Repetitive equivalences and good Wakamatsu-tilting modules *

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Abstract

Let R be a ring and T be a good Wakamatsu-tilting module with $S = \mathrm{End}_R T$. We prove that T induces an equivalence between stable categories of repetitive algebras $\hat{R}$ and $\hat{S}$.

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1 Introduction

Tilting theory plays an important role in the representation theory of artin algebras. The classical tilting modules were introduced in the early eighties by Brenner-Butler [6], Bongartz [5] and Happel and Ringel [17]. Beginning with Miyashita [21] and Happel [17], the defining conditions for a classical tilting module were relaxed to tilting modules of arbitrary finite projective dimension, and further were relaxed to arbitrary rings and infinitely generated modules by many authors such as Colby and Fuller [10], Colpi and Trlifaj [12], Angeleri-Hügel and Coelho [1], Bazzoni [4] etc..

One important result in tilting theory is the famous Brenner-Butler Theorem which shows that a tilting module induces some equivalences between certain subcategories. In this sense, tilting theory may be viewed as a far-reaching way of generalization of Morita theory of equivalences between module categories. More interesting, when considering the derived category of an algebra, which contains module category of the algebra as a full subcategory, Happel [16] and later Cline, Parshall and Scott [8] proved that a tilting module of finite projective dimension induces an equivalence between the bounded derived category of the ordinary algebra and the derived category of the endomorphism algebra of the tilting module.

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This leads to the study of Morita theory for derived categories, which were completely solved by Rickard [22] through the notion of tilting complexes and by Keller [19] through dg-categories.

A further generalization of tilting modules to tilting modules of possibly infinite projective dimension was given by Wakamatsu [23]. Following [15], such tilting modules of possibly infinite projective dimension are called Wakamatsu-tilting modules. It is known that Wakamatsu-tilting modules also induce some equivalences between certain subcategories of module categories [24]. But Wakamatsu-tilting modules don't induce derived equivalences in general.

However, we will show in this paper that Wakamatsu-tilting modules make more sense when we consider a more general category than the derived category of an algebra, namely, the stable module category of the repetitive algebra of an algebras. To be compared, let us call the later category the repetitive category of the algebra. The repetitive category is a triangulated category. Moreover, by Happel's result [16], for an artin algebra R, there is a fully faithful triangle embedding of the bounded derived category of R into the repetitive category of R. Moreover, this embedding is an equivalence if and only if the global dimension of R is finite.

We say that two algebras are repetitive equivalent if there is an equivalence between their repetitive categories. It should be noted that repetitive equivalences are more general than derived equivalences. In fact, by results in [2, 7, 22] etc., if two algebras are derived equivalent, then their repeptitive algebras are derived equivalent, and hence stably equivalent. Thus derived equivalences always induce repetitive equivalences.

The following is our main theorem.

Main Theorem *Let R be an artin algebra. If T is a good Wakamatsu-tilting R-module with $S = \mathrm{End}(T_R)$, i.e., bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between a complete hereditary cotorsion piar $(\mathcal{B}, \mathcal{A})$ in $\mathrm{mod}R$ and a complete hereditary cotorsion piar $(\mathcal{G}, \mathcal{K})$ in $\mathrm{mod}S$, then R and S are repetitive equivalent. The equivalence restricts to the equivalence between $\mathcal{A}$ and $\mathcal{G}$.*

We refer to subsection 3.2 for more details on good Wakamatsu-tilting modules. Note that that examples of good Wakamatsu-tilting modules contain tilting modules of finite projective dimension and cotilting modules of finite injective dimension.

Though a good Wakamatsu-tilting module induces a repetitive equivalence, unfortunately we can't say anything about whether or not the equivalence is a triangle equivalence now. However, if a repetitive equivalence is a triangle equivalence, we have the following result.

Proposition *Let R and S are artin algebras. Assume that there is a triangle equivalence between their repetitive categories and that this equivalence restricts to an equivalence between a covariantly finite coresolving subcategory $\mathcal{A}$ in $\mathrm{mod}R$ and a contravariantly finite resolving subcategory $\mathcal{G}$ in $\mathrm{mod}R$. Let T be the preimage in $\mathrm{mod}R$ of S. Then T is a good Wakamatsu-*

tilting R-module with $S \simeq \mathrm{End}(T_R)$.

The paper is organized as follows. After the introduction, we provide basic knowledge on Wakamatsu-tilting modules and repetitive categories in Section 2. Then in Section 3 we introduce good Wakamatsu-tilting modules through cotorsion pair counter equivalences. Some properties and characterizations of good Wakamatsu-tilting modules are presented. Section 4 is devoted to the proof of the main theorem and the proposition in the introduction. Though the proof of the theorem is a little complicated, the main idea is inspired by constructions in [16, Lemma 4.1 in Chapter 3] and [24, Section 1]. Finally, we provide some examples in the last section. In particular, it is shown that every Wakamatsu-tilting module over an algebra of finite representation type is a good Wakamatsu-tilting module and hence induces a repetitive equivalence.

Conventions Throughout this paper, we always work over artin algebras and right modules unless we claim otherewise. For an algebra R, we denote by mod$R$ the category of all finitely generated R-modules, and by proj$R$ (resp., inj$R$) the category of finitele generated projective (resp., injective) R-modules. We denote the usual duality over an artin algebra R by D.

For two functors $\mathrm{F}:\mathcal{A}\to\mathcal{B}$ and $\mathrm{G}:\mathcal{B}\to\mathcal{C}$, we use GF to denote their composition. While we use $f\circ g$, or simply just fg, to denote the composition of two homomorphisms $f: A\to B$ and $g: B\to C$.

