S \bigoplus T)\rangle = {}^{\perp_{i>0}}T \bigcap \langle \mathrm{Adp}_{\mathcal{D}}T\rangle = \widetilde{\mathrm{Adp}_{\mathcal{D}}T}.$$

Hence $S \in \widetilde{\mathrm{Adp}_{\mathcal{D}}T} \bigcap (\mathrm{Adp}_{\mathcal{D}}T)^{\perp_{i>0}} = \mathrm{Adp}_{\mathcal{D}}T$ by Proposition 2.1.

(2) Obviously, $\bigoplus_{i=0}^n Q_i \in \mathrm{Adp}_{\mathcal{D}}Q$ is prod-semi-selforthogonal. It is not difficult to see that $\mathrm{Adp}_{\mathcal{D}}T \subseteq \langle \bigoplus_{i=0}^n Q_i\rangle$ by Proposition 2.1. It follows that

$$\langle \mathrm{Adp}_{\mathcal{D}}Q\rangle = \mathcal{K}^b(\mathrm{Inj}R) = \langle \mathrm{Adp}_{\mathcal{D}}T\rangle \subseteq \langle \mathrm{Adp}_{\mathcal{D}}(\bigoplus_{i=0}^{n} Q_i)\rangle \subseteq \langle \mathrm{Adp}_{\mathcal{D}}Q\rangle.$$

Hence $\langle \mathrm{Adp}_{\mathcal{D}}(\bigoplus_{i=0}^n Q_i)\rangle = \langle \mathrm{Adp}_{\mathcal{D}}Q\rangle$. Clearly, $(\bigoplus_{i=0}^n Q_i) \bigoplus Q$ is cosilting. It follows from (1) that $\mathrm{Adp}_{\mathcal{D}}(\bigoplus_{i=0}^n Q_i) = \mathrm{Adp}_{\mathcal{D}}Q$.

(3) It is easy to verify that both $(\bigoplus_{i=0}^m T_i) \oplus T$ and $\bigoplus_{i=0}^m T_i$ are cosilting by Theorem 2.5. Consequently, we have that $\mathrm{Adp}_{\mathcal{D}}(\bigoplus_{i=0}^m T_i) = \mathrm{Adp}_{\mathcal{D}}T$ by (1). □

Proposition 2.8 *Let $T \in \mathcal{D}^{\geqslant}$ be a cosilting complex and $n \geqslant 0$. Then $T \in (\widetilde{\mathrm{Adp}_{\mathcal{D}}Q})_n$ if and only if $Q \in (\widehat{\mathrm{Adp}_{\mathcal{D}}T})_n$.*

Proof. $\Rightarrow$ We have that $Q \in (\widehat{\mathrm{Adp}_{\mathcal{D}}T})_m$ for some m, by Theorem 2.5. If $m \leqslant n$, then the conclusion holds clearly. Suppose that $m > n$. There are triangles $Q_{i+1} \to T_i \to Q_i \to$ with

$T_i \in \mathrm{Adp}_{\mathcal{D}}T$ for $0 \leqslant i \leqslant m$, where $Q_0 = Q$, $Q_{m+1} = 0$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, Q_m)$ to these triangles, we can obtain that

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}}(Q_{m-1}, Q_m[1]) &\simeq \mathrm{Hom}_{\mathcal{D}}(Q_{m-2}, Q_m[2]) \\ &\simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}}(Q_0, Q_m[m]) = \mathrm{Hom}_{\mathcal{D}}(Q, Q_m[m]).\end{aligned}$$

It is not difficult to verify that $\mathrm{Adp}_{\mathcal{D}}T \subseteq ({}^{\perp_{i>0}}Q)^{\perp_{i>n}}$ since $T \in (\widetilde{\mathrm{Adp}_{\mathcal{D}}Q})_n$. Then we have that $\mathrm{Hom}_{\mathcal{D}}(Q, Q_m[t]) = 0$ for $t > n$ since $Q \in {}^{\perp_{i>0}}Q$ and $Q_m = T_m \in \mathrm{Adp}_{\mathcal{D}}T$. Consequently, $\mathrm{Hom}_{\mathcal{D}}(Q_{m-1}, Q_m[1]) = 0$ and the triangle $T_m = Q_m \to T_{m-1} \to Q_{m-1} \to$ is split. Hence $Q_{m-1} \in \mathrm{Adp}_{\mathcal{D}}T$ and $Q \in (\widehat{\mathrm{Adp}_{\mathcal{D}}T})_{m-1}$. By continuing this process, we can finally obtain that $Q \in (\widehat{\mathrm{Adp}_{\mathcal{D}}T})_n$.

$\Leftarrow$ The proof is just the dual of above statement. □

The following is an easy observation.

Lemma 2.9 *Suppose that $T \in \mathcal{D}$ is prod-semi-selforthogonal, then ${}^{\perp_{i>0}}T = \mathcal{X}_{\mathrm{Adp}_{\mathcal{D}}T}$.*

Proof. Clearly, $\mathcal{X}_{\mathrm{Adp}_{\mathcal{D}}T} \subseteq {}^{\perp_{i>0}}T$.

Take any $M \in {}^{\perp_{i>0}}T$ and consider the triangle $M \to^{\alpha} T^X \to M_1 \to$, where α is the canonical evaluation map. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(-, T)$ to this triangle, we can obtain that $\mathrm{Hom}_{\mathcal{D}}(M_1, T[i]) = 0$ for all $i > 0$. i.e., $M_1 \in {}^{\perp_{i>0}}T$. Continuing this process, we get triangles $M_j \to T_j \to M_{j+1}$ with $T_j \in \mathrm{Adp}_{\mathcal{D}}T$ and $M_j \in {}^{\perp_{i>0}}T$ for all $j \geqslant 0$, where $M_0 = M$. Consequently, $M \in \mathcal{X}_{\mathrm{Adp}_{\mathcal{D}}T}$ by the definition. So ${}^{\perp_{i>0}}T \subseteq \mathcal{X}_{\mathrm{Adp}_{\mathcal{D}}T}$ and the conclusion holds. □

We say a complex is partial cosilting, if it satisfies the first two conditions in Definition 2.4.

Proposition 2.10 *If $T \in \mathcal{D}^{\geqslant}$ is partial cosilting. Then T is cosilting if and only if ${}^{\perp_{i>0}}T \subseteq \mathcal{D}^{\geqslant}$.*

Proof. $\Rightarrow$ By Theorem 2.5 (3), there are triangles $Q_{i+1} \to T_i \to Q_i \to$ with $T_i \in \mathrm{Adp}_{\mathcal{D}}T$ for all $0 \leqslant i \leqslant n$, where $Q_0 = Q, Q_{n+1} = 0$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(M, -)$ to these triangles, where $M \in {}^{\perp_{i>0}}T$, we can get that $M \in {}^{\perp_{i>0}}Q$. Hence ${}^{\perp_{i>0}}T \subseteq {}^{\perp_{i>0}}Q = \mathcal{D}^{\geqslant}$.

$\Leftarrow$ It is not difficult to verify that $Q \in {}^{\perp_{i>n}}T$ for some n, so $Q[-n] \in {}^{\perp_{i>0}}T$. Then there are triangles $Q[-i-1] \to 0 \to Q[-i] \to$ for all $0 \leqslant i \leqslant n-1$. Hence $Q \in (\widehat{{}^{\perp_{i>0}}T})_n$. We can obtain a triangle $Q \to X \to Y \to$ with $X \in (\widehat{\mathrm{Adp}_{\mathcal{D}}T})_n, Y \in {}^{\perp_{i>0}}T$ by Corollary 2.3 and Lemma 2.9. Note that ${}^{\perp_{i>0}}T \subseteq \mathcal{D}^{\geqslant}$. So $Y \in {}^{\perp_{i>0}}Q$ and the triangle $Q \to X \to Y \to$ is split. Hence $Q \in (\widehat{\mathrm{Adp}_{\mathcal{D}}T})_n$ since $(\widehat{\mathrm{Adp}_{\mathcal{D}}T})_n$ is closed under direct summands. Consequently, T is a cosilting complex. □

We say that a complex $T \in \mathcal{D}^{\geqslant}$ is n-cosilting if it is a cosilting complex such that $Q \in (\widehat{\mathrm{Adp}_{\mathcal{D}}T})_n$. A characterization of n-cotilting modules says that an R-module T is n-cotilting if and only if $\mathrm{Cogen}^n T = \mathrm{KerExt}_R^{i>0}(-, T)$, see [7] for more details. We will present a similar characterization of n-cosilting complexes in the following.

We need the following subcategory of $\mathcal{D}$. Let $T \in \mathcal{D}$ and $n > 0$, we denote

$$\begin{aligned}\mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T) = \{M \in \mathcal{D} \mid &\text{ there exist some triangles } M_i \to T_i \to M_{i+1} \to \text{ with} \\ &T_i \in \mathrm{Adp}_{\mathcal{D}}T \text{ for all } 0 \leqslant i < n, \text{ where } M_n \in \mathcal{D}^{\geqslant} \text{ and } M_0 = M\ \}.\end{aligned}$$

It is not difficult to verify that $\mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T)$ is closed under products. The following result gives more properties about this subcategory.

Lemma 2.11 (1) $\mathcal{D}^{\geqslant}[-n] \subseteq \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T)$.

(2) *If* $T \in \mathcal{D}^{\geqslant}$, *then* $\mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T) \subseteq \mathcal{D}^{\geqslant}$.

Proof. Since $0 \in \mathrm{Adp}_{\mathcal{D}}T$ and $\mathcal{D}^{\geqslant}$ is resolving, it is easy to verify the conclusions by the definitions. $\square$

Proposition 2.12 *Assume that* $T \in \mathcal{D}^{\geqslant}$ *is* n*-cosilting. Then* ${}^{\perp_{i>0}}T = \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T)$.

