Quasi-cotilting modules and torsion-free classes*

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Abstract

We prove that all quasi-cotilting modules are pure-injective and cofinendo. It follows that the class Cogen$M$ is always a covering class whenever M is a quasi-cotilting module. Some characterizations of quasi-cotilting modules are given. As a main result, we prove that there is a bijective correspondence between the equivalent classes of quasi-cotilting modules and torsion-free covering classes.

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1 Introduction and Preliminaries

Quasi-tilting modules were introduced by Colpi, Deste, and Tonolo in [9] in the study of contexts of $*$-modules and tilting modules, where it was shown that these modules are closely relative to torsion theory counter equivalences. Recently, quasi-tilting modules turn out to be new interesting after the study of support τ-tilting modules by Adachi, Iyama and Reiten [1]. The second author proved that a finitely generated module over an artin algebra is support τ-tilting if and only if it is quasi-tilting [19]. Moreover, Angeleri-Hügel, Marks and Vitória [2] proved that finendo quasi-tilting modules are also closely related to silting modules (which is a generalizations of support τ-tilting modules in general rings).

It is natural to consider the dual of quasi-tilting modules, i.e., quasi-cotilting modules. As cotilting modules possess their own interesting properties not dual to tilting modules, we show in this paper that quasi-cotilting modules are not only to be the dual of quasi-tilting modules and they have their own interesting properties too. We prove that all quasi-cotilting modules are pure-injective and cofinendo. These are clearly new important properties of quasi-cotilting modules. Note that not all quasi-tilting modules are finendo. As a corollary, we obtain that the class Cogen$M$ is a covering class whenever M is quasi-cotilting. We give variant characterizations of quasi-cotilting modules and cotilting modules. As the main result, we prove that there is a bijection between the equivalent classes of quasi-cotilting modules and torsion-free covering classes.

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Throughout this paper, R is always an associative ring with identity and subcategories are always full and closed under isomorphisms. We denote by R-Mod the category of all left R-modules. By Proj$R$ we denote the class of all projective R-module.

Let $M \in R$-Mod, we use following notations throughout this paper.

Adp$M := \{N \in R\text{-Mod} \mid$ there is a module L such that $N \oplus L = M^X$ for some $X\}$;

Cogen$M := \{N \in R\text{-Mod} \mid$ there is an exact sequence $0 \to N \to M_0$ with $M_0 \in$ Adp$M\}$;

Copres$M := \{N \in R\text{-Mod} \mid$ there is an exact sequence $0 \to N \to M_0 \to M_1$ with $M_0, M_1 \in$ Adp$M\}$;

${}^{\perp_1}M := \{N \in R\text{-Mod} \mid \mathrm{Ext}^1_R(N, M) = 0\}$;

$M^{\perp_1} := \{N \in R\text{-Mod} \mid \mathrm{Ext}^1_R(M, N) = 0\}$;

${}^{\circ}M := \{N \in R\text{-Mod} \mid \mathrm{Hom}_R(N, M) = 0\}$;

$M^{\circ} := \{N \in R\text{-Mod} \mid \mathrm{Hom}_R(M, N) = 0\}$.

Note that Cogen$M$ is clearly closed under submodules, direct products and direct sums.

Let $\mathcal{T}$ be a class of R-modules, we denoted by Fac$(\mathcal{T})$ the classes formes by the factor modules of all modules in $\mathcal{T}$. An R-module $M \in \mathcal{T}$ is called Ext-injective in $\mathcal{T}$ if $\mathcal{T} \subseteq {}^{\perp_1}M$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two subcategories of R-Mod. The pair $(\mathcal{X},\mathcal{Y})$ is said to be a torsion pair if it satisfies the following three condition (1) $\mathrm{Hom}(\mathcal{X}, \mathcal{Y}) = 0$; (2) if $\mathrm{Hom}(M, \mathcal{Y}) = 0$, then $M \in \mathcal{X}$; (3) if $\mathrm{Hom}(\mathcal{X}, N) = 0$, then $N \in \mathcal{Y}$. In the case, $\mathcal{X}$ is called a torsion class and $\mathcal{Y}$ is called a torsion-free class. Note that a subcategory $\mathcal{X}$ of R-Mod is torsion-free if and only if $\mathcal{X}$ is closed under direct products, submodules and extensions, see [11].

Lemma 1.1 *If an R-module M is Ext-injective in* Cogen$M$*, then* $({}^{\circ}M, \mathrm{Cogen}M)$ *is a torsion pair.*

Proof. It is easy to verify that ${}^{\circ}M = {}^{\circ}(\mathrm{Cogen}M)$. We only need to prove that $({}^{\circ}M)^{\circ} = \mathrm{Cogen}M$. If $T \in \mathrm{Cogen}M$, then there is an injective homomorphism i: $T \to M^X$ for some set X. For any $N \in {}^{\circ}M$ and $f \in \mathrm{Hom}_R(N, T)$, then $\mathrm{Hom}_R(N, M^X) = (\mathrm{Hom}_R(N, M))^X = 0$, hence $if = 0$. Then $f = 0$ since i is injective. So $\mathrm{Cogen}M \subseteq ({}^{\circ}M)^{\circ}$.