Let $\mathrm{F}:\mathcal{A}\to\mathcal{B}$ be a functor, we use KerF to denote the subcategory of $A\in\mathcal{A}$ such that $\mathrm{F}(A)=0$. Moreover, if $\mathrm{F}_i:\mathrm{A}\to\mathcal{B}$, $i\in I$, is a class of functors, we denote $\mathrm{KerF}_I = \bigcap_{i\in I}\mathrm{KerF}$. For instance, $\mathrm{KerExt}_R^{\geq 1}(T,-)$ is the subcategory of all $M\in \mathrm{mod}R$ such that $\mathrm{Ext}_R^i(T,M)=0$ for all $i\geq 1$.

We write the elements of direct sums as row vectors.

2 Wakamatsu-tilting modules and repetitive categories

2.1 Wakamatsu-tilting modules

Recall that an R-module T is *Wakamatsu-tilting* [23] provided that

(1) $\mathrm{End}_S T\simeq R$, where $S:=\mathrm{End}_R T$ and,
(2) $\mathrm{Ext}_R^i(T,T)=0=\mathrm{Ext}_S^i(T,T)=0$ for all $i>0$.

These two conditions are also equivalent to the following two conditions [23].

(1) $\mathrm{Ext}_R^i(T,T)=0$ for all $i>0$ and,
(2) There is an exact sequence $0\to R\to T_0\to T_1\to\cdots$, where $T_i\in\mathrm{add}_R T$ for all i, which stays exact after applying the functor $\mathrm{Hom}_R(-,T)$.

Note that if T is Wakamatsu-tilting and $S = \mathrm{End}(T_R)$, then ${}_ST$ is a Wakamatsu-tilting left S-module. In this case, we say that T is a Wakamatsu-tilting S-R-bimodule. It is easy to see that DT is a Wakamatsu-tilting R-S-bimodule in the mean time.

2.1.1 *Auslander class and co-Auslander class*

Let T be a Wakamatsu-tilting module with $S = \mathrm{End}(T_R)$. There are the following two interesting classes associated with Wakamatsu-tilting modules.

The *Auslander class* in mod$R$ with respect to the Wakamatsu-tilting module T_R, denoted by $\mathcal{X}_T$, is defined as follows [3].

$$\mathcal{X}_T := \{M \in \mathrm{mod}R \mid \text{ there is an infinite exact sequence } 0 \to M \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \xrightarrow{f_2} \cdots \text{ such that } \mathrm{Im} f_i \in \mathrm{KerExt}_R^{\geq 1}(-, T) \text{ for each } i \geq 0\}.$$

Obviously, it hold that $\mathcal{X}_T \subseteq \mathrm{KerExt}_R^{\geq 1}(-, T)$. Moreover, the two classes coincide with each other provided that T is a cotilting R-module.

Dually, the *co-Auslander class* in mod$R$ with respect to the Wakamatsu-tilting R-module T, denoted by ${}_T\mathcal{X}$, is defined as follows.

$${}_T\mathcal{X} := \{M \in \mathrm{mod}R \mid \text{ there is an infinite exact sequence } \cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} M \to 0 \text{ such that } \mathrm{Im} f_i \in \mathrm{KerExt}_R^{\geq 1}(T, -) \text{ for each } i \geq 0\}.$$

Similarly, we have that ${}_T\mathcal{X} \subseteq \mathrm{KerExt}_R^{\geq 1}(T, -)$ and they coincide with each other provided that T is a tilting R-module.

The following result collects some properties about Auslander class and co-Auslander class for a Wakamatsu-tilting module [3, 20, 24, 25].

Proposition *Let T be a Waksmatsu-tilting R-module with $S = \mathrm{End}_R T$.*

(1) *The Auslander class $\mathcal{X}_T$ is a resolving subcategory, i.e., it contains all projective R-modules and is closed under extensions, kernels of epimorphisms and direct summands.*

(2) *The co-Auslander class ${}_T\mathcal{X}$ is a coresolving subcategory, i.e., it contains all injective R-modules and is closed under extensions, cokernels of monomorphisms and direct summands.*

(3) $\mathrm{KerExt}_R^1(\mathcal{X}_T, -) = \mathrm{KerExt}_R^{\geq 1}(\mathcal{X}_T, -) \subseteq {}_T\mathcal{X}$.

(4) $\mathrm{KerExt}_R^1(-, {}_T\mathcal{X}) = \mathrm{KerExt}_R^{\geq 1}(-, {}_T\mathcal{X}) \subseteq \mathcal{X}_T$.

(5) *The functors $\mathrm{Hom}_R(T, -)$ and $- \otimes_S T$ induce an equivalence between the co-Auslander class ${}_T\mathcal{X}$ in mod$R$ and the Auslander class $\mathcal{X}_{DT}$ in mod$S$. The equivalence restricts to an equivalence between the class $\mathrm{KerExt}_R^{\geq 1}(\mathcal{X}_T, -)$ and the class $\mathrm{KerExt}_S^{\geq 1}(-, {}_{DT}\mathcal{X})$.*

Proof. (1) and (2) follows from [3, Section 5], see also [20].

(3) and (4) follows from [25, Lemma 1.4 and Proposition 1.6].

(5) follows from [24, Proposition 2.14]. □

We remark that in case $T = R$, the class $\mathcal{X}_T = \mathcal{X}_R$ is just the class of all Gorenstein projective R-modules. Dually, in case $T = DR$, the class ${}_T\mathcal{X} = {}_{DR}\mathcal{X}$ is just the class of all Gorenstein injective modules. We refer to [13] for more on Gorenstein projective and Gorenstein injective modules.