Proof. By Proposition 2.10 and Lemma 2.9, we get that ${}^{\perp_{i>0}}T \subseteq {}^{\perp_{i>0}}Q = \mathcal{D}^{\geqslant}$ and ${}^{\perp_{i>0}}T = \mathcal{X}_{\mathrm{Adp}_{\mathcal{D}}T}$. In particular, ${}^{\perp_{i>0}}T \subseteq \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T)$.

Now we prove that $\mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T) \subseteq {}^{\perp_{i>0}}T$. For any $M \in \mathcal{D}^{\geqslant} = {}^{\perp_{i>0}}Q$, it is not difficult to verify that $M \in {}^{\perp_{i>n}}T$. Take any $N \in \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T)$, then there are triangles $N_i \to T_i \to N_{i+1} \to$ with $T_i \in \mathrm{Adp}_{\mathcal{D}}T$ for all $0 \leqslant i < n$, where $N_n \in \mathcal{D}^{\geqslant}$ and $N_0 = N$. Applying the functor $\mathrm{Hom}_{\mathcal{D}}(T, -)$ to these triangles, we have that

$$\mathrm{Hom}_{\mathcal{D}}(N_0, T[i]) \cong \mathrm{Hom}_{\mathcal{D}}(N_1, T[i+1]) \cong \cdots \cong \mathrm{Hom}_{\mathcal{D}}(N_n, T[i+n]) = 0,\ i > 0.$$

Hence $\mathrm{Hom}_{\mathcal{D}}(N, T[i]) = \mathrm{Hom}_{\mathcal{D}}(N_0, T[i]) = 0$, $i > 0$ and $N \in {}^{\perp_{i>0}}T$. i.e., $\mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T) \subseteq {}^{\perp_{i>0}}T$. The proof is then completed. $\square$

It is well known that $T \in \mathcal{K}^b(\mathrm{Inj}R)$ if and only if $\mathcal{D}^{+} \subseteq {}^{\perp_{i\gg 0}}T$.

Proposition 2.13 *Assume that* $T \in \mathcal{D}^{\geqslant}$. *If* ${}^{\perp_{i>0}}T = \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T)$, *then* T *is* n*-cosilting.*

Proof. Note that $T \in \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T) = {}^{\perp_{i>0}}T$. Since $\mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T)$ is closed under products, we have that $\mathrm{Adp}_{\mathcal{D}}T \subseteq {}^{\perp_{i>0}}T$. Hence T is prod-semi-selforthogonal.

From Lemma 2.11, we know that $\mathcal{D}^{\geqslant}[-n] \subseteq \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T) = {}^{\perp_{i>0}}T$. So $\mathcal{D}^{\geqslant} \subseteq {}^{\perp_{i>n}}T$. In particular, $\mathcal{D}^{+} \subseteq {}^{\perp_{i\gg 0}}T$. Consequently, $T \in \mathcal{K}^b(\mathrm{Inj}R)$. It follows that $T \in (\mathrm{Adp}_{\mathcal{D}}Q)_m$ for some m from the argument above and Proposition 2.1.

We note that ${}^{\perp_{i>0}}T = \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T) \subseteq \mathcal{D}^{\geqslant}$ by Lemma 2.11 (2). Hence, by Proposition 2.10, T is m-cosilting complex, where m is the integer given in the last paragraph. Since $T \in \mathrm{Adp}_{\mathcal{D}}Q$. then there are triangles $T_i \to Q_i \to T_{i+1} \to$ with $Q_i \in \mathrm{Adp}_{\mathcal{D}}Q$ for all $0 \leqslant i \leqslant m$, where $T_{m+1} = 0$ and $T_0 = T$. Applying the functor $\mathrm{Hom}(Q_m, -)$ to these triangles, we can obtain that

$$\mathrm{Hom}_{\mathcal{D}}(Q_m, T_{m-1}[1]) \cong \mathrm{Hom}_{\mathcal{D}}(Q_m, T_{m-2}[2]) \cong \cdots \cong \mathrm{Hom}_{\mathcal{D}}(Q_m, T_0[m]) = \mathrm{Hom}_{\mathcal{D}}(Q_m, T[m]).$$

Noted that $Q[-n] \in \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T) = {}^{\perp_{i>0}}T$ since $0 \in \mathrm{Adp}_{\mathcal{D}}T$. Hence $\mathrm{Hom}_{\mathcal{D}}(Q, T[i+n]) = 0$, for any $i > 0$. If $m \leqslant n$, then T is clearly n-cosilting. If $m > n$, it follows from the discussion above that $T_{m-1} \to Q_{m-1} \to Q_m \to$ is split. i.e., $T \in (\mathrm{Adp}_{\mathcal{D}}Q)_{m-1}$. Repeating this process, we finally get that $T \in (\mathrm{Adp}_{\mathcal{D}}Q)_n$. Consequently, T is n-cosilting. $\square$

Combining Proposition 2.12 and Proposition 2.13, we obtain the following characterization of n-cosilting complexes.

Theorem 2.14 *Assume that* $T \in \mathcal{D}^{\geqslant}$. *Then the following are equivalent:*

(1) T *is* n*-cosilting,*

(2) ${}^{\perp_{i>0}}T = \mathrm{Copres}^n_{\mathcal{D}^{\geqslant}}(\mathrm{Adp}_{\mathcal{D}}T)$.

In [6], Auslander and Reiten showed that there is a one-one correspondence between isomorphism classes of basic cotilting modules and certain contravariantly finite resolving subcategories. Extending this result, Buan [11] showed that there is a one-to-one correspondence between basic cotilting complexes and certain contravariantly finite subcategories of the bounded derived category of an artin algebra. In the following, we aim to extend such a result to cosilting complexes.

We need the following definitions. Let $\mathcal{X} \subseteq \mathcal{Y}$ be two subcategories of $\mathcal{D}$. $\mathcal{X}$ is said to be contravariantly finite in $\mathcal{Y}$, if for any $Y \in \mathcal{Y}$, there is a homomorphism $f : X \to Y$ for some $X \in \mathcal{X}$ such that $\mathrm{Hom}_{\mathcal{D}}(X', f)$ is surjective for any $X' \in \mathcal{X}$. Moreover, $\mathcal{X}$ is said to be specially contravariantly finite in $\mathcal{Y}$, if for any $Y \in \mathcal{Y}$, there is triangle $U \to X \to Y \to$ with some $X \in \mathcal{X}$ such that $\mathrm{Hom}_{\mathcal{D}}(X', U[1]) = 0$ for any $X' \in \mathcal{X}$. Note that in the later case, one has that $U \in \mathcal{X}^{\perp_{i>0}}$ if $\mathcal{X}$ is closed under $[-1]$.

Proposition 2.15 *Assume that $T \in \mathcal{D}^{\geqslant}$ is cosilting. Then $\widehat{{}^{\perp_{i>0}}T} = \mathcal{D}^+$ and ${}^{\perp_{i>0}}T \subseteq \mathcal{D}^{\geqslant}$ is specially contravariantly finite in $\mathcal{D}^+$.*

Proof. We have proved that ${}^{\perp_{i>0}}T \subseteq \mathcal{D}^{\geqslant}$ in Proposition 2.10, so we get that $\widehat{{}^{\perp_{i>0}}T} \subseteq \mathcal{D}^+$. Now we take any $X \in \mathcal{D}^+$. It is easy to see that $X \in {}^{\perp_{i>m}}T$ for some m since $T \in \mathcal{K}^b(\mathrm{Inj}R)$. Consequently, $X[-m] \in {}^{\perp_{i>0}}T$. Note that $0 \in \mathrm{Adp}_{\mathcal{D}}T$, so we have $X \in \widehat{{}^{\perp_{i>0}}T}$ by the definition. Hence $\widehat{{}^{\perp_{i>0}}T} = \mathcal{D}^+$.

By Lemma 2.9, we have that ${}^{\perp_{i>0}}T = \mathcal{X}_{\mathrm{Adp}_{\mathcal{D}}T}$. Taking any $X \in \mathcal{D}^+ = \widehat{{}^{\perp_{i>0}}T} = \widehat{\mathcal{X}_{\mathrm{Adp}_{\mathcal{D}}T}}$, by Corollary 2.3, we obtain a triangle $U \to V \to X \to$ with $U \in \widehat{\mathrm{Adp}_{\mathcal{D}}T}$ and $V \in {}^{\perp_{i>0}}T$. Note that $\widehat{\mathrm{Adp}_{\mathcal{D}}T} \subseteq ({}^{\perp_{i>0}}T)^{\perp_{i>0}}$, so particularly we get that $\mathrm{Hom}_{\mathcal{D}}(M, U[1]) = 0$ for any $M \in {}^{\perp_{i>0}}T$. It follows that ${}^{\perp_{i>0}}T$ is specially contravariantly finite in $\mathcal{D}^+$. □

Proposition 2.16 *Assume that $\mathcal{T} \subseteq \mathcal{D}^{\geqslant}$ is specially contravariantly finite in $\mathcal{D}^+$ and is resolving such that $\widehat{\mathcal{T}} = \mathcal{D}^+$. If $\mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$ is closed under products, then there is a cosilting complex T such that $\mathcal{T} = {}^{\perp_{i>0}}T$.*

Proof. It is not difficult to verify that $\mathcal{D}^+ = \widehat{\mathcal{T}} \subseteq {}^{\perp_{i\gg 0}}(\mathcal{T}^{\perp_{i>0}})$. Hence we can obtain that $\mathcal{T}^{\perp_{i>0}} \subseteq \mathcal{K}^b(\mathrm{Inj}R)$.