For any $T \in ({}^{\circ}M)^{\circ}$, consider the evaluation map α: $T \to M^{\mathrm{Hom}_R(T,M)}$ with $K = \ker \alpha$. Take canonical resolution of α, i.e. $\alpha = i\pi$ with π: $T \to \mathrm{Im}\alpha$ and i: $\mathrm{Im}\alpha \to M^{\mathrm{Hom}_R(T,M)}$. Clearly, $\mathrm{Hom}_R(\pi, M)$ is surjective from the definition of α. Applying the functor $\mathrm{Hom}_R(-, M)$ to the exact sequence $0 \to K \to T \to Im\alpha \to 0$, we have an exact sequence

$$0 \to \mathrm{Hom}_R(Im\alpha, M) \to \mathrm{Hom}_R(T, M) \to \mathrm{Hom}_R(K, M) \to \mathrm{Ext}^1_R(Im\alpha, M).$$

Since $Im\alpha \in \mathrm{Cogen}M \subseteq {}^{\perp_1}M$, $\mathrm{Ext}^1_R(Im\alpha, M) = 0$. As $\mathrm{Hom}_R(\pi, M)$ is surjective by above discussion, $\mathrm{Hom}_R(K, M) = 0$, and hence, $K \in {}^{\circ}M$. Since $T \in ({}^{\circ}M)^{\circ}$, we have that $\mathrm{Hom}_R(K, T) = 0$ and α is injective. Thus $K = 0$, i.e. $T \in \mathrm{Cogen}(M)$. □

Definition 1.2 [3, Definition1.4] (1) *Let Q be an injective cogenerator of R-Mod. A module M is called Q-cofinendo if there exist a cardinal γ and a map f: $M^{\gamma} \to Q$ such that for any cardinal α, all maps $M^{\alpha} \to Q$ factor through f.*

(2) *A module M is cofinendo if there is an injective cogenerator Q of R-Mod such that M is Q-cofinendo.*

Let $\mathcal{T}$ be a class of R-modules and M be an R-module. Then $f\colon X \to M$ with $X \in \mathcal{T}$ is a $\mathcal{T}$-precover of M provided that $\mathrm{Hom}(Y, f)$ is surjective for any $Y \in \mathcal{T}$. A $\mathcal{T}$-precover of M is called $\mathcal{T}$-cover of M if any $g\colon X \to X$ such that $f = fg$ must be an isomorphism. A class $\mathcal{T}$ of R-modules is said to be a precover class (cover class) provided that each module has a $\mathcal{T}$-precover ($\mathcal{T}$-cover).

Lemma 1.3 [3, Proposition1.6] *The following are equivalent for a module M:*

(1) *M is cofinendo;*

(2) *there is an $\mathrm{Adp}M$-precover of an injective cogenerator Q of R-Mod;*

(3) $\mathrm{Cogen}M$ *is a precover class.*

2 Quasi-cotilting modules

In this section, we introduce notion of quasi-cotilting modules and give some characterization of quasi-cotilting modules. In particular, we prove that all quasi-cotilting modules are pure injective and cofinendo.

Definition 2.1 (1) *An R-module M is said to be a costar module if* $\mathrm{Cogen}M = \mathrm{Copres}M$ *and* $\mathrm{Hom}_R(-, M)$ *preserves exactness of any short exact sequence in* $\mathrm{Cogen}M$*.*

(2) *An R-module M is called a quasi-cotilting module, if it is a costar module and M is Ext-injective in* $\mathrm{Cogen}M$*.*

Remark (1) Costar modules defined above had been studied in [16]. Moreover, their general version, i.e., n-costar modules, were studied by He [14] and Yao and Chen [22] respectively.

(2) Colby and Fuller [7] had defined another notion of costar modules which can be viewed as a special case of the above-defined costar modules.

For convenience, we say that a short exact sequence $0 \to A \to B \to C \to 0$ is $\mathrm{Hom}_R(-, M)$-exact (($M \bigotimes_R -$, resp.)-exact) if the functor $\mathrm{Hom}_R(-, M)$ ($M \bigotimes_R -$, resp.) preserves exactness of this exact sequence.

The following result is well-known.

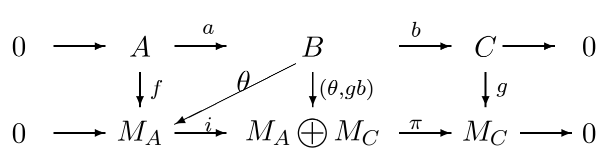
Lemma 2.2 *Suppose that two short exact sequences* $0 \to A \to B \to C \to 0$ *and* $0 \to A \to B' \to C' \to 0$ *are* $\mathrm{Hom}(-, M)$*-exact with* $B, B' \in \mathrm{Apd}M$*. Then* $B \bigoplus C' \cong B' \bigoplus C$

Proof. It is dual to Lemma 2.2 in [20]. □

The following result presents a useful property of costar modules.