2.1.2 The following is a characterization of Auslander class and co-Auslander class, by [24, Section 2] .

Lemma *Let T be a Wakamatsu-tilting R-module with $S = \mathrm{End}(T_R)$. Assume $X \in \mathrm{mod}R$*

(1) $X \in {}_T\mathcal{X}$ *if and only if* $X \in \mathrm{KerExt}_R^{\geq 0}(T, -)$, $\mathrm{Hom}_R(T, X) \in \mathrm{KerTor}_{>0}^S(-, T)$ *and* $\mathrm{Hom}_R(T, X) \otimes_S T \simeq X$ *canonically.*

(2) $X \in \mathcal{X}_T$ *if and only if* $X \in \mathrm{KerExt}_R^{\geq 0}(-, T)$, $\mathrm{Hom}_R(X, T) \in \mathrm{KerExt}_S^{\geq 0}(-, T)$ *and* $\mathrm{Hom}_R(\mathrm{Hom}_R(X, T), T) \simeq A$ *canonically.*

2.1.3 *Useful isomorphisms*

Let T be a Wakamatsu-tilting S-R-bimodule. Then we have the following isomorphisms of bimodules.

$${}_SDS_S \simeq {}_ST \otimes_R DT_S \text{ and } {}_RDR_R \simeq {}_RDT \otimes_S T_R.$$

Given an adjoint pair (F, G) of functors, we denote by $\mathbf{\Gamma}$ the natural adjoint isomorphism

$$\mathbf{\Gamma}: \ \mathrm{Hom}(\mathrm{F}(-), -) \simeq \mathrm{Hom}(-, \mathrm{G}(-)).$$

Moreover, for a homomorphism $f : \mathrm{F}(X) \to Y$, we denote by $\mathbf{\Gamma}(f) : X \to \mathrm{G}(Y)$ the image of f under the isomorphism $\mathbf{\Gamma}$.

In particular, associated with a S-R-bimodule T, we have the following adjoint isomorphism

$$\mathbf{\Gamma}^T: \ \mathrm{Hom}_R(- \otimes_S T, -) \simeq \mathrm{Hom}_S(-, \mathrm{Hom}_R(T, -)).$$

We denote by η^T and ϵ^T the unit and counit of this adjoint pair respectively, i.e.,

$$\eta_X^T = \mathbf{\Gamma}^T(1_{X \otimes_S T}) : X \to \mathrm{Hom}_R(T, X \otimes_S T) \text{ and}$$

$$\epsilon_Y^T = (\mathbf{\Gamma}^T)^{-1}(1_{\mathrm{Hom}_R(T,Y)}) : \mathrm{Hom}_R(T, Y) \otimes_S T \to Y$$

for $X \in \mathrm{mod}S$ and $Y \in \mathrm{mod}R$ respectively.

By the naturality of the isomorphism $\mathbf{\Gamma}$, for all homomorphisms $f : X_1 \to X_2$, $g : \mathrm{F}(X_2) \to Y_1$ and $h : Y_1 \to Y_2$, it holds that $\mathbf{\Gamma}(\mathrm{F}(f) \circ g \circ h) = f \circ \mathbf{\Gamma}(g) \circ \mathrm{G}(h)$.

2.2 Reprtitive algebras and Repetitive categories

2.2.1 We recall some basic facts on repetitive algebras mainly from [16].

Let R be an artin algebra. The repetitive algebra $\widehat{R}$ of R was first introduced in [18]), which is defined to be the direct sum $\widehat{R}=\bigoplus_{n\in\mathbb{Z}}R\oplus\bigoplus_{n\in\mathbb{Z}}DR$ with the multiplication given by

$$(a_n,\varphi_n)(b_n,\psi_n)_n=(a_nb_n,a_{n+1}\psi_n+\varphi_nb_n)_n.$$

The repetitive algebra $\widehat{R}$ can be interpreted as the following infinite matrix algebra (without the identity)

$$\begin{pmatrix} \ddots & & & & \\ \ddots & R & & & \\ & DR & R & & \\ & & DR & R & \\ & & & \ddots & \ddots \end{pmatrix}.$$

2.2.2 Denote by mod$\widehat{R}$ the category of finitely generated $\widehat{R}$-modules. Then mod$\widehat{R}$ is equivalent to the following two equivalent categories.

(1) $\mathcal{RC}^{\otimes}(R):=\{X=\{X_i,\delta_i^{\otimes}(X)\}_{i\in\mathbb{Z}}\mid X_i\in \text{mod}R$ such that almost all X_i are 0 and that $\delta_i^{\otimes}(X):X_i\otimes_R DR\to X_{i-1}$ satisfying $(\delta_{i+1}^{\otimes}(X)\otimes_R DR)\circ\delta_i^{\otimes}(X)=0$, for each $i\}$.

(2) $\mathcal{RC}^{\text{H}}(R):=\{X=\{X_i,\delta_i^{\text{H}}(X)\}_{i\in\mathbb{Z}}\mid X_i\in\text{mod}R$ such that almost all X_i are 0 and that $\delta_i^{\text{H}}(X):X_i\to\text{Hom}_R(DR,X_{i-1})$ satisfying $\delta_{i+1}^{\text{H}}(X)\circ\text{Hom}_R(DR,\delta_i^{\text{H}}(X))=0$, for each $i\}$.

We will freely use these equivalences. In particular, we often view $X\in\text{mod}\widehat{R}$ as the following form with almost terms $X_i=0$

$$\cdots\overset{\delta_{i+1}}{\rightsquigarrow}X_i\overset{\delta_i}{\rightsquigarrow}X_{i-1}\overset{\delta_{i-1}}{\rightsquigarrow}\cdots.$$

and we call it a (bounded chain) repe-complex with the repe-difference δ. We denote by $\mathcal{RC}(R)$ the category of all such repe-complexes. Thus, $\mathcal{RC}(R)=\text{mod}\widehat{R}$. Note that there is an obvious automorphism $[1]:\mathcal{RC}(R)\to\mathcal{RC}(R)$ defined by $(X[1])_i=X_{i-1}$ for each i.