Taking any $M \in \mathcal{D}^+$, since $\mathcal{T}$ is specially contravariantly finite in $\mathcal{D}^+$ and is resolving, there are triangles $M_{j+1} \to T_j \to M_j \to$ with $T_j \in \mathcal{T}$ for all $j \geqslant 0$, where $M_0 := M$ and each $M_j \in \mathcal{T}^{\perp_{i>0}}$ for $j \geqslant 1$. It follows that $T_j \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$ for all $j \geqslant 1$. Since $\mathcal{T}^{\perp_{i>0}} \subseteq \mathcal{K}^b(\mathrm{Inj}R)$ and $M \in \mathcal{D}^+$, it is easy to see that $M \in {}^{\perp_{i>n}}(\mathcal{T}^{\perp_{i>0}})$ for some n depending on M. Applying $\mathrm{Hom}_{\mathcal{D}}(-, M_{n+1})$ to the triangles above, we obtain that $\mathrm{Hom}_{\mathcal{D}}(M_n, M_{n+1}[1]) \simeq \cdots \simeq \mathrm{Hom}_{\mathcal{D}}(M, M_{n+1}[n+1]) = 0$. Thus, the triangle $M_{n+1} \to T_n \to M_n \to$ is split and so M_n is a direct summand of T_n. Note that the conditions $\mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$ is closed under products and $\mathcal{T}$ is resolving imply that $\mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$ is closed under direct summands, so we have that $M_n \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$.

Recall that Q is an injective cogenerator in $\mathrm{Mod}R$. Note that $Q \in \mathcal{T}^{\perp_{i>0}}$ since $\mathcal{T} \subseteq \mathcal{D}^{\geqslant} = {}^{\perp_{i>0}}Q$. Specially the object M in the above to be Q, we obtain triangles $Q_{j+1} \to T'_j \to Q_j \to$ with $Q_j \in \mathcal{T}^{\perp_{i>0}}$ and $T'_j \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$ for all $0 \leqslant j \leqslant n$, where $Q_0 = Q$ and $Q_{n+1} = 0$. Taking $T = \bigoplus_{j=0}^{n} T'_j$. We will show that T is cosilting. It is easy to see that T is precosilting since $\mathcal{T}^{\perp_{i>0}} \subseteq \mathcal{K}^b(\mathrm{Inj}R)$ and $\mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$ is closed under products. Moreover, the argument above shows that $Q \in \widehat{\mathrm{Adp}_{\mathcal{D}}T}$ too. Hence T is cosilting.

Now we need only prove that $\mathcal{T} = {}^{\perp_{i>0}}T$. Obviously, we have that $\mathcal{T} \subseteq {}^{\perp_{i>0}}T$ since $T = \bigoplus_{j=0}^{n} T'_j$ and $T'_j \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$ for all $0 \leqslant j \leqslant n$. Taking any $N \in {}^{\perp_{i>0}}T$. Similar to the discussion above, there are triangle $N_{j+1} \to T''_j \to N_j \to$ with $N_j \in \mathcal{T}^{\perp_{i>0}}$, $T''_0 \in \mathcal{T}$ and $T''_j \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$ for all $1 \leqslant j \leqslant m$, where $N_0 = N$ and $N_{m+1} = 0$. Note that all objects in these triangles are in ${}^{\perp_{i>0}}T$. For any $L \in \mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}}$, it is easy to verify that $T \bigoplus L$ is also cosilting, hence $L \in \mathrm{Adp}_{\mathcal{D}}T$ by Proposition 2.7 (1). It follows that $\mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}} \subseteq \mathrm{Adp}_{\mathcal{D}}T$. Now it is easy to see that $\mathcal{T} \bigcap \mathcal{T}^{\perp_{i>0}} = \mathrm{Adp}_{\mathcal{D}}T$. So the above triangles imply that $N_1 \in \widehat{\mathrm{Adp}_{\mathcal{D}}T}$ and consequently, $N_1 \in {}^{\perp_{i>0}}T \bigcap \widehat{\mathrm{Adp}_{\mathcal{D}}T} = \mathrm{Adp}_{\mathcal{D}}T$. Thus, the triangle $N_1 \to T_0 \to N \to$ is split. It follows that $N \in \mathcal{T}$ from the facts that $T_0, N_1 \in \mathcal{T}$ and that $\mathcal{T}$ is resolving. So we obtain that ${}^{\perp_{i>0}T} \subseteq \mathcal{T}$. The proof is then completed. □

By Propositions 2.15 and 2.16, we obtain the following desired result. Here, we say two complexes M and N are equivalent if $\mathrm{Adp}_{\mathcal{D}}M = \mathrm{Adp}_{\mathcal{D}}N$.

Theorem 2.17 *There is a one-one correspondence, given by $u : T \mapsto {}^{\perp_{i>0}}T$, between equivalent class of cosilting complexes in $\mathcal{D}^{\geqslant}$ and subcategories $\mathcal{T} \subseteq \mathcal{D}^{\geqslant}$ which is specially contravariantly finite in $\mathcal{D}^{+}$, resolving and closed under products such that $\widehat{\mathcal{T}} = \mathcal{D}^{+}$.*

Proof. It follows from Propositions 2.15 and 2.16 that the correspondence is well-defined. Moreover, u is surjective by Proposition 2.16. If both T_1 and T_2 are cosilting with ${}^{\perp_{i>0}}T_1 = {}^{\perp_{i>0}}T_2$, it is easy to varify that $T_1 \bigoplus T_2$ is also cosilting by the definition. So we have that $\mathrm{Adp}_{\mathcal{D}}T_1 = \mathrm{Adp}_{\mathcal{D}}T_2$ by Proposition 2.7, i.e., T_1 and T_2 are equivalent. Hence, u is bijective. □

3 Quasi-cotilting modules and cosilting modules

In this section, we introduce cosilting modules which is the dual of silting modules introduced in [4]. We study their relationship with quasi-cotilting modules and provide some characterizations of cosilting modules. In particular, we obtain that all cosilting modules are pure-injective and cofinendo and that every presilting module has a Bongartz complement.

Let R be a ring and $\mathcal{U}$ be a class of R-modules. Following [7], we denote by $\mathrm{Gen}^n\mathcal{U}$ the class of all modules M such that there is an exact sequence $U_n \to \cdots \to U_1 \to M \to 0$ with each $U_i \in \mathcal{U}$. Note that $\mathrm{Gen}^1\mathcal{U}$ and $\mathrm{Gen}^2\mathcal{U}$ are often denoted by $\mathrm{Gen}\mathcal{U}$ and $\mathrm{Pres}\mathcal{U}$ respectively. Dually, we denote by $\mathrm{Cogen}^n\mathcal{U}$ the class of all modules M such that there is an exact sequence $0 \to M \to U_1 \to \cdots \to U_n$ with each $U_i \in \mathcal{U}$. Also we have $\mathrm{Cogen}\mathcal{U} = \mathrm{Cogen}^1\mathcal{U}$ and $\mathrm{Copres}\mathcal{U} = \mathrm{Cogen}^2\mathcal{U}$.

We simply denote $\mathrm{Gen}^n\mathcal{U}$ with Gen^nT in case that $\mathcal{U} = \mathrm{Add}T$ for some R-module T, where $\mathrm{Add}T$ denotes the class of modules which is a direct summand of some direct sums of copies of T. Dually, we simply denote $\mathrm{Cogen}^n\mathcal{U}$ with Cogen^nT in case that $\mathcal{U} = \mathrm{Adp}T$ for some R-module T, where $\mathrm{Adp}T$ denotes the class of modules which is a direct summand of some direct products of copies of T. We have similar simple notations $\mathrm{Gen}T$, $\mathrm{Pres}T$, $\mathrm{Cogen}T$, $\mathrm{Copres}T$.

Now let T be an R-module. Recall that T is an n-star module if $\mathrm{Gen}^nT = \mathrm{Gen}^{n+1}T$ and $\mathrm{Hom}_R(T, -)$ preserve the exactness of exact sequences in Gen^nT [26]. Dually, one call that T is an n-costar module if $\mathrm{Cogen}^nT = \mathrm{Cogen}^{n+1}T$ and $\mathrm{Hom}_R(-, T)$ preserve the exactness of exact sequences in Cogen^nT [18].

Let $\mathcal{U}$ be a class of R-modules. We denote by $\mathrm{KerExt}^1_R(\mathcal{U},-)$ the class of all R-modules M such that $\mathrm{Ext}^1_R(U,M) = 0$ for all $U \in \mathcal{U}$. We have similar notations such as $\mathrm{KerExt}^1_R(-,\mathcal{U})$.

Recall that an R-module E is Ext-injective in $\mathcal{U}$ if $E \in \mathcal{U} \bigcap \mathrm{KerExt}^1_R(\mathcal{U},-)$, and dually, an R-module E is Ext-projective in $\mathcal{U}$ if $E \in \mathcal{U} \bigcap \mathrm{KerExt}^1_R(-,\mathcal{U})$.

We have the following definitions [4, 13, 30].

Definition 3.1 (1) *An R-module M is called a quasi-tilting module, if it is a 1-star module and is Ext-projective in* $\mathrm{Gen}M$.

(2) *An R-module M is called a quasi-cotilting module, if it is a 1-costar module and is Ext-injective in* $\mathrm{Cogen}M$.

We will say that two quasi-cotilting modules M, N are equivalent if $\mathrm{Adp}M = \mathrm{Adp}N$.

Let Q be an injective cogenerator of $\mathrm{Mod}R$. Following [5], an R-module M is called Q-cofinendo if there exist a cardinal γ and a map $f\colon\ M^{\gamma} \to Q$ such that for any cardinal α, all maps $M^{\alpha} \to Q$ factor through f. An R-module M is cofinendo if there is some injective cogenerator Q of $\mathrm{Mod}R$ such that M is Q-cofinendo.