Lemma 2.3 *Let M be a co-star module. Suppose that the short exact sequence* $0 \to A \to B \to C \to 0$ *is* $\mathrm{Hom}_R(-, M)$*-exact and* $A \in \mathrm{Cogen}M$*, then* $B \in \mathrm{Cogen}M$ *if and only if* $C \in \mathrm{Cogen}M$*.*

Proof. $\Leftarrow$ Since A and C are in $\mathrm{Cogen}M$, we have two monomorphisms $f\colon A \to M_A$ and $g\colon C \to M_C$ with M_A and M_C in $\mathrm{Adp}M$. We consider the following diagram:



where i and π is canonical injective and canonical projective respectively. Since the first row in above diagram is $\mathrm{Hom}_R(-, M)$-exact, there exists a morphism θ such that $f = \theta a$. It further induces a commutative diagram as above. It follows from Snake Lemma that (θ, gb) is injective, i.e. $B \in \mathrm{Cogen}M$.

$\Rightarrow$ Since $B \in \mathrm{Cogen}M$ and M is a costar module, we have that $0 \to B \to M_0 \to L' \to 0$ with $M_0 \in \mathrm{Adp}M$ and $L' \in \mathrm{Cogen}M$. There is a module M_0' such that $M_0 \oplus M_0' = M^X$. So we obtain a new short exact sequence $0 \to B \to M^X \to L \to 0$ with $L = L' \oplus M_0' \in \mathrm{Cogen}M$. Consider the pushout of $B \to C$ and $B \to M^X$:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow 1 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & M^X & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & L & \xrightarrow{1} & L & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since the first row and second column are $\mathrm{Hom}_R(-, M)$-exact in above diagram, it is easy to see that the second row is also $\mathrm{Hom}_R(-, M)$-exact. Since $A \in \mathrm{Cogen}M$, similar to B, there is a short exact sequence $0 \to A \to M^Y \to K \to 0$ with $K \in \mathrm{Cogen}M$. By Lemma 2.2, we have that $M^Y \oplus N \cong M^X \oplus K$, then it is easy to see that $N \in \mathrm{Cogen}M$. Consequently, $C \in \mathrm{Cogen}M$. □

Now we give some characterizations of quasi-cotilting modules.

Proposition 2.4 *Let M be an R-Module, the following statements are equivalent:*

(1) *M is a quasi-cotilting module;*

(2) $\mathrm{Cogen}M = \mathrm{Copres}M$ *and M is Ext-injective in* CogenM*;*

(3) *M is a costar module and* $\mathrm{Cogen}M$ *is a torsion-free class;*

(4) $\mathrm{Cogen}M = \mathrm{Fac}(\mathrm{Cogen}M) \bigcap {}^{\perp_1}M$.

Proof. (1)$\Rightarrow$ (2) By the involved definitions.

(2)$\Rightarrow$ (1),(3) If M is Ext-injective in $\mathrm{Cogen}M$, then it is easy to see that the functor $\mathrm{Hom}_R(-, M)$ preserves the exactness of any short exact sequences in $\mathrm{Cogen}M$. So M is a costar module by the assumption. By Lemma 1.1, we also have that $\mathrm{Cogen}M$ is a torsion-free class.

(3)$\Rightarrow$ (4) Clearly, $\mathrm{Cogen}M \subseteq \mathrm{Fac}(\mathrm{Cogen}M)$. To see that $\mathrm{Cogen}M \subseteq {}^{\perp_1}M$, we take any extension $0 \to M \to N \to L \to 0$ with $L \in \mathrm{Cogen}M$. Since $\mathrm{Cogen}M$ is a torsion-free class, it is closed under extensions. It follows that $N \in \mathrm{Cogen}M$. But M is a costar module, the exact sequence is then $\mathrm{Hom}_R(-, M)$-exact. Thus, we can obtain that the exact sequence is actually split. Consequently we have that $L \in {}^{\perp_1}M$ for any $L \in \mathrm{Cogen}M$, i.e., $\mathrm{Cogen}M \subseteq {}^{\perp_1}M$.

On the other hand, take any $N \in \mathrm{Fac}(\mathrm{Cogen}M) \bigcap {}^{\perp_1}M$. Then there is a module L in $\mathrm{Cogen}M$ and an epimorphism $f\colon L \to N$. Set $K = \ker f$, then $K \in \mathrm{Cogen}M$ since $L \in \mathrm{Cogen}M$. Then there is a short exact sequence $0 \to K \to M_0 \to A \to 0$ with $A \in \mathrm{Cogen}M$ and $M_0 \in \mathrm{Adp}M$, as $\mathrm{Cogen}M = \mathrm{Copres}M$. Since $N \in {}^{\perp_1}M$, we have the following communicative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & L & \xrightarrow{f} & N & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow \alpha & & \downarrow \beta & & \\ 0 & \longrightarrow & K & \longrightarrow & M_0 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

By Snake Lemma, we obtain that $\ker\beta = \ker\alpha \in \text{Cogen}M = \text{Copres}M$ since $L \in \text{Cogen}M$. From the third column in above diagram, we obtain a new short exact sequence $0 \to \ker\beta \to N \to \text{Im}\beta \to 0$ with $\text{Im}\beta \in \text{Cogen}M$ (since $A \in \text{Cogen}M$). By the assumption, $\text{Cogen}M$ is a torsion-free class, so we have $N \in \text{Cogen}M$. Hence, $\text{Cogen}M = \text{Fac}(\text{Cogen}M) \bigcap {}^{\perp_1}M$.