We say that a repe-complex $X=\{X_i,\delta_i\}\in\mathcal{RC}(R)$ is trivial if each $\delta_i=0$. The full subcategory of all trivial repe-complexes is denoted by $\mathcal{RC}^{\text{tr}}(R)$. Note that there is a natural forgetting functor from $\mathcal{RC}(R)$ to $\mathcal{RC}^{\text{tr}}(R)$ by forgetting the repe-difference.

Let $\mathcal{C}$ be a class of R-modules, we denote by $\mathcal{RC}(\mathcal{C})$ the class of repe-complexes with terms in $\mathcal{C}$. The notation $\mathcal{RC}^{\text{tr}}(\mathcal{C})$ is defined similarly.

2.2.3 As shown in [16], $\widehat{R}$ is a selfinjective algebra and the category $\mathcal{RC}(R)(=\text{mod}\widehat{R})$ is a Frobenius category, where the projective (and also injective) objects are of the form

$$\cdots\overset{\delta_{i+1}}{\rightsquigarrow}P_i\oplus I_i\overset{\delta_i}{\rightsquigarrow}P_{i-1}\oplus I_{i-1}\overset{\delta_{i-1}}{\rightsquigarrow}\cdots,$$

where $P_i\in\text{proj}R$, $I_i\in\text{inj}R$ and $\delta_i=\begin{pmatrix}0&\delta_i'\\0&0\end{pmatrix}$ such that $\delta_i':P_i\otimes_R DR\to I_{i-1}$ is an isomorphism (considered in $\mathcal{RC}^{\otimes}(R)$), or equivalently, $\delta_i':P_i\to\text{Hom}_R(DR,I_{i-1})$ is an isomorphism (considered in $\mathcal{RC}^{\text{H}}(R)$). Thus its stable category $\underline{\mathcal{RC}}(R)$ is a triangulated category. To compare with the derived category of an algebra, we will call it the *repetitive category* of mod$R$ (or simply, of R).

It was shown in [16] that there is a fully faithful triangle embedding from the derived category $\mathcal{D}^b(\mathrm{mod}R)$ to the repetitive category $\underline{\mathcal{RC}}(R)$. Moreover, there is a triangle equivalence between $\mathcal{D}^b(\mathrm{mod}R)$ and $\underline{\mathcal{RC}}(R)$ if and only if R has finite global dimension.

For basic knowledge on triangulated categories, derived categories and the tilting theory, we refer to [16].

3 Cotorsion pairs and good Wakamatusu-tilting modules

3.1 Cotorsion pair counter equivalences

A pair of subcategories $(\mathcal{B}, \mathcal{A})$ in mod$R$ is called a cotorsion pair, if $\mathcal{B} = \mathrm{KerExt}^1_R(-, \mathcal{A})$ and $\mathcal{A} = \mathrm{KerExt}^1_R(\mathcal{B}, -)$. A cotorsion pair $(\mathcal{B}, \mathcal{A})$ is called hereditary provided that $\mathcal{B}$ is resolving, or equivalently, $\mathcal{A}$ is coresolving. Moreover, a cotorsion pair $(\mathcal{B}, \mathcal{A})$ is called complete provided that, for each $X \in \mathrm{mod}R$, there exist exact sequences $0 \to X \to A \to B \to 0$ and $0 \to A' \to B' \to X \to 0$ for some $A, A' \in \mathcal{A}$ and $B, B' \in \mathcal{B}$. We refer to the book [14] for the general results on cotorsion pairs.

Let $(\mathcal{B}, \mathcal{A})$ be a cotorsion pair in mod$R$ and $(\mathcal{G}, \mathcal{K})$ be a cotorsion pair in mod$S$. Similarly to torsion theory counter equivalences studied in [9, 11], we say that there is a *cotorsion pair counter equivalence* between $(\mathcal{B}, \mathcal{A})$ and $(\mathcal{G}, \mathcal{K})$ provided that there are an equivalence $\mathrm{H} : \mathcal{A} \rightleftarrows \mathcal{G} : \mathrm{T}$ and an equivalence $\mathrm{H}' : \mathcal{K} \rightleftarrows \mathcal{B} : \mathrm{T}'$. Moreover, we say that two bimodules ${}_SV_R$ and ${}_RV'_S$ represent the cotorsion pair counter equivalence if $\mathrm{H} = \mathrm{Hom}_R(V, -)$, $\mathrm{T} = - \otimes_S V$ and $\mathrm{H}' = \mathrm{Hom}_S(V', -)$, $\mathrm{T} = - \otimes_R V'$.

There are close relations between Wakamatsu-tilting modules and cotorsion pairs, as shown in the following proposition.

Proposition *Let T be a Waksmatsu-tilting R-module with $S = \mathrm{End}(T_R)$.*

(1) *Both pairs $(\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}), {}_T\mathcal{X})$ and $(\mathcal{X}_T, \mathrm{KerExt}^1_R(\mathcal{X}_T, -))$ are hereditary cotorsion pairs.*

(2) *The bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the cotorsion pair $(\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}), {}_T\mathcal{X})$ in mod$R$ and the cotorsion pair $(\mathcal{X}_{DT}, \mathrm{KerExt}^1_S(\mathcal{X}_{DT}, -))$ in mod$S$.*

(3) *The bimodules ${}_RDT_S$ and ${}_ST_R$ represent a cotorsion pair counter equivalence between the cotorsion pair $(\mathrm{KerExt}^1_S(-, {}_{DT}\mathcal{X}), {}_{DT}\mathcal{X})$ in mod$S$ and the cotorsion pair $(\mathcal{X}_T, \mathrm{KerExt}^1_R(\mathcal{X}_T, -))$ in mod$R$.*

Proof. (1) follows from [20, Proposition 3.1] and Propsition 2.1.1.

(2) follows from Propositon 2.1.1 (5).