A short exact sequence $0 \to A \to B \to C \to 0$ is called pure exact if the induced sequence $0 \to N \otimes_R A \to N \otimes_R B \to N \otimes_R C \to 0$ is still exact for any right R-module N. An R-module M is called pure injective if $\mathrm{Hom}_R(-,M)$ preserves the exactness of all pure exact sequences. We note that a remarkable properties of cotilting modules is that they are pure-injective [8, 24].

Let $\mathcal{U}$ be a class of R-modules and N be an R-module. Recall that a homomorphism $f : U \to N$ is called a precover, or a right $\mathcal{U}$-approximation, of N if $U \in \mathcal{U}$ and $\mathrm{Hom}_R(U', f)$ is surjective for any $U' \in \mathcal{U}$. A $\mathcal{U}$-precover $f : U \to N$ of N is called a $\mathcal{U}$-cover, or a minimal right $\mathcal{U}$-approximation, of M if any $g : U \to U$ such that $f = fg$ must be an isomorphism. A $\mathcal{U}$-precover $f : U \to N$ of N is called special if $\mathrm{Ker}f \in \mathrm{KerExt}^1_R(\mathcal{U},-)$. A class $\mathcal{U}$ of R-modules is said to be a precover class, or contrvariantly finite, provided that every R-module has a $\mathcal{U}$-precover. Cover classes and special precover classes are defined similarly.

Recall that a class $\mathcal{U}$ of R-modules is torsion-free if $\mathcal{U}$ is closed under direct products, submodules and extensions, see [15].

We collect some import results on quasi-cotilting modules from [30] in the following proposition.

Proposition 3.2 *Let M be an R-module and Q be an injective cogenerator of* $\mathrm{Mod}R$.

(1) *If* $\mathrm{Cogen}M \subseteq \mathrm{KerExt}^1_R(-,M)$, *then* $(\mathrm{KerHom}_R(-,M), \mathrm{Cogen}M)$ *is a torsion pair.*

(2) *All quasi-cotilting modules are pure-injective and cofinendo;*

(3) *M is quasi-cotilting module if and only if M is Ext-injective in* $\mathrm{Cogen}M$ *and there is an exact sequence $0 \to M_1 \to M_0 \to^{\alpha} Q$ with $M_0, M_1 \in$* $\mathrm{Adp}M$ *such that α is a* $\mathrm{Cogen}M$*-precover.*

(4) *M is 1-cotilting if and only if M is quasi-cotilting and $Q \in$* $\mathrm{Gen}(\mathrm{Cogen}M)$.

(5) *There are one-one correspondences between the following three classes*

(i) equivalent classes of quasi-cotilting modules,

(ii) torsionfree cover classes and,

(iii) torsionfree specially precover classes.

We now turn to cosilting modules.

Firstly, let us recall the definition of silting modules given in [4]. Let $\sigma : P_1 \to P_0$ be in $\mathrm{Proj}R$ and $\mathcal{D}_\sigma$ be the class of all R-modules N such that $\mathrm{Hom}_R(\sigma, N)$ is surjective. Then an

R-module M is said to be presilting if there is some σ in Proj$R$ such that $M = \mathrm{Ker}\sigma \in \mathcal{D}_\sigma$ and that $\mathcal{D}_\sigma$ is a torsion class. Moreover, an R-module M is said to be silting if there is some σ in Proj$R$ such that $M = \mathrm{Ker}\sigma$ and $\mathcal{D}_\sigma = \mathrm{Gen}M$. Note that silting modules are always presilting [4].

As for the duall case, we take a homomorphism $\sigma : E_0 \to E_1$ in Inj$R$, and consider the associated class of R-modules

$$\mathcal{F}_\sigma = \{M \in \mathrm{Mod}R \mid \mathrm{Hom}_R(M, \sigma) \text{ is surjective}\}.$$

The following result gives some useful properties of $\mathcal{F}_\sigma$. Here we say that a homomorphism $\sigma : E_0 \to E_1$ in Inj$R$ is an injective copresentation of M if $M \simeq \mathrm{Ker}\sigma$.

Lemma 3.3 *Let σ be a homomorphism in* Inj$R$ *with $K = \mathrm{Ker}\sigma$. Then the following assertions hold:*

(1) *$\mathcal{F}_\sigma$ is closed under submodules, extensions and direct sums;*

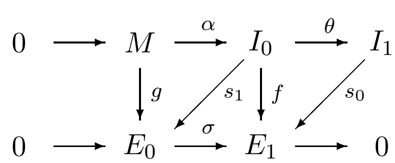
(2) *$\mathcal{F}_\sigma \subseteq \mathrm{KerExt}^1_R(-, K)$;*

(3) *An R-module M belongs to $\mathcal{F}_\sigma$ if and only if $\mathrm{Hom}_{\mathcal{D}}(\theta[-1], \sigma) = 0$, for any injective copresentation θ of M (here θ, σ are considered as a complex with terms concentred on 0-th and 1-th positions).*

Proof. (1) Easily.

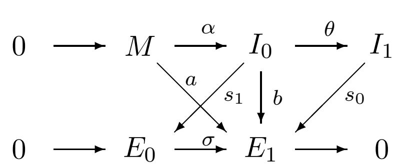
(2) Factor σ: $E_0 \to E_1$ canonically as $\sigma = i\pi$ with i: $C \to E_1$ and π: $E_0 \to C$, where $C = \mathrm{Im}\sigma$. For any $M \in \mathcal{F}_\sigma$, applying $\mathrm{Hom}_R(M, -)$ to the exact sequence $0 \to K \to E_0 \to^{\pi} C \to 0$, we have an induced exact sequence $\mathrm{Hom}_R(M, E_0) \to^{\mathrm{Hom}_R(M,\pi)} \mathrm{Hom}_R(M, C) \to \mathrm{Ext}^1_R(M, K) \to 0$. It is easy to verify that $\mathrm{Hom}_R(M, \pi)$ is surjective since that $\mathrm{Hom}_R(M, \sigma)$ is surjective implies $\mathrm{Hom}_R(M, i)$ is isomorphic. So $\mathrm{Ext}^1_R(M, K) = 0$.

(3) ($\Rightarrow$) Set θ: $I_0 \to I_1$ to be an injective copresentation of M and write $\theta = i\pi$ with i: $\mathrm{Im}\theta \to I_1$ and π: $I_0 \to \mathrm{Im}\theta$. For any morphism f: $I_0 \to E_1$, We consider the following commutative diagram, where the second row is a complex.



Since $M \in \mathcal{F}_\sigma$, we have a morphism g such that $f\alpha = \sigma g$. There is a morphism s_1 such that $g = s_1\alpha$ since E_0 is injective. And $(f - \sigma s_1)\alpha = 0$, thus, there is a morphism h: $\mathrm{Im}\theta \to E_1$ such that $f - \sigma s_1 = h\pi$. Since E_1 is injective and i is monomorphic, we have a homomorphism s_0 such that $h = s_0 i$. It is easy to see that $f = \sigma s_1 + s_0\theta$.

($\Leftarrow$) For any morphism a: $M \to E_1$, consider the following commutative diagram



Since E_1 is injective, we have a morphism b such that $a = b\alpha$. Hence we have s_1 and s_0 such that $b = \sigma s_1 + s_0\theta$ by the assumption that $\mathrm{Hom}_{\mathcal{D}}(\theta[-1], \sigma) = 0$. It is not difficult to verify that $a = \sigma s_1\alpha$. Thus $M \in \mathcal{F}_\sigma$. □

We also need the following result. The proof is simple, so we left to the reader.

Lemma 3.4 (1) *Let a and b be a morphism in* Inj$R$*, then* $\mathcal{F}_{a\oplus b} = \mathcal{F}_a \bigcap \mathcal{F}_b$.

(2) *Let α: $I_0 \to I$ and β: $I_1 \to I$ be two morphisms in* Inj$R$*, then* $\mathcal{F}_\alpha \subseteq \mathcal{F}_{(\alpha,\beta)}$.

We now give the definition of cosilting modules.

Definition 3.5 (1) *An R-module M is called precosilting if there is an injective copresentation σ of M such that $M \in \mathcal{F}_\sigma$ and $\mathcal{F}_\sigma$ is a torsion-free class.*

(2) *An R-module M is called cosilting if there is an injective copresentation σ of M such that* $\mathcal{F}_\sigma = \mathrm{Cogen}M$.

Remark 3.6 (1) *If M is precosilting, then* $\mathrm{Cogen}M \subseteq \mathcal{F}_\sigma \subseteq \mathrm{KerExt}^1_R(-, M)$ *by Lemma 3.3;*

(2) *If M is cosilting, then* $\mathrm{Cogen}M = \mathcal{F}_\sigma \subseteq \mathrm{KerExt}^1_R(-, M)$. *In particular, we have a torsion pair* $(\mathrm{KerHom}_R(-, M), \mathrm{Cogen}M)$ *by Proposition 3.2. So all cosilting modules are precosilting.*

We say that an R-module M is cosincere if $\mathrm{Hom}(M, I) \neq 0$ for any $0 \neq I \in \mathrm{Inj}R$.

Recall that an R-module M is said to 1-cotilting, if it satisfies the following three conditions: (i) $\mathrm{id}M \leqslant 1$, i.e., the injective dimension of M is not more than 1, (ii) $\mathrm{Ext}^1_R(M^X, M) = 0$ for every set X, and (iii) there is an exact sequence $0 \to M_1 \to M_0 \to Q \to 0$ with $M_0, M_1 \in \mathrm{Adp}M$, where Q is some injective cogenerator. An R-module M is called partial 1-cotilting if it just satisfies the first two conditions above.