(4)$\Rightarrow$ (2) We need only to prove that $\text{Cogen}M \subseteq \text{Copres}M$. Take any $N \in \text{Cogen}M$ and consider the evaluation map u: $N \to M^X$ with $X = \text{Hom}_R(N, M)$. It is easy to verify that u is injective. Thus we have a short exact sequence $0 \to N \to M^X \to C \to 0$, so it is enough to prove that $C \in \text{Cogen}M$. Applying the functor $\text{Hom}_R(-, M)$ to the sequence, we have an exact sequence $0 \to \text{Hom}_R(C, M) \to \text{Hom}_R(M^X, M) \to^{\gamma} \text{Hom}_R(N, M) \to \text{Ext}^1_R(C, M) \to 0$. It follows from u is the evaluation map that γ is surjective. So that $\text{Ext}^1_R(C, M) = 0$, i.e. $C \in {}^{\perp_1}M$. It follows that $C \in \text{Fac}(\text{Cogen}M) \bigcap {}^{\perp_1}M = \text{Cogen}M$ by (4). □

The above result suggests the following definition.

Definition 2.5 *The torsion-free class $\mathcal{T}$ is called a quasi-cotilting class if $\mathcal{T} = \text{Cogen}M$ for some quasi-cotilting module.*

Let M be an R-module. We denoted by $\text{Ann}M$ the ideal of R consisting of all elements $r \in R$ such that $rM = 0$. If $\text{Ann}M = 0$, then M is called faithful.

Lemma 2.6 *The following statements are equivalent for an R-module M.*

(1) *M is faithful;*

(2) *$R \in \text{Cogen}M$;*

(3) *$\text{Proj}R \subseteq \text{Cogen}M$;*

(4) *$\text{Fac}(\text{Cogen}M) = R{-}\text{Mod}$.*

(5) *$Q \in \text{Fac}(\text{Adp}M)$.*

(6) *$Q \in \text{Fac}(\text{Cogen}M)$.*

Proof. (1)$\Leftrightarrow$(2) Note that $\text{Ann}M$ is just the kernel of the evaluation map $R \to M^{\text{Hom}_R(R,M)}$, so the result follows from the universal property of the evaluation map.

(2)$\Leftrightarrow$(3) This is followed from the fact that $\text{Cogen}M$ is closed under direct sums and direct summands.

(3)$\Rightarrow$(4) Using the fact that every module is a quotient of a projective module.

(4)$\Rightarrow$(3) If P is any projective module and $\text{Fac}(\text{Cogen}M) = R{-}\text{Mod}$, then there is an exact sequence $0 \to P_1 \to C \to P \to 0$ with $C \in \text{Cogen}M$. But P is projective implies that the exact sequence is split, it follows that P is a direct summand of C and consequently, $P \in \text{Cogen}M$. Thus, (3) follows.

(4)$\Rightarrow$(5)$\Rightarrow$(6) Obviously.

(6)$\Rightarrow$(4) Clearly, $\text{Fac}(\text{Cogen}M) \subseteq R$-Mod. On the other hand, since $Q \in \text{Fac}(\text{Cogen}M)$, there exists an $H \in \text{Cogen}M$ such that f: $H \to Q$ is surjective. For any $L \in R$-Mod, we have a monomorphism $L \to Q^X$ for some X, since Q an injective cogenerator. Then f^X is surjective and $Q^X \cong H^X/K$ with $K = \ker f^X$. Consequently, there exists a submodule H_1 of H^X such that $L \cong H_1/K$. It is easy to see that $H_1 \in \text{Cogen}M$, so $L \in \text{Fac}(\text{Cogen}M)$ and (4) holds. □

Recall that an R-module M is called (1-)cotilting if it satisfies the following three conditions: (1) the injective dimension of M is not more than 1, i.e., $\mathrm{id}M \leqslant 1$; (2) $\mathrm{Ext}_R^1(M^\lambda, M) = 0$ for any set λ; (3) There is an exact sequence $0 \to M_1 \to M_0 \to Q \to 0$ where $M_0, M_1 \in \mathrm{Adp}M$ and Q is an injective cogenerator. Note that M is cotilting is equivalent to that $\mathrm{Cogen}M = {}^{\perp_1}M$, see for instance [3]. We will freely use these two equivalent definition of cotilting modules.

A torsion-free class $\mathcal{T}$ is called a cotilting class if $\mathcal{T} = \mathrm{Cogen}M$ for some cotilting module M.