(3) is obtained from (2) by replacing ${}_ST_R$ with ${}_RDT_S$. □

3.2 Good Wakamatsu-tilting modules

3.2.1 In general case, the two cotorsion pairs in Proposition 3.1 (1) are not complete. For instance, consider the case $T = R$. Then $\mathcal{X}_R$ is the class of all Gorensten projective modules. It is well known that this class is not a precovering class in general, see for instance [26]. Thus, the cotorsion pair $(\mathcal{X}_R, \mathrm{KerExt}^1_R(\mathcal{X}_R, -))$ cannot be complete. Dually, in case $T = DR$, the cotorsion pair $(\mathrm{KerExt}^1_R(-, {}_{DR}\mathcal{X}), {}_{DR}\mathcal{X})$ is not complete in general.

However, the other cotorsion pair of the two cotorsion pairs in Proposition 3.1 (1) for the above examples is complete respectively. This leads to the following general definition.

Definition *A Wakamatsu-tilting bimodule ${}_ST_R$ is said to be* **good** *if the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between a compete hereditary cotorsion pair $(\mathcal{B}, \mathcal{A})$ in* mod$R$ *and a compete hereditary cotorsion pair $(\mathcal{G}, \mathcal{K})$ in* mod$S$*. Furthermore, an R-module T is said to be a good Wakamatsu-tilting module if ${}_ST_R$ is a good Wakamatsu-tilting bimodule with $S = \mathrm{End}(T_R)$.*

For example, R and DR are good Wakamatsu-tilting modules.

3.2.2 By the definition, we have the following property about good Wakamatsu-tilting bimodules.

Proposition *Let ${}_ST_R$ be a good Waksmatsu-tilting bimodule. Assume that $(\mathcal{B}, \mathcal{A})$ is a compete hereditary cotorsion pair in* mod$R$ *and $(\mathcal{G}, \mathcal{K})$ is a compete hereditary cotorsion pair in* mod$S$ *such that the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between them. Then*

(1) *There is an equivalence*

$$\mathrm{Hom}_R(T, -): \quad \mathcal{A} \rightleftarrows \mathcal{G} \quad : - \otimes_S T$$

and an equivalence

$$\mathrm{Hom}_S(DT, -): \quad \mathcal{K} \rightleftarrows \mathcal{B} \quad : - \otimes_R DT.$$

(2) *${}_RDT_S$ is also a good Waksmatsu-tilting bimodule.*

(3) *$\mathcal{B} \subseteq \mathcal{X}_T$, $\mathcal{A} \in {}_T\mathcal{X}$ and $\mathcal{G} \subseteq \mathcal{X}_{DT}$, $\mathcal{K} \subseteq {}_{DT}\mathcal{X}$.*

(4) *$\mathrm{add}_R T = \mathcal{B} \bigcap \mathcal{A}$ and $\mathrm{add}_S DT = \mathcal{G} \bigcap \mathcal{K}$.*

Proof. (1) follows from the definition of good Wakamatsu-tilting bimodules.

(2) Replacing the Waksmatsu-tilting bimodule ${}_ST_R$ with the Wakamatsu-tilting bimodule ${}_RDT_S$ and noting that $DDT = T$, one can obtain (2) directly.

(3) Firstly, we show that $\mathrm{add}_R T \subseteq \mathcal{B} \bigcap \mathcal{A}$ and $\mathrm{add}_S DT \subseteq \mathcal{G} \bigcap \mathcal{K}$.

Note that all the involved subcategories in (1) are closed under finite direct sums and direct summands. Since $\mathcal{G}$ is resolving, we have that $S \in \mathcal{G}$. By the first equivalence in (1), we obtain that $T = S \otimes_S T \in \mathcal{A}$. It follows that $\mathrm{add}_R T \subseteq \mathcal{A}$. Dually, since $\mathcal{K}$ is coresolving, we have that $DS \in \mathcal{K}$. It follows from the second equivalence in (1) that

$T = \mathrm{Hom}_S(S, T)) = \mathrm{Hom}_S(DT, DS) \in \mathcal{B}$. Hence, $\mathrm{add}_R T \subseteq \mathcal{B}$ too. Thus, we obtain that $\mathrm{add}_R T \subseteq \mathcal{B} \bigcap \mathcal{A}$. Dually, one also has $\mathrm{add}_S DT \subseteq \mathcal{G} \bigcap \mathcal{K}$.

Clearly, $\mathcal{B} = \mathrm{KerExt}^1_S(-, \mathcal{A}) = \mathrm{KerExt}^{\geq 1}_S(-, \mathcal{A}) \subseteq \mathrm{KerExt}^{\geq 1}_S(-, T)$ follows from $\mathrm{add}_R T \subseteq \mathcal{B} \bigcap \mathcal{A}$ and the fact that $(\mathcal{B}, \mathcal{A})$ is a complete and hereditary cotorsion pair. Take any $B \in \mathcal{B}$, then $B \otimes_R DT \in \mathcal{K}$. Take an exact sequence $0 \to B \otimes_R DT \to I \to Y \to 0$ with $I \in \mathrm{inj}S = \mathrm{add}_S DS$. Since $\mathcal{K}$ is coresolving, we have that $I, Y \in \mathcal{K}$ too. Applying the functor $\mathrm{Hom}_S(DT, -)$, we obtain an induced exact sequence $0 \to \mathrm{Hom}_S(DT, B \otimes_R DT) \to \mathrm{Hom}_S(DT, I) \to \mathrm{Hom}_S(DT, Y) \to 0$, since $\mathcal{K} = \mathrm{KerExt}^1_S(\mathcal{G}, -) \subseteq \mathrm{KerExt}^1_S(DT, -)$ by the above argument. Note that $B \simeq \mathrm{Hom}_S(DT, B \otimes_R DT)$, $\mathrm{Hom}_S(DT, I) \in \mathrm{add}_R T$ and $\mathrm{Hom}_S(DT, Y) \in \mathcal{B}$, so one can easily see that $B \in \mathcal{X}_T$. Thus $\mathcal{B} \subseteq \mathcal{X}_T$. By the equivalence in Proposition 3.1 (3), we also obtain that $\mathcal{K} \subseteq {}_{DT}\mathcal{X}$.