Proposition 3.7 (1) *An R-module M is partial 1-cotilting (resp., 1-cotilting) if and only if M is a precosilting (resp., cosilting) module with respect to a surjective injective copresentation.*

(2) *Suppose that* $\mathrm{id}M \leqslant 1$. *Then M is 1-cotilting if and only if M is a cosincere cosilting module.*

Proof. (1) If M is a 1-precotilting module, then $\mathrm{id}M \leqslant 1$. So we have a short exact sequence $0 \to M \to I_0 \to^\sigma I_1 \to 0$ with I_0 and I_1 injective. It is easy to verify that $\mathcal{F}_\sigma = \mathrm{KerExt}^1_R(-, M)$ in the case. Since M is 1-precotilting, we have $M \in \mathrm{KerExt}^1_R(-, M) = \mathcal{F}_\sigma$. So M is precosilting. If M is 1-cotilting, we have $\mathrm{Cogen}M = \mathrm{KerExt}^1_R(-, M) = \mathcal{F}_\sigma$, thus M is cosilting. The converse is similar.

(2) It is easy to see that all cotilting modules are cosincere. By (1), all 1-cotilting modules are also cosilting.

Assume that M is cosilting with respect to some $\sigma : I_0 \to I_1$ in Inj$R$, we have an exact sequence $0 \to M \to I_0 \to^\sigma I_1 \to^\pi C \to 0$ with I_0 and I_1 injective. Set $K = \mathrm{Im}\sigma$, then K is injective since $\mathrm{id}M \leqslant 1$. It follows that the exact sequence $0 \to K \to I_1 \to C \to 0$ is split. Thus, C is injective. For any morphism g: $M \to C$, there is a morphism f: $M \to I_1$ such that $g = \pi f$. Note that f factors through σ since $M \in \mathcal{F}_\sigma$, we have $g = 0$. So $C = 0$ since M is cosincere. Cosequently, M is 1-cotilting by (1). □

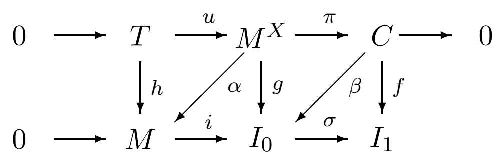
The following result gives some relations among cosilting modules, quasi-cotilting modules and 1-cotilting modules. In particular, It shows all cosilting modules are pure-injective and cofinendo.

Proposition 3.8 (1) *All cosilting modules are quasi-cotilting. In particular, all cosilting modules are pure-injective and cofinendo.*

(2) *Let M be an R-module and Q be an injective cogenerator. If* $Q \in \mathrm{Gen}(\mathrm{Adp}M)$*, then the following statements are equivelent:*

(i) M is cotilting;
(ii) M is cosilting;
(iii) M is quasi-cotilting.

Proof. (1) Let M be a cosilting module with respect to a homomorphism σ in Inj$R$. Then Cogen$M = \mathcal{F}_\sigma \subseteq \mathrm{KerExt}^1_R(-, M)$. Then we only need to prove that Cogen$M \subseteq$ Copres$M$ by the definition. For any $T \in$ Cogen$M$, we have a short exact sequence $0 \to T \to^u M^X \to C \to 0$ with u the canonical evaluation map. It is enough to prove that $C \in \mathcal{F}_\sigma =$ Cogen$M$. For any morphism $f\colon C \to I_1$, we consider the following commutative diagram:



Since $M^X \in \mathcal{F}_\sigma$, there is a morphism g such that $f\pi = \sigma g$. As u is the canonical evaluation map, we have a morphism α such that $h = \alpha u$. It is easy to verify that $(g - i\alpha)u = 0$. Thus there exists β such that $g - i\alpha = \beta\pi$. And then $f\pi = \sigma g = \sigma\beta\pi$, $f = \sigma\beta$. Consequently, $C \in \mathcal{F}_\sigma$. In particular, we get that all cosilting modules are pure-injective and cofinendo, by Proposition 3.2.

(2) By Proposition 3.7, (1) and Proposition 3.2. □

We now give some characterizations of cosilting modules.

Proposition 3.9 *Let M be an R-module and Q be an injective cogenerator. The following conditions are equivalent:*

(1) *M is a cosilting module with respect to σ.*
(2) *M is a precosilting module with respect to σ and there exists an exact sequence $0 \to M_1 \to M_0 \to^\varphi Q$ with M_0 and M_1 in* Adp$M$ *such that φ is an $\mathcal{F}_\sigma$-precover.*
(3) Cogen$M \subseteq \mathcal{F}_\sigma$ *and there exists an exact sequence $0 \to M_1 \to M_0 \to^\varphi Q$ with M_0 and M_1 in* Adp$M$ *such that φ is an $\mathcal{F}_\sigma$-precover.*

Proof. (3) ⇒ (1) Clearly, we only need to prove that $\mathcal{F}_\sigma \subseteq$ Cogen$M$. For any $T \in \mathcal{F}_\sigma$, there exists a monomorphism $f\colon T \to Q^Y$ since Q is an injective cogenerator. As φ is an $\mathcal{F}_\sigma$-precover, we have a morphism $g\colon T \to M_0^Y$ such that $f = \varphi^Y g$. Then g is injective since f is injective. Thus $T \in$ Cogen$M$.

(1) ⇒ (2) ⇒ (3) By Remark 3.6, Proposition 3.8 and Proposition 3.2. □

It is well known that every partial 1-cotilting module can be completed to a 1-cotilting module and the complement is usually called Bongartz complement. The following result shows that every precosilting modules has also a Bongartz complement.

Proposition 3.10 *Every precosilting module M with respect to an injective representation σ is a direct summand of a cosilting module $\overline{M} = M \bigoplus N$ with same associated torsion-free class, that is,* Cogen$\overline{M} = \mathcal{F}_\sigma$.

Proof. Set $\sigma\colon I_0 \to I_1$. Taking the canonical evaluation map $u\colon Q \to I_1^X$ with Q an injective cogenerator. Consider the pullback diagram of σ^X and u:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & M^X & \longrightarrow & N & \xrightarrow{\phi} & Q \\
 & & \downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\
0 & \longrightarrow & M^X & \longrightarrow & I_0^X & \xrightarrow{\sigma^X} & I_1^X
\end{array}
$$

Next, we prove that $N \in \mathcal{F}_\sigma$. For any morphism $f\colon N \to I_1$, we consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & \longrightarrow & M^X & \xrightarrow{\alpha} & N & \xrightarrow{\phi} & Q \\
 & & & & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle h} & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle b} & \\
0 & \longrightarrow & M & \longrightarrow & I_0 & \xrightarrow{\sigma} & I_1 & &
\end{array}
$$

Similarly to discussion in the proof of Proposition 3.8 (1), we have that $f\alpha = \sigma g$, $g = h\alpha$ and $f = \sigma h + b\phi$. From the pullback diagram above, we have $b\phi = \sigma v'$, where b and v' are component maps u and v respectively. Thus $f = \sigma(h + v')$ and $N \in \mathcal{F}_\sigma$.

It is easy to see that ϕ is an $\mathcal{F}_\sigma$-precover from the universal property of the pullback. Set $\overline{M} = M \bigoplus N$. The pullback diagram above gives an exact sequence $0 \to N \to I_0^X \bigoplus Q \stackrel{(\sigma^X,u)}{\longrightarrow} I_1^X$. So we obtain an injective representation ρ of $\overline{M}$, with $\rho = \sigma \bigoplus (\sigma^X, u)$. Now we get that $\mathcal{F}_\rho = \mathcal{F}_\sigma$ and $\overline{M} \in \mathcal{F}_\rho$, by Lemma 3.4. So $\overline{M}$ is precosilting. Combining with Proposition 3.9, we know that $\overline{M}$ is cosilting. □

Remark 3.11 Most results in this section are also independently obtained by Breaz and Pop [9] recently.

4 AIR-cotilting modules

In this section, we introduce AIR-cotilting modules and give precise relations between them and cosilting modules and quasi-cotilting modules. Moreover, it is shown that they are intimately related to 1-cosilting complexes.

Let R be a ring and M be an R-module. Recall from [28] that M is a **large support τ-tilting** module if it satisfies the following two conditions: (1) there is an exact sequence $P_1 \to^f P_0 \to T \to 0$ with P_1, P_0 projective such that $\mathrm{Hom}(f, T^{(X)})$ is surjective for any set X and, (2) there is an exact sequence $R \to^g T_0 \to T_1 \to 0$ with $T_0, T_1 \in \mathrm{Add}T$ such that $\mathrm{Hom}(g, T^{(X)})$ is surjective for any set X.

We will say that M is an **AIR-tilting** module if it is large support τ-tilting.

It is easy to see that 1-tilting modules, support τ-tilting modules over artin algebras [1] and silting modules [4] are all AIR-tilting modules. From the proof of the main theorem in [28], we also know that an AIR-tilting module M can always be completed to an equivalent silting module $\overline{M}$ in sense that there is some $M' \in \mathrm{Add}M$ such that $\overline{M} = M \oplus M'$ is a silting module. It is known that both silting modules and AIR-tilting modules coincide with support τ-tilting modules in the scope of the category of finitely generated modules over artin algebras. But it is a question if silting modules and AIR-tilting modules coincide with each other in general. It is also known that AIR-tilting modules are finendo quasi-tilting. But the converse is not true in general [25].

We introduce the following dual definition.