We have the following characterizations of cotilting modules. Some of them were obtained in [16] (in Chinese). For reader's convenience, we include here a complete proof.

Proposition 2.7 *Let M be an R-module and Q be an injective cogenerator of $R-\mathrm{Mod}$, then the following statements are equivalent:*

(1) *M is a cotilting module;*

(2) *M is a quasi-cotilting module and $Q \in \mathrm{Fac}(\mathrm{Adp}M)$;*

(3) *M is a quasi-cotilting module and $\mathrm{Proj}R \subseteq \mathrm{Cogen}M$;*

(4) *M is a quasi-cotilting module and $\mathrm{Fac}(\mathrm{Cogen}M) = R-\mathrm{Mod}$;*

(5) *M is a faithful quasi-cotilting module.*

(6) *M is a quasi-cotilting module and $\mathrm{Cogen}M$ is a cotilting torsion-free class.*

(7) *M is a costar module and $Q \in \mathrm{Fac}(\mathrm{Adp}M)$;*

(8) *M is a costar module and $\mathrm{Proj}R \subseteq \mathrm{Cogen}M$;*

(9) *M is a costar module and $\mathrm{Fac}(\mathrm{Cogen}M) = R-\mathrm{Mod}$;*

(10) *M is a faithful costar module.*

(11) *M is a costar module and $\mathrm{Cogen}M$ is a cotilting torsion-free class.*

Proof. (2)⇔(3)⇔(4)⇔(5)⇒(10)⇔(9)⇔(8)⇔(7) By Lemma 2.6 and the definitions.

(1)⇒(2) Since M is cotilting, we have that and $\mathrm{Cogen}M = {}^{\perp_1}M$ and $\mathrm{Fac}(\mathrm{Cogen}M) = R-\mathrm{Mod}$ by the above arguments. It follows $\mathrm{Cogen}M = \mathrm{Fac}(\mathrm{Cogen}M) \bigcap {}^{\perp_1}M$. Hence, M is quasi-cotilting by Proposition 2.4.

(8)⇒(1) Since $\mathrm{Proj}R \subseteq \mathrm{Cogen}M$, we have that $\mathrm{Fac}(\mathrm{Cogen}M)$=$R$-Mod. For any $T \in \mathrm{Cogen}M$, there is a short exact sequence $0 \to K \to P_0 \to T \to 0$ with $P_0 \in \mathrm{Proj}R$. Note that the sequence is indeed in $\mathrm{Cogen}M$, so $\mathrm{Hom}_R(-, M)$ preserves exactness of this short exact sequence, as M is a costar module. It follows that $\mathrm{Ext}_R^1(T, M) = 0$ since $\mathrm{Ext}_R^1(P_0, M) = 0$. Thus $\mathrm{Cogen}M \subseteq {}^{\perp_1}M$ and $\mathrm{Cogen}M$ is a torsion-free class by Lemma 1.1. Furthermore, we have that M is a quasi-cotilting module by Proposition 2.4. Consequently, $\mathrm{Cogen}M = {}^{\perp_1}M$ by Proposition 2.4 again, since $\mathrm{Fac}(\mathrm{Cogen}M)$=$R$-Mod. So M is a cotilting module.

(1)⇒(6)⇒(11) It is obvious now.

(11)⇒(8). If $\mathrm{Cogen}M$ is a cotilting torsion-free class, i.e., $\mathrm{Cogen}M = \mathrm{Cogen}T$ is for some cotilting module T, Then $\mathrm{Proj}R \subseteq \mathrm{Cogen}T$ by the above argument. Thus (8) follows. □

The above result yields the following characterization of costar modules.

Proposition 2.8 *The following statements are equivalent for an R-module M.*

(1) *M is a costar R-module;*

(2) *M is a costar $\bar{R}$-module, where $\bar{R} = R/\mathrm{Ann}M$;*

(3) *M is a costar R/I-module for any ideal I of R such that $IM = 0$;*

(4) *M is a cotilting $\bar{R}$-module.*

Proof. Note that the category of R/I-modules can be identified to the full subcategory $\{M \in R\text{-Mod} \mid IM = 0\}$. Under this identification, it is not difficult to verify that $\text{Cogen}_{R/I}M = \text{Cogen}_R M = \text{Cogen}_{\bar{R}}M$. Therefore, (1)⇔(2)⇔(3) is obvious from the definitions.

(2)⇒(4) Note that M is always faithful as an $\bar{R}$-module, so the conclusion follows from Proposition 2.7.

(4)⇒(2) By Proposition 2.7. □

A short exact sequence $0 \to A \to B \to C \to 0$ is called pure exact if it is $(M \otimes_R -)$-exact for any right R-module M. In this case, C is called a pure quotient module of B. A module M is called pure injective if any pure exact sequence is $\text{Hom}_R(-, M)$-exact.