Now consider the good Wakamatsu-tilting module ${}_R DT_S$ and apply the above result, we can obtain that $\mathcal{G} \in \mathcal{X}_{DT}$ and that $\mathcal{A} \in {}_T\mathcal{X}$.

(4) If $X \in \mathcal{B} \bigcap \mathcal{A}$, then $X \in \mathcal{B}$. Following from the proof of (3), we obtain that there is an exact sequence $0 \to X \to T_X \to X' \to 0$ with $T_X \in \mathrm{add}_R T$ and $X' \in \mathcal{B}$. Since $X \in \mathcal{A}$ too, we have that $\mathrm{Ext}^1_R(X', X) = 0$. It follows that the exact sequence splits. Hence $X \in \mathrm{add}_R T$. Together with the first claim in the proof of (3), we obtain that $\mathrm{add}_R T = \mathcal{B} \bigcap \mathcal{A}$. Dually, we also have that $\mathrm{add}_S DT = \mathcal{G} \bigcap \mathcal{K}$. □

3.2.3 Recall that a subcategory $\mathcal{A} \in \mathrm{mod}R$ is covariantly finite (or, a preenveloping calss) if for any $X \in \mathrm{mod}R$, there is an object $A_X \in \mathcal{A}$ and a homomorphism $u_X : X \to A_X$ such that $\mathrm{Hom}_R(u_X, A)$ is surjective for any object $A \in \mathcal{A}$, see for instance [3]. Dually, a subcategory $\mathcal{B} \in \mathrm{mod}R$ is contravariantly finite (or, a precovering calss) if for any $X \in \mathrm{mod}R$, there is an object $B_X \in \mathcal{A}$ and a homomorphism $v_X : B_X \to X$ such that $\mathrm{Hom}_R(B, v_X)$ is surjective for any object $B \in \mathcal{B}$.

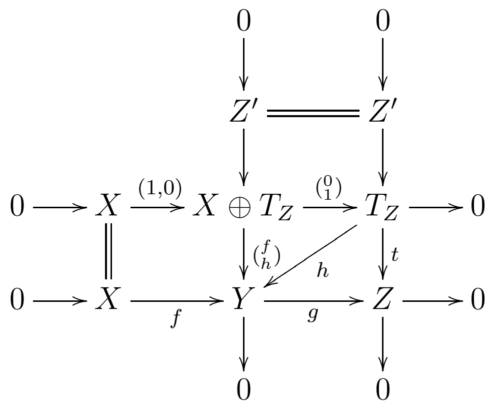
Let $\mathcal{A}$ be a subcategory of $\mathrm{mod}R$. An R-module T is said to be Ext-projective if $T \in \mathcal{A} \bigcap \mathrm{KerExt}^1_R(-, \mathcal{A})$. Moreover, it is said to be an Ext-projective generator if, for any $A \in \mathcal{A}$, there exists an exact sequence $0 \to A' \to T_A \to A \to 0$ with $T_A \in \mathrm{add}_R T$ and $A' \in \mathcal{A}$. Dually, an R-module T is said to be an Ext-injective cogenerator if $T \in \mathcal{A} \bigcap \mathrm{KerExt}^1_R(\mathcal{A}, -)$ and that, for any $A \in \mathcal{A}$, there exists an exact sequence $0 \to A \to T_A \to A' \to 0$ with $T_A \in \mathrm{add}_R T$ and $A' \in \mathcal{A}$.

Lemma *Let $\mathcal{A}$ be a subcategory closed under extensions and direct summands.*

(1) *Assume that $\mathcal{A}$ has an Ext-projective generator T. If $0 \to X \to Y \to Z \to 0$ is an exact sequence which stays exact after the functor $\mathrm{Hom}_R(T, -)$, where $Y, Z \in \mathcal{A}$, then $X \in \mathcal{A}$ too.*

(2) *Assume that $\mathcal{A}$ has an Ext-injective cogenerator T. If $0 \to X \to Y \to Z \to 0$ is an exact sequence which stays exact after the functor $\mathrm{Hom}_R(-, T)$, where $X, Y \in \mathcal{A}$, then $Z \in \mathcal{A}$ too.*

Proof. (1) By the assumptions, we can construct the following commutative diagram, where $T_Z \in \mathrm{add}_R T$ and $Z' \in \mathcal{A}$.



Since $\mathcal{A}$ is closed under extensions and direct summands, we have that $X \in \mathcal{A}$ from the middle column.

(2) Dually. $\square$

3.2.4 Lemma *Let T be a Wakamatsu-tilting R-module with $S = \mathrm{End}(T_R)$. Assume that $\mathrm{Hom}_R(T,-) : \mathcal{A} \rightleftarrows \mathcal{G} : - \otimes_S T$ define an equivalence. Then the following are equivalent.*

(1) *$\mathcal{A}$ is coresolving and T is an Ext-projective generator in $\mathcal{A}$.*

(2) *$\mathcal{G}$ is resolving and DT is an Ext-injective cogenerator in $\mathcal{G}$.*

Proof. (1) $\Rightarrow$ (2) The condition that T is an Ext-projective generator in $\mathcal{A}$ says that $T \in \mathcal{A} \subseteq \mathrm{KerExt}_R^1(T,-)$ and that every $A \in \mathcal{A}$ admits an exact sequence $0 \to A' \to T_A \to A \to 0$ with $T_A \in \mathrm{add}_R T$ and $A' \in \mathcal{A}$. This implies that $\mathcal{A} \subseteq \mathrm{KerExt}_R^{\geq 1}(T,-)$, since $\mathcal{A}$ is coresolving. In particular, $\mathcal{A} \subseteq {}_T\mathcal{X}$.