Definition 4.1 *Let R be a ring and M be an R-module. M is called AIR-cotilting module if it satisfies the following conditions:*

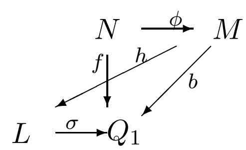
(1) *there exists an exact sequence* $0 \to M \to Q_0 \to^f Q_1$ *such that* $\mathrm{Hom}_R(M^X, f)$ *is surjective for any set X, where Q_0 and Q_1 are in* $\mathrm{Inj}R$.

(2) *there exists an exact sequence* $0 \to M_1 \to M_0 \to^g Q$ *such that* $\mathrm{Hom}_R(M^X, g)$ *is surjective for any set X, where Q is an injective cogenerator of* $\mathrm{Mod}R$ *and* M_1, $M_0 \in \mathrm{Adp}M$.

An R-module M will be called **partial AIR-cotilting** if it satisfies the first condition in the above definition.

Lemma 4.2 *Let $\sigma : L \to Q_1$ be a homomorphism with Q_1 injective. Assume that M is an R-module such that* $\mathrm{Hom}_R(M, \sigma)$ *is surjective. Then every submodule N of M has the property that* $\mathrm{Hom}_R(N, \sigma)$ *is surjective.*

Proof. Let N be a submodule of M and $\phi : N \to M$ be the canonical embedding. Take any $f \in \mathrm{Hom}_R(N, Q_1)$ and consider the following diagram.



Since Q_1 is injective, f lifts to a homomorphism $b : M \to Q_1$ such that $f = b\phi$. By the assumption, b further lifts to a homomorphism $h : M \to L$ such that $b = \alpha h$. Then $f = b\phi = \sigma h\phi$, i.e., f factors through σ. Hence we see that $\mathrm{Hom}_R(N, \sigma)$ is surjective. □

The following proposition gives a characterization of partial AIR-cotilting modules in terms of $\mathcal{F}_\sigma$.

Proposition 4.3 *An R-module M is partial AIR-cotilting if and only if there is an exact sequence* $0 \to M \to Q_0 \to^\sigma Q_1$ *such that* $\mathrm{Cogen}M \subseteq \mathcal{F}_\sigma$. *In particular,* $\mathrm{Cogen}M \subseteq \mathrm{KerExt}^1_R(-, M)$ *if M is partial AIR-cotilting.*

Proof. ($\Rightarrow$) Suppose that M is partial AIR-cotilting, i.e., there exists an exact sequence $0 \to M \to Q_0 \to^\sigma Q_1$ such that $\mathrm{Hom}(M^X, \sigma)$ is surjective for any set X, where Q_0 and Q_1 are in $\mathrm{Inj}R$. Let $N \in \mathrm{Cogen}M$. Then there a monomorphism $\phi : N \to M^X$ for some X. Now by Lemma 4.2 we obtain that $\mathrm{Hom}_R(N, \sigma)$ is surjective. Thus, $\mathrm{Cogen}M \subseteq \mathcal{F}_\sigma$.

($\Leftarrow$) Obviously. □

We have the following easy corollary, which implies that, for an R-module M of injective dimension not more than 1, M is partial AIR-cotilting if and only if M is partial 1-cotilting.

Corollary 4.4 *Let M be an R-module and* $\mathrm{id}M \leqslant 1$. *Then the following statements are equivalent:*

(1) *M is partial AIR-cotilting;*

(2) $\mathrm{Cogen}M \subseteq \ker \mathrm{Ext}^1_R(-, M)$;

(3) $\mathrm{Ext}^1_R(M^X, M) = 0$ *for any set X.*

Proof. (1)$\Rightarrow$(2) By Proposition 4.3.

(2)$\Rightarrow$(3) Obviously.

(3)⇒(1) Since $\mathrm{id}M \leqslant 1$, there is a short exact sequence $0 \to M \to Q_0 \to^{\alpha} Q_1 \to 0$ with Q_0 and Q_1 in $\mathrm{Inj}R$. Applying $\mathrm{Hom}_R(M^X, -)$ to this short exact sequence, we have

$$0 \to \mathrm{Hom}_R(M^X, M) \to \mathrm{Hom}_R(M^X, Q_0) \to \mathrm{Hom}_R(M^X, Q_1) \to \mathrm{Ext}^1_R(M^X, M) = 0.$$

So $\mathrm{Hom}_R(M^X, \alpha)$ is surjective. □

We also have the following result.

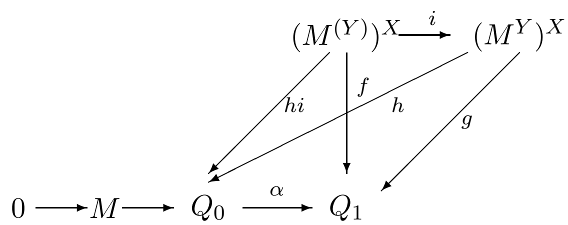
Proposition 4.5 *Let Q be an injective cogenerator of* $\mathrm{Mod}R$ *and M be an R-module such that $Q \in \mathrm{Gen}M$.*

(1) *If M is partial AIR-cotilting, then* $\mathrm{id}M \leqslant 1$.

(2) *If M is AIR-cotilting, then M is 1-cotilting.*

Proof. Note that there is an exact sequence $M^{(Y)} \to^{\gamma} Q \to 0$, since $Q \in \mathrm{Gen}M$.

(1) By the definition, we have an exact sequence $0 \to M \to Q_0 \to^{\alpha} Q_1$ such that $\mathrm{Hom}_R(M^X, \alpha)$ is surjective for any set X, where Q_0 and Q_1 are in $\mathrm{Inj}R$. As Q is an injective cogenerator, Q_1 is a summand of Q^X for some X, and then we have a canonical projective π: $Q^X \to Q_1$. Hence we have a surjection $f = \pi\gamma^X$: $(M^{(Y)})^X \to Q_1$. Consider the following commutative diagram, where i is a canonical embedding.



There is a morphism g such that $f = gi$ since Q_1 is injective. Following from the property of the morphism α, there exists a morphism h such that $g = \alpha h$. Hence, $f = gi = (\alpha h)i$. Then we obtain that α is surjective since f surjective. Thus, $\mathrm{id}M \leqslant 1$.

(2) Since M is AIR-cotilting, there exists an exact sequence $0 \to M_1 \to M_0 \to^{\beta} Q$ such that $\mathrm{Hom}_R(M^X, \beta)$ is surjective for any X, where M_1 and M_0 are in $\mathrm{Adp}M$. Clearly, $\mathrm{Hom}_R(M^{(Y)}, \beta)$ is also surjective in the case, so we have a morphism δ: $M^{(Y)} \to M_0$ such that $\gamma = \beta\delta$. Thus β surjective since γ is surjective. It follows that M is 1-cotilting from (1), Proposition 4.4 and the definition of 1-cotilting modules. □

Next we will consider the relations between 2-term cosilting complexes (i.e., 1-cosilting complexes in Section 2) and AIR-cotilting modules. We need some preparations.

Lemma 4.6 *Let $I^\bullet$: $0 \to I_0 \to^{\alpha} I_1 \to 0$ be a 2-term complex of injective modules and $J^\bullet$: $0 \to J_0 \to J_1 \to \cdots$ be a complex. If $K = H^0(J^\bullet)$, then the following statements are equivalent:*

(1) *$K \in \mathcal{F}_\alpha$, i.e.,* $\mathrm{Hom}_R(K, \alpha)$ *is surjective;*

(2) $\mathrm{Hom}_{\mathcal{D}}(J^\bullet, I^\bullet[1]) = 0$.

In particular, an R-module $K \in \mathcal{F}_\alpha$ if and only if $\mathrm{Hom}_{\mathcal{D}}(K, I^\bullet[1]) = 0$.

Proof. The proof is dual to Lemma 3.4 in [1]. □

By Lemma 4.6, we can easily obtain the following corollary:

Corollary 4.7 *Suppose that there is an exact sequence* $0 \to M \to I_0 \to^{\alpha} I_1$ *with* I_0 *and* I_1 *in* Inj$R$*. Then the following statements are equivalent:*

(1) $\mathrm{Hom}_R(M^X, \alpha)$ *is surjective for any set* X*;*

(2) *The complex* $I^\bullet$*:* $0 \to I_0 \to I_1 \to 0$ *is partial cosilting.*

The following result shows that we can obtain AIR-cotilting modules and cosilting modules from 2-term cosilting complexes.

Proposition 4.8 *Let the 2-term complex* $I^\bullet$*:* $0 \to I_0 \to^{\alpha} I_1 \to 0$ *be cosilting and set* $K = H^0(I^\bullet)$*. Then*

(1) K *is AIR-cotilting.*

(2) K *is a cosilting module.*

Proof. (1) By Corollary 4.7, we only need to check the condition (2) in Definition 4.1. Since $I^\bullet$ is cosilting, there is a triangle $I_1^\bullet \to I_0^\bullet \to^{\beta^\bullet} Q \to$ with $I_1^\bullet$ and $I_0^\bullet$ in Adp$I^\bullet$, where Q injective cogenerator (see Section 2, Theorem 2.5 and Proposition 2.8). By taking homologies, we can obtain an exact sequence $0 \to H^0(I_1^\bullet) \to H^0(I_0^\bullet) \to H^0(Q)(= Q)$. Set $K_1 = H^0(I_1^\bullet)$, $K_0 = H^0(I_0^\bullet)$ and $\beta = H^0(\beta^\bullet)$, then we have an exact sequence $0 \to K_1 \to K_0 \to^{\beta} Q$. It is easy to see that K_1 and K_0 are in Adp$K$.