Lemma 2.9 *The following statements are equivalent:*

(1) *M is a pure injective R-module;*

(2) *For any X, the short exact sequence $0 \to M^{(X)} \to M^X \to C \to 0$ is $\text{Hom}_R(-, M)$ exact;*

(3) *M is a pure injective $\bar{R}$-module, where $\bar{R} = R/\text{Ann}M$.*

Proof. (1) ⇔ (2) by [5, Lemma 2.1]. Similar to (1) ⇔ (2), it is easy to see that (2) ⇔ (3) since $M^{(X)}$, M^X and C are in $\bar{R}$-Mod. □

From the above discussion, we can get the following important properties of quasi-cotilting modules.

Proposition 2.10 *All costar modules are pure injective and cofinendo. Specially, all quasi-cotilting modules are pure injective and cofinendo.*

Proof. Let M be a costar R-module. We obtain that M is a cotilting $\bar{R}$-module by Proposition 2.8. It follows that M is a pure injective $\bar{R}$-module since all cotilting modules are pure injective [5]. Thus M is a pure injective R-module by Lemma 2.9.

To prove that M is cofinendo, we only need to prove that $\text{Cogen}M$ is closed under direct sums and pure quotient modules by Lemma 2.11 and Lemma 1.3. Obviously, $\text{Cogen}M$ is closed under direct sums. Suppose the short exact sequence $0 \to A \to B \to C \to 0$ is pure exact and $B \in \text{Cogen}M$. It follows that $A \in \text{Cogen}M$ and that $0 \to A \to B \to C \to 0$ is $\text{Hom}_R(-, M)$ exact from M is pure injective. So $C \in \text{Cogen}M$ by Proposition 2.3, i.e. $\text{Cogen}M$ is closed under pure quotient modules. □

Lemma 2.11 *[15, Theorem 2.5] If a class $\mathcal{A}$ is closed under pure quotient modules, then the following statements are equivalent:*

(1) *$\mathcal{A}$ is closed under arbitrary direct sums;*

(2) *$\mathcal{A}$ is a precover class;*

(3) *$\mathcal{A}$ is a cover class.*

Corollary 2.12 *If M is a costar module, then $\text{Cogen}M$ is a cover class. In particular, $\text{Cogen}M$ is a cover class for any quasi-cotilting module M.*

Proof. By the proof of Proposition 2.10, we obtain that $\text{Cogen}M$ is closed under pure quotient and direct sums. Thus $\text{Cogen}M$ is a cover class by Lemma 2.11. □

3 Quasi-cotilting torsion-free class

Lemma 3.1 *If M is a quasi-cotilting module, then* $\mathrm{Adp}M = \ker \mathrm{Ext}^1_R(\mathrm{Cogen}M, -) \bigcap \mathrm{Cogen}M$.

Proof. Suppose that $N \in \mathrm{Adp}M$. Clearly we get $N \in \ker \mathrm{Ext}^1_R(\mathrm{Cogen}M, -) \bigcap \mathrm{Cogen}M$, from (3) in Proposition 2.4,. For the inverse inclusion, take any $L \in \ker \mathrm{Ext}^1_R(\mathrm{Cogen}M, -) \bigcap \mathrm{Cogen}M$. Then there is a short exact sequence $0 \to L \to M_0 \to C \to 0$ with $M_0 \in \mathrm{Adp}M$ and $C \in \mathrm{Cogen}M$ since $L \in \mathrm{Cogen}M = \mathrm{Copres}M$. Since $\mathrm{Ext}^1(C, L) = 0$, we can obtain that this exact sequence is split and hence $L \in \mathrm{Adp}M$. $\square$

Theorem 3.2 *Let Q be an injective cogenerator of R-Mod. The following statements are equivalent:*

(1) *M is a quasi-cotilting module;*

(2) *M is Ext-injective in* $\mathrm{Cogen}M$ *and there is an exact sequence*

$$0 \to M_1 \to M_0 \to^{\alpha} Q$$

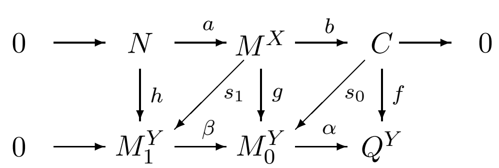
with M_0 and M_1 in $\mathrm{Adp}M$ *and α a* $\mathrm{Cogen}M$*-precover.*

Proof. (1)⇒(2) By Proposition 2.10, M is a cofinendo module. Then there is a morphism α: $M_0 \to Q$ with α an $\mathrm{Adp}M$-precover by Lemma 1.3. It can be shown that α is also a $\mathrm{Cogen}M$-precover. Indeed, suppose that $M' \in \mathrm{Cogen}M$, we proof that f can factor through α for any f: $M' \to Q$. There exists a monomorphism i: $M' \to M^X$ since $M' \in \mathrm{Cogen}M$. We have a morphism g: $M^X \to Q$ such that $f = gi$ since Q is injective. It follows from α is $\mathrm{Adp}M$-precover that there is a morphism h: $M^X \to M_0$ such that $g = \alpha h$. Thus $f = gi = \alpha hi$. So α is a $\mathrm{Cogen}M$-precover. Now set $M_1 = \ker \alpha$, we only need to prove that $M_1 \in \mathrm{Adp}M$. Consider the exact sequence $0 \to M_1 \to M_0 \to^{\pi} \mathrm{Im}\alpha \to 0$, it is easy to prove that π is also $\mathrm{Cogen}M$-precover by the definition. For any $N \in \mathrm{Cogen}M$, applying the functor $\mathrm{Hom}_R(N, -)$ to this exact sequence, we have that