Note that, for any $X \in \mathcal{A}$, there is an exact sequence $0 \to X \to I \to X' \to 0$ with $I' \in \mathrm{inj}R \subseteq \mathcal{A}$ and $X' \in \mathcal{A}$. Applying the functor $\mathrm{Hom}_R(T,-)$, we have an exact sequence $0 \to \mathrm{Hom}_R(T,X) \to \mathrm{Hom}_R(T,I) \to \mathrm{Hom}_R(T,X') \to 0$. Since $\mathrm{Hom}_R(T,I) \in \mathrm{add}_S DT$ and $\mathrm{Ext}_S^1(\mathrm{Hom}_R(T,X),DT) \simeq \mathrm{Ext}_S^1(\mathrm{Hom}_R(T,X),\mathrm{Hom}_R(T,DR)) = 0$, we obtain that DT is an Ext-injective cogenerator in $\mathrm{Hom}_R(T,\mathcal{A}) = \mathcal{G}$.

It is clear that $\mathcal{G}$ is closed under direct summands. Assume now there is an exact sequence $(\flat):\ 0 \to X \to Y \xrightarrow{g} Z \to 0$ with $Z \in \mathcal{G}$, then $Z \in \mathrm{Hom}_R(T,\mathcal{A}) \subseteq \mathrm{KerTor}_1^S(-,T)$. Applying the functor $-\otimes_S T$, we obtain an induce exact sequence $(\flat \otimes_S T):\ 0 \to X \otimes_S T \to Y \otimes_S T \xrightarrow{g \otimes_S T} Z \otimes_S T \to 0$.

Assume first $X \in \mathcal{G}$ too, then $X \otimes_S T \in \mathcal{A}$. It follows that $Y \otimes_S T \in \mathcal{A}$ too, since $\mathcal{A}$ is closed under extensions. Note now there is an exact sequence $0 \to \mathrm{Hom}_R(T, X \otimes_S T) \to \mathrm{Hom}_R(T, Y \otimes_S T) \to \mathrm{Hom}_R(T, Z \otimes_S T) \to 0$, so we have that $\mathrm{Hom}_R(T, Y \otimes_S T) \simeq Y$ since $\mathrm{Hom}_R(T, X \otimes_S T) \simeq X$ and $\mathrm{Hom}_R(T, Z \otimes_S T) \simeq Z$. Thus $Y \in \mathrm{Hom}_R(T,\mathcal{A}) = \mathcal{G}$. This shows that $\mathcal{G}$ is closed under extensions.

Assume now $Y \in \mathcal{G}$, then $\mathrm{Hom}_R(T, g \otimes_S T) \simeq g$. In particular, we have that $\mathrm{Hom}_R(T, X \otimes_S$

$T) \simeq X$ and the homomorphism $\mathrm{Hom}_R(T, g \otimes_S T)$ is surjective. It follows that the exact sequence $(\flat \otimes_S T)$ stays exact after the functor $\mathrm{Hom}_R(T, -)$. By Lemma 3.2.3, we obtain that $X \otimes_S T \in \mathcal{A}$. Hence $X \in \mathrm{Hom}_R(T, \mathcal{A}) = \mathcal{G}$. This shows that $\mathcal{G}$ is closed under kernels of epimorphisms.

(2) Dually. □

3.2.5 Proposition *Let ${}_ST_R$ be a Wakamatsu-tilting module. Assume that $(\mathcal{B}, \mathcal{A})$ is a hereditary cotorsion pair in* mod$R$ *and that T is an Ext-projective generator in $\mathcal{A}$, then $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$ is a hereditary cotorsion pair in* mod$S$*. In particular, the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between $(\mathcal{B}, \mathcal{A})$ and $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$ in the case.*

Proof. Since T is an Ext-projective generator in $\mathcal{A}$, we see that $T \in \mathrm{KerExt}^1_R(-, \mathcal{A}) \bigcap \mathcal{A} = \mathcal{B} \bigcap \mathcal{A}$ and that, for any $A \in \mathcal{A}$, there is an exact sequence $0 \to A' \to T_A \to A \to 0$ with $T_A \in \mathrm{add}_R T$ and $A' \in \mathcal{A}$. In particular, for any $X \in \mathcal{A} \bigcap \mathcal{B}$, there is an exact sequence $0 \to X' \to T_X \to X \to 0$ with $T_X \in \mathrm{add}_R T$ and $X' \in \mathcal{A}$, which is clearly split. Hence, $X \in \mathrm{add}_R T$. It follows that $\mathrm{add}_R T = \mathcal{B} \bigcap \mathcal{A}$. Moreover, by an argument similar to the one used in the proof of [20, Proposition 2.13(b)], we have that T is also an Ext-injective cogenerator in $\mathcal{B}$. Note that these imply that $\mathcal{A} \subseteq {}_T\mathcal{X}$ and that $\mathcal{B} \subseteq \mathcal{X}_T$.

By Lemma 3.2.4, we see that $\mathrm{Hom}_R(T, \mathcal{A})$ is resolving and that $\mathcal{B} \otimes_R DT$ is coresolving. It is also clear that the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a counter equivalence between two pairs $(\mathcal{B}, \mathcal{A})$ and $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$, by assumptions. So, it just remains to show that $(\mathrm{Hom}_R(T, \mathcal{A}), \mathcal{B} \otimes_R DT)$ is a cotorsion pair.

We divide the remained proof into three steps.

Step 1 $\mathrm{Ext}^i_S(\mathrm{Hom}_R(T, A), B \otimes_R DT) = 0$, for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$ and for any $i \geq 0$.