For any $\gamma \in \mathrm{Hom}_R(K^X, Q)$ with X a set, we see that γ lifts to a homomorphism $\gamma^\bullet \in \mathrm{Hom}_{\mathcal{D}}((I^\bullet)^X, Q)$, since Q is injective. By the assumption, $I^\bullet$ is prod-semi-selforthogonal, so we have that $\mathrm{Hom}_{\mathcal{D}}((I^\bullet)^X, \beta^\bullet)$ is surjective. Thus, there is a morphism $\delta^\bullet : (I^\bullet)^X \to I_0^\bullet$ such that $\gamma^\bullet = \beta^\bullet\delta^\bullet$. Then we obtain that $H^0(\gamma^\bullet) = H^0(\beta^\bullet)H^0(\delta^\bullet)$. That is, $\gamma = \beta\delta$, where $\delta = H^0(\delta^\bullet)$. Hence $\mathrm{Hom}_R(K^X, \beta)$ is surjective.

(2) It is easy to see that Cogen$K \subseteq \mathcal{F}_\alpha$ by Corollary 4.7 and Lemma 3.3. We claim that the exact sequence $0 \to K_1 \to K_0 \to^{\beta} Q$ obtained in (1) satisfies that β is an $\mathcal{F}_\alpha$-precover. Thus K is cosilting by Proposition 3.9.

In fact, take any $M \in \mathcal{F}_\alpha$ and any homomorphism $f : M \to Q$. We can consider these objects and homomorphisms in the derived category. By Lemma 4.6, one has that $\mathrm{Hom}_{\mathcal{D}}(M, I^\bullet[1]) = 0$. Thus, by applying the functor $\mathrm{Hom}_{\mathcal{D}}(M, -)$ to the triangle $I_1^\bullet \to I_0^\bullet \to^{\beta^\bullet} Q \to$, we get that $\mathrm{Hom}_{\mathcal{D}}(M, \beta^\bullet)$ is surjective, i.e., there exists some $\eta^\bullet : M \to I_0^\bullet$ such that $f = \beta^\bullet\eta^\bullet$. Then we obtain that $H^0(f) = H^0(\beta^\bullet)H^0(\eta^\bullet)$. That is, $f = \beta\eta$, where $\eta = H^0(\delta^\bullet)$. Hence $\mathrm{Hom}_R(M, \beta)$ is surjective and β is an $\mathcal{F}_\alpha$-precover. □

We will say that two AIR-cotilting modules M, N are equivalent, denoted by $M \sim N$, provided that Adp$M$ = Adp$N$. The next result shows that AIR-cotilting modules also give 2-term cosilting complexes.

Proposition 4.9 *Let* T *be an AIR-cotilting module. Then*

(1) *there is a 2-term cosilting complex* $M^\bullet$ *such that* $H^0(M^\bullet) \simeq T \oplus T'$ *with* $T' \in$ Adp$T$*. In particular,* $H^0(M^\bullet) \sim T$*;*

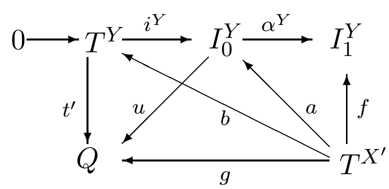
(2) *the cosilting complexs in* (1) *is unique up to equivalences.*

Proof. (1) Since T is an AIR-cotilting module, there exist two exact sequences $0 \to T \to^{i} I_0 \to^{\alpha} I_1$ and $0 \to T_1 \to^{s} T_0 \to^{t} Q$ such that $\mathrm{Hom}_R(T^X, \alpha)$ and $\mathrm{Hom}_R(T^X, t)$ are surjective respectively for any set X, where $I_0, I_1 \in$ Inj$R$ and $T_1, T_0 \in$ Adp$T$. Assume that $T_0' \oplus T_0 = T^Y$ for some R-module T_0', then we have an exact sequence $0 \to T_1' \to^{s'} T^Y \to^{t'} Q$ with $T_1' = T_0' \oplus T_1$, $s' = \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix}$ and $t' = (t, 0)$. It is easy to see that $\mathrm{Hom}_R(T^X, t')$ is surjective. Since Q is injective, there exists a morphism $u\colon I_0^Y \to Q$ such that $t' = u \cdot i^Y$. Set $I^\bullet\colon 0 \to I_0 \to^\alpha I_1 \to 0$. Clearly, u induces a map of complex $u^\bullet\colon (I^\bullet)^Y \to Q$ with $H^0(u^\bullet) = t'$. Then we get a triangle $(I^\bullet)^Y \to^{u^\bullet} Q \to \mathrm{Con}(u^\bullet) \to$, where $\mathrm{Con}(u^\bullet)\colon 0 \to I_0^Y \to^\theta I_1^Y \bigoplus Q \to 0$ with $\theta = \begin{pmatrix} \alpha^Y \\ u \end{pmatrix}$ is a complex with terms fixed in (-1)-th and 0-th positions. By taking homology of this triangle, we have an exact sequence $0 \to H^{-1}(\mathrm{Con}(u^\bullet)) \to T^Y \to^{t'} Q$, thus $H^{-1}(\mathrm{Con}(u^\bullet)) \cong T_1'$.

We assert that $M^\bullet = I^\bullet \oplus \mathrm{Con}(u^\bullet)[-1]$ is the desired cosilting complex. Note that $H^0(M^\bullet) = H^0(I^\bullet) \bigoplus H^{-1}(\mathrm{Con}(u^\bullet)) = T \bigoplus T_1' \sim T$, since $T_1' \in \mathrm{Adp}T$. Obviously, $M^\bullet \in K^b(\mathrm{Inj}R)$ and $Q \in \langle \mathrm{Adp}M^\bullet \rangle$. It remains to prove that $M^\bullet$ is prod-semi-selforthogonal by Definition 2.4. This is proceeded as follows.

(1) $\mathrm{Hom}_{\mathcal{D}}((M^\bullet)^X, I^\bullet[1]) = 0$ for any X. This is by Lemma 4.6.

(2) $\mathrm{Hom}_{\mathcal{D}}((M^\bullet)^X, \mathrm{Con}(u^\bullet)[-1][1]) = 0$ for any X. Indeed, we only need to prove that $\mathrm{Hom}_R(K, \theta)$ is surjective by Lemma 4.6, where $K = H^0((M^\bullet)^X) = (T \oplus T')^X$. Note that K is a direct summand of $T^{X'}$ for some X', it is sufficient to prove that $\mathrm{Hom}_R(T^{X'}, \theta)$ is surjective. For any morphism $\begin{pmatrix} f \\ g \end{pmatrix}\colon T^{X'} \to I_1^Y \bigoplus Q$. Consider the following diagram, where t', i^Y, α^Y, u was defined as above.



Since $\mathrm{Hom}_R(T^{X'}, \alpha^Y)$ is surjective, we can obtain a morphism a such that $f = \alpha^Y a$. Further, since $\mathrm{Hom}_R(T^{X'}, t')$ is surjective, we can obtain some morphism $b, c : T^{X'} \to T^Y$ such that $g = t'b$ and $ua = t'c$. Note that $t' = u \cdot i^Y$ and $\alpha^Y i^Y = 0$, it is easy to see that $\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha^Y \\ u \end{pmatrix}(a - i^Y c + i^Y b)$.

Thus $M^\bullet$ is just the desired cosilting complex.

(2) Suppose that $M^\bullet$ and $N^\bullet$ are 2-term cosilting complexes satisfying $H^0(M^\bullet) \sim H^0(N^\bullet)$. Then $\mathrm{Hom}_{\mathcal{D}}((M^\bullet)^X, N^\bullet[1]) = 0$ and $\mathrm{Hom}_{\mathcal{D}}((N^\bullet)^X, M^\bullet[1]) = 0$, by Lemma 4.6. Now it is easy to verify that $M^\bullet \bigoplus N^\bullet$ is a cosilting complex, therefore, we have that $M^\bullet \in \mathrm{Adp}_{\mathcal{D}} N^\bullet$ and $N^\bullet \in \mathrm{Adp}_{\mathcal{D}} M^\bullet$ by Proposition 2.7. It follows that $\mathrm{Adp}_{\mathcal{D}} N^\bullet = \mathrm{Adp}_{\mathcal{D}} N^\bullet$, i.e., $M^\bullet$ and $N^\bullet$ are equivalent. □

As a direct corollary, we obtain the following relation between cosilting modules and AIR-cotilting modules, dual to the tilting case.

Corollary 4.10 *Let M be AIR-cotilting. Then there is some $M' \in \mathrm{Adp}M$ such that $\overline{M} = M \oplus M'$ is a cosilting module.*

Lemma 4.11 *If two cosilting complexes $U^\bullet$ and $V^\bullet$ are equivalent, then $\mathrm{Adp}H^k(U^\bullet) = \mathrm{Adp}H^k(V^\bullet)$ for any integer k.*

Proof. It is enough to show that $\mathrm{Adp}H^k(U^\bullet) \subseteq \mathrm{Adp}H^k(V^\bullet)$. For any $M \in \mathrm{Adp}H^k(U^\bullet)$, we have $M \bigoplus N = [H^k(U^\bullet)]^X \cong H^k((U^\bullet)^X)$ for some N. Since $\mathrm{Adp}_{\mathcal{D}} U^\bullet = \mathrm{Adp}_{\mathcal{D}} V^\bullet$, we have $(U^\bullet)^X \bigoplus W^\bullet = (V^\bullet)^Y$ for some $W^\bullet$. Thus, $H^k((U^\bullet)^X) \bigoplus H^k(W^\bullet) \cong H^k((V^\bullet)^Y)$ and $M \bigoplus N \bigoplus H^k(W^\bullet) \cong [H^k(V^\bullet)]^Y$. So $M \in \mathrm{Adp}H^k(V^\bullet)$. □

Now we obtain the following theorem.