$$0 \to \mathrm{Hom}_R(N, M_1) \to \mathrm{Hom}_R(N, M_0) \to \mathrm{Hom}_R(N, \mathrm{Im}\alpha) \to \mathrm{Ext}^1_R(N, M_1) \to \mathrm{Ext}^1_R(N, M_0) = 0.$$

It follows from π is $\mathrm{Cogen}M$-precover that $\mathrm{Ext}^1_R(N, M_1) = 0$. Obviously, $M_1 \in \mathrm{Cogen}M$. Thus $M_1 \in \mathrm{Adp}M$ by Lemma 3.1.

(2)⇒(1) We only need to prove that $\mathrm{Cogen}M = \mathrm{Copres}M$ by Proposition 2.4. Suppose that $N \in \mathrm{Cogen}M$. Consider the evaluation map a: $N \to M^X$ with $X = \mathrm{Hom}_R(N, M)$, which is injective since $N \in \mathrm{Cogen}M$. Set $C = \mathrm{coker}\ a$, next we prove that $C \in \mathrm{Cogen}M$. There is a monomorphism f: $C \to Q^Y$ for some Y since Q is an injective cogenerator. Now consider the following commutative diagram:



Since α is $\mathrm{Cogen}M$-precover, there is a morphism g such that $\alpha g = fb$, and then we have that $\beta h = ga$. Since a is evaluation map and $M_1^Y \in \mathrm{Adp}M$, we have s_1 such that $h = s_1 a$. It is easy to see that $(g - \beta s_1)a = 0$, and then we have s_0 such that $g - \beta s_1 = s_0 b$. So $\alpha s_0 b = \alpha(g - \beta s_1) = \alpha g = fb$, thus $f = \alpha s_0$ since b is surjective. Since f is injective, s_0 is injective. Consequently, $C \in \mathrm{Cogen}M$ and $\mathrm{Cogen}M = \mathrm{Copres}M$. Then the proof is completed.

□

Let $\mathcal{D}$ be a class of R-modules. A module $M \in \mathcal{D}$ is called an injective object of $\mathcal{D}$ if for any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{D}$ is $\mathrm{Hom}_R(-, M)$-exact. A module $M \in \mathcal{D}$ is called a cogenerator of $\mathcal{D}$ if for any N in $\mathcal{D}$, there is a monomorphism i: $N \to M^X$ for some set X.

Lemma 3.3 *Let $\mathcal{T}$ be a class of R-modules and Q be an injective cogenerator of R-Mod. Suppose that an exact sequence $0 \to A \to B \to^{\alpha} Q \to^{\pi} K \to 0$ satisfies that α is a $\mathcal{T}$-precover and that A is* Ext*-injective in $\mathcal{T}$. Denote $\mathcal{D}$=$\{N \in R\text{-}Mod \mid \mathrm{Hom}_R(N, \pi) = 0\}$. Then $\mathcal{T} \subseteq \mathcal{D}$ and $M := \mathrm{Im}\alpha$ is an injective cogenerator in $\mathcal{D}$.*

Proof. For any $C \in \mathcal{T}$ and any morphism f: $C \to Q$, there is a morphism g: $C \to B$ such that $f = \alpha g$ since α is a $\mathcal{T}$-precover. Thus $\mathrm{Hom}_R(C, \pi) = 0$ and $\mathcal{T} \subseteq \mathcal{D}$.

Take any short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathcal{D}$ and any morphism h: $X \to M$. Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{a} & Y & \longrightarrow & Z & \longrightarrow & 0 \\ & & \downarrow h & \swarrow \beta & \downarrow c & & & & \\ 0 & \longrightarrow & M & \xrightarrow{b} & Q & \xrightarrow{\pi} & K & \longrightarrow & 0 \end{array}$$

It follows that from Q is injective that there is a morphisms c such that $bh = ca$. Since $Y \in \mathcal{D}$, we have a morphism β such that $c = b\beta$. Thus $bh = b\beta a$ and $h = \beta a$ since b is injective. So the exact sequence is $\mathrm{Hom}(-, M)$-exact, i.e. M is injective in $\mathcal{D}$. For any $N \in \mathcal{D}$, we have a monomorphism i: $N \to Q^I$ since Q is an injective cogenerator. Consider the exact sequence $0 \to M^I \to Q^I \to K^I \to 0$. Since $N \in \mathcal{D}$, we have that $\mathrm{Hom}_R(N, \pi^I) \cong (\mathrm{Hom}_R(N, \pi))^I = 0$, i.e., $\pi^I i = 0$. So there is a morphism γ: $N \to M^I$ such that $i = b^I\gamma$ and γ is injective since i is injective. Consequently, M is a cogenerator in $\mathcal{D}$. □

The following result is usually called Wakamatsu's lemma, see for instance [12].