Note that there is an isomorphism $D\mathrm{Hom}_S(S, B \otimes_S DT) \simeq \mathrm{Hom}_R(B, S \otimes_S T)$ and that it induces an isomorphism $D\mathrm{Hom}_S(S_i, B \otimes_S DT) \simeq \mathrm{Hom}_R(B, S_i \otimes_S T)$, for any $S_i \in \mathrm{add}_S S$.

Now take $A \in \mathcal{A}$, since T is an Ext-projective generator, there is a long exact sequence

$$\cdots \to T_n \to \cdots \to T_1 \to T_0 \to A \to 0 \qquad (\dagger)$$

with each $T_i \in \mathrm{add}_R T$ and each image in $\mathcal{A}$. Since $B \in \mathrm{KerExt}^1_R(-, A)$, we have an induced exact sequence $\mathrm{Hom}_R(B, \dagger)$:

$$\cdots \to \mathrm{Hom}_R(B, T_n) \to \cdots \to \mathrm{Hom}_R(B, T_1) \to \mathrm{Hom}_R(B, T_0) \to \mathrm{Hom}_R(B, A) \to 0.$$

On the other hand, by applying the functor $D\mathrm{Hom}_S(\mathrm{Hom}_R(T, -), B \otimes_R DT)$, we have a complex $D\mathrm{Hom}_S(\mathrm{Hom}_R(T, \dagger), B \otimes_R DT)$:

$$\cdots \to D\mathrm{Hom}_S(\mathrm{Hom}_R(T, T_n), B \otimes_R DT) \to \cdots \to D\mathrm{Hom}_S(\mathrm{Hom}_R(T, T_1), B \otimes_R DT)$$
$$\to D\mathrm{Hom}_S(\mathrm{Hom}_R(T, T_0), B \otimes_R DT) \to D\mathrm{Hom}_S(\mathrm{Hom}_R(T, A), B \otimes_R DT) \to 0.$$

Since $\mathrm{Hom}_R(T, \dagger)$ is exact, one sees that the functor $D\mathrm{Hom}_S(\mathrm{Hom}_R(T, -), B \otimes_R DT)$ is right exact. By the above isomorphism, we obtain the following isomorphisms of complexes

$$D\mathrm{Hom}_S(\mathrm{Hom}_R(T, \dagger), B \otimes_R DT) \simeq \mathrm{Hom}_R(B, \mathrm{Hom}_R(T, \dagger) \otimes_S T) \simeq \mathrm{Hom}_R(B, \dagger).$$

But the later is exact, so we obtain that

$$\operatorname{Ext}^i_S(\operatorname{Hom}_R(T,A),B\otimes_R DT)\simeq \operatorname{H}^i(\operatorname{Hom}_S(\operatorname{Hom}_R(T,\dagger),B\otimes_R DT))$$
$$\simeq D\operatorname{H}^i(D\operatorname{Hom}_S(\operatorname{Hom}_R(T,\dagger),B\otimes_R DT))\simeq D\operatorname{Hom}_R(B,\dagger)=0.$$

Thus, step 1 is established. In particular, we obtain that $\operatorname{Hom}_R(T,\mathcal{A})\subseteq \operatorname{KerExt}^1_S(-,\mathcal{B}\otimes_R DT)$ and that $\operatorname{Hom}_R(T,\mathcal{A})\subseteq \operatorname{KerExt}^1_S(-,\mathcal{B}\otimes_R DT)$, due to the arbitrarity of $A\in\mathcal{A}$ and $B\in\mathcal{B}$.

Step 2 $\operatorname{KerExt}^1_S(-,\mathcal{B}\otimes_R DT)\subseteq \operatorname{Hom}_R(T,\mathcal{A})$.

Take any $Y\in \operatorname{KerExt}^1_S(-,\mathcal{B}\otimes_R DT)$ and a projective resolution of Y:

$$\cdots \xrightarrow{f_{n+1}} S_n \xrightarrow{f_n} \cdots \xrightarrow{f_1} S_1 \xrightarrow{f_1} S_0 \xrightarrow{f_0} Y\to 0. \qquad (\sharp)$$

Note that $DT=R\otimes_R DT\in\mathcal{B}\otimes_R DT$ and that $\mathcal{B}\otimes_R DT$ is coresolving, so we obtain that

$$Y\in \operatorname{KerExt}^1_S(-,\mathcal{B}\otimes_R DT)=\operatorname{KerExt}^{\geqslant 0}_S(-,\mathcal{B}\otimes_R DT)\subseteq \operatorname{KerExt}^{\geqslant 0}_S(-,DT)=\operatorname{KerTor}^S_{>0}(-,T).$$

Then we have an induced exact sequence:

$$\cdots\to S_n\otimes_S T\to\cdots\to S_1\otimes_S T\to S_0\otimes_S T\to Y\otimes_S T\to 0. \qquad (\sharp\otimes_S T)$$

For any $B\in\mathcal{B}$, applying the left exact functor $\operatorname{Hom}_R(B,-)$, we obtain a complex $\operatorname{Hom}_R(B,\sharp\otimes_S T)$:

$$\cdots\to \operatorname{Hom}_R(B,S_n\otimes_S T)\to\cdots\to \operatorname{Hom}_R(B,S_1\otimes_S T)$$
$$\to \operatorname{Hom}_R(B,S_0\otimes_S T)\to \operatorname{Hom}_R(B,Y\otimes_S T)\to 0.$$

Applying the right exact functor $D\operatorname{Hom}_S(-,B\otimes_S DT)$ to the sequence $(\sharp)$, we obtain a complex $D\operatorname{Hom}_S(\sharp,B\otimes_S DT)$:

$$\cdots\to D\operatorname{Hom}_S(S_n,B\otimes_S DT)\to\cdots\to D\operatorname{Hom}_S(S_1,B\otimes_S DT)$$
$$\to D\operatorname{Hom}_S(S_0,B\otimes