Theorem 4.12 *Let R be a ring. There is a bijective correspondence between the equivalent classes of AIR-cotilting modules and two-terms cosilting complexes.*

Proof. By Proposition 4.8, 4.9 and Lemma 4.11. $\square$

We now turn to study the precise relations between quasi-cotilting modules, cosilting modules and AIR-cotilting modules. To this aim, the following is a key result.

Lemma 4.13 *An R-module M is partial AIR-cotilting if and only if* $\mathrm{Cogen}M \subseteq \mathrm{KerExt}^1_R(-, M)$.

Proof. ($\Rightarrow$) By Proposition 4.3.

($\Leftarrow$) Let $0 \to M \to E_0 \to^f E_1 \to E_2$ be the minimal injective resolution of M, where each E_i is in $\mathrm{Inj}R$. We need only to prove that $\mathrm{Hom}_R(M^X, f)$ is surjective for any X.

Take any morphism g: $M^X \to E_1$ and consider the following commutative diagram.

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & K & \xrightarrow{k} & M^X & \xrightarrow{\theta g} & E_2 \\ & & & & \downarrow{\scriptstyle a} & \overset{t}{\swarrow} & \downarrow{\scriptstyle g} & \overset{n}{\swarrow} & \downarrow{\scriptstyle 1} \\ 0 & \longrightarrow & M & \longrightarrow & E_0 & \xrightarrow{f} & E_1 & \xrightarrow{\theta} & E_2 \end{array}$$

Set $K = \mathrm{Ker}(\theta g)$ and factor $f = h\pi$ canonically, where $h : \mathrm{Im}f \to E_1$ and $\pi : E_0 \to \mathrm{Im}f$. There exists a morphism $\alpha : K \to \mathrm{Im}f$ such that $gk = h\alpha$, since $\theta gk = 0$. As $K \in \mathrm{CogenM} \subseteq \mathrm{KerExt}^1_R(-, M)$, we have that $\mathrm{Hom}_R(K, \pi)$ is surjective. Hence, there is some $a : K \to E_0$ such that $\alpha = \pi a$. Then we get $gk = h\alpha = h\pi a = fa$.

Since E_0, E_1 are injective, a canonical argument shows that there are two morphisms t and n such that $a = tk$ and $g = n\theta g + ft$. Setting $\beta = g - ft$, then we have that $n\theta\beta = n\theta(g - ft) = n\theta g = g - ft = \beta$, that is, $\beta = n\theta\beta$. Now we claim that $\mathrm{Im}h \bigcap \mathrm{Im}\beta = 0$. Indeed, for any $e \in \mathrm{Im}h \bigcap \mathrm{Im}\beta$, we have $h(x) = e = \beta(y) = n\theta\beta(y) = n\theta h(x) = 0$, since $\theta h = 0$. Thus $e = 0$. Since h is an injective envelope by assumption, we have that $\mathrm{Im}h$ is an essential submodule of E_1. This implies that $\mathrm{Im}\beta = 0$, i.e., $\beta = 0$. Then we have $g = ft$. So $\mathrm{Hom}(M^X, f)$ is surjective for any set X. $\square$

The following is a direct corollary.

Corollary 4.14 *A direct summand of a partial AIR-cotilting module is again partial AIR-cotilting.*

The following result is well-known.

Lemma 4.15 (1) *Suppose that f: $M \to E$ is an injective envelope of M and g: $M \to E'$ is a monomorphism with E' injective. Then $E' \simeq E \oplus E''$ and $g \simeq \begin{pmatrix} f \\ 0 \end{pmatrix}$.*

(2) *Let $0 \to M \to^\alpha E_0 \to^\beta E_1$ be a minimal injective resolution of M and $0 \to M \to^\delta I_0 \to^\sigma I_1$ be any injective resolution of M. Then $I_0 \simeq E_0 \oplus E'_0$ and $I_1 \simeq E_1 \oplus E'_0 \oplus E'_1$ and, moreover, the complex $0 \to M \to^\delta I_0 \to^\sigma I_1$ is isomorphic to the direct sums of three complexes $0 \to M \to^\alpha E_0 \to^\beta E_1$, $0 \to 0 \to E'_0 \to^{1_{E'_0}} E'_0$ and $0 \to 0 \to 0 \to E'_1$.*

The following result gives a characterization of partial AIR-cotilting modules in term of its minimal injective copresentation.

Proposition 4.16 *Let M be an R-module and $0 \to M \to E_0 \to^{\delta} E_1$ be its minimal injective copresentation. Then M is partial AIR-cotilting if and only if* $\text{Cogen}M \subseteq \mathcal{F}_{\delta}$.

Proof. ($\Leftarrow$) By the definition.

($\Rightarrow$) If M is partial AIR-cotilting, then there exists an exact sequence $0 \to M \to I_0 \to^{\sigma} I_1$ such that $\text{Hom}_R(M^X, \sigma)$ is surjective for any set X, where I_0 and I_0 are in $\text{Inj}R$. By Lemma 4.15, we see that $\sigma : I_0 \to I_1$ is isomorphic to the direct sums of $\beta : E_0 \to E_1$ and $1_{E_0'} : E_0' \to E_0'$ and $0 \to E_2'$ for some injective modules E_0' and E_1'. Thus, δ is a direct summand of σ. It follows that $\text{Hom}_R(M^X, \delta)$ is surjective from the assumption that $\text{Hom}_R(M^X, \sigma)$ is surjective. Thus, $M^X \in \mathcal{F}_{\delta}$ for any X. It follows that $\text{Cogen}M \subseteq \mathcal{F}_{\delta}$ by Lemma 3.3. □

Moreover, we have the following easy observation by Lemma 4.15 and the involved definition.

Lemma 4.17 *Let $\sigma : I_0 \to I_1$ be a homomorphism of injective R-modules. Assume that $\alpha : I_0 \to I$ is the composition of the canonical map $\pi : I_0 \to \text{Im}\sigma$ and the injective envelope map $i : \text{Im}\sigma \to I$. Then σ is isomorphic to the direct sum of α and the zero map $0 \to I'$ for some injective module I'. In the case, it holds that* $\mathcal{F}_{\sigma} = \mathcal{F}_{\alpha} \bigcap \text{KerHom}_R(-, I')$.

Now we are in the position to give our main result in the paper.

Theorem 4.18 *Let M be an R-module. Then the following statements are equivalent.*

(1) *M is AIR-cotilting.*

(2) *M is quasi-cotilting.*

(3) *M is cosilting.*

Proof. (1) $\Rightarrow$ (2) By Lemma 4.13, we have that $\text{Cogen}M \subseteq \text{KerExt}^1_R(-, M)$. Then it is easy to see that M is Ext-injective in $\text{Cogen}M$. Since M is AIR-cotilting, we also have an exact sequence $0 \to M_1 \to M_0 \to^{g} Q$ such that $\text{Hom}_R(M^X, g)$ is surjective, where Q is an injective cogenerator and $M_1, M_0 \in \text{Adp}M$. It is easy to see that g is a $\text{Cogen}M$-precover by Lemma 4.2. Hence, M is quasi-cotilting by Proposition 3.2.

(2) $\Rightarrow$ (1) By Lemma 4.13 and Proposition 3.2.

(3) $\Rightarrow$ (2) By Proposition 3.8.

(1) $\Rightarrow$ (3) By Corollary 4.10, there exists some $M' \in \text{Adp}M$ such that $\overline{M} \simeq M \oplus M'$ is a cosilting module. Let σ, α, β be the minimal injective copresentations of $\overline{M}, M, M'$ respectively. Then we have that $\sigma = \alpha \oplus \beta$. Now we will show that $\mathcal{F}_{\sigma} = \mathcal{F}_{\alpha}$. Since $M' \in \text{Adp}M$, there is a morphism γ in $\text{Inj}R$ satisfiing $\beta \oplus \gamma = \alpha^X$ for some X, so $\mathcal{F}_{\alpha} = \mathcal{F}_{\alpha^X} = \mathcal{F}_{\beta\oplus\gamma} = \mathcal{F}_{\beta} \bigcap \mathcal{F}_{\gamma}$. In particular, $\mathcal{F}_{\alpha} \subseteq \mathcal{F}_{\beta}$. Thus, $\mathcal{F}_{\sigma} = \mathcal{F}_{\alpha\oplus\beta} = \mathcal{F}_{\alpha} \bigcap \mathcal{F}_{\beta} = \mathcal{F}_{\alpha}$.

Since $\overline{M}$ is a cosilting module, there is an injective copresentation η of $\overline{M} : I_0 \to I_1$, where I_0, I_1 are injective, such that $\text{Cogen}\overline{M} = \mathcal{F}_{\eta}$. By Lemma 4.17, we have that $\eta \simeq \sigma \oplus (0 \to I')$ for some injective R-module I', and then $\mathcal{F}_{\eta} = \mathcal{F}_{\sigma} \bigcap \text{KerHom}_R(-, I')$. Let $\xi = \alpha \oplus (0 \to I')$, then ξ is an injective copresentation of M. Note that $\mathcal{F}_{\eta} = \mathcal{F}_{\xi}$ by the argument above and that $\text{Cogen}M = \text{Cogen}\overline{M}$, so we obtain that $\text{Cogen}M = \text{Cogen}\overline{M} = \mathcal{F}_{\eta} = \mathcal{F}_{\xi}$, i.e., M is a cosilting module. □

Combining results in the Proposition 3.2 and Theorems 4.12 and 4.18, we obtain the following result.

Theorem 4.19 *There are bijections between*

(1) *equivalent classes of AIR-cotilting (resp., cosilting, quasi-cotilting) modules,*

(2) *equivalent classes of 2-term cosilting complexes,*

(3) *torsion-free cover classes and,*

(4) *torsion-free special precover classes.*

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