Lemma 3.4 *(*Wakamatsu's lemma*) If $\mathcal{T}$ is a class of modules closed under extensions and if φ: $T \to M$ is a $\mathcal{T}$-cover, then $\ker\varphi \in T^{\perp_1}$.*

Theorem 3.5 *Let $\mathcal{T}$ be a torsion-free classes in R-Mod. The following statements are equivalent:*

(1) *$\mathcal{T}$ is quasi-cotilting torsion-free. i.e. there exists a quasi-cotilting module M such that $\mathcal{T} = \mathrm{Cogen}M$;*

(2) *$\mathcal{T}$ is a cover class;*

(3) *For any R-module N, there is an exact sequence*

$$0 \to A \to B \to^{\alpha} N$$

with α a $\mathcal{T}$-precover and A Ext*-injective in $\mathcal{T}$.*

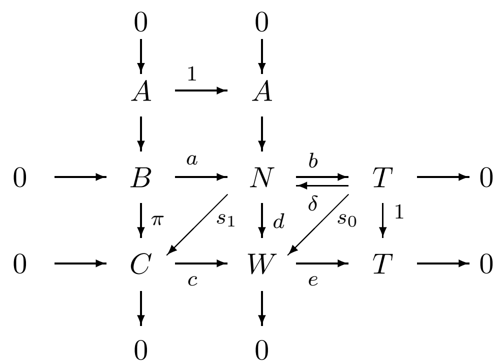
Proof. (1)⇒ (2) By Corollary 2.12.

(2)⇒ (3) For any R-module N, there is an exact sequence

$$0 \to A \to B \to^{\alpha} N$$

with α $\mathcal{T}$-cover by (2). By Wakamatsu's lemma, we have that $A \in \mathcal{T}^{\perp_1}$. Since $B \in \mathcal{T}$ and $\mathcal{T}$ is a torsion-free class, A is in $\mathcal{T}$. Thus A is Ext-injective in $\mathcal{T}$.

(3)⇒ (1) Take $N = Q$ with Q an injective cogenerator of R-Mod, then we have an exact sequence $0 \to A \to B \to^{\alpha} Q$ with α a $\mathcal{T}$-precover and A Ext-injective in $\mathcal{T}$ by assumption. Set $M = A \oplus B$. Then Cogen$M \subseteq \mathcal{T}$ since $\mathcal{T}$ is a torsion-free class. On the other hand, for any $L \in \mathcal{T}$, we have a monomorphism f: $L \to Q^X$. There is a morphism g: $L \to B^X$ such that $f = \alpha^X g$ since α^X is clearly a $\mathcal{T}$-precover. It follows from f is injective that g is injective. Thus $L \in$ Cogen$M$ and Cogen$M = \mathcal{T}$. It remains to show that M is Ext-injective in Cogen$M$ by Theorem 3.2. In fact, by assumption, we have to verify this only to B. Take any exact sequence $0 \to B \to N \to T \to 0$ with $T \in$ Cogen$M$. Then N is in Cogen$M$ since Cogen$M = \mathcal{T}$ is a torsion-free class. Set $C = \mathrm{Im}\alpha$, consider the pushout of $B \to N$ and $B \to C$:



By Lemma 3.3, the exact sequence $0 \to B \to N \to T \to 0$ is Hom$(-, C)$-exact. So there is a morphism s_1 such that $\pi = s_1 a$. It is easy to see that $(d - cs_1)a = 0$. So we have $d = cs_1 + s_0 b$ in the above diagram for some s_0. Since A is Ext-injective in Cogen$M$, there is a morphism δ: $T \to N$ such that $s_0 = d\delta$. So $b = ed = e(s_0 b + cs_1) = es_0 b$ and $es_0 = 1$ since b is surjective. But $b\delta = ed\delta = es_0 = 1$, thus, b is split. Consequently, B is Ext-injective in Cogen$M$. □

We say that two quasi-cotilting modules M_1 and M_2 are equivalent if AdpM_1 = AdpM_2.

Corollary 3.6 *There are bijections between*

(1) *equivalence classes of quasi-cotilting modules;*

(2) *torsion-free cover classes;*

(3) *torsion-free classes $\mathcal{T}$ in R-Mod such that every module has a $\mathcal{T}$-precover with Ext-injective kernel.*

Proof. Let M_1 and M_2 be two quasi-cotilting modules, it is easy to prove that AdpM_1 = AdpM_2 if and only if CogenM_1 = CogenM_2 by Lemma 3.1. Now this correspondences can be defined as follows:

(1)→(2): $M \longmapsto$ Cogen$M$

(2)→(3): $\mathcal{T} \longmapsto \mathcal{T}$

(3)→(1): $\mathcal{T} \longmapsto M$ with Cogen$M = \mathcal{T}$. □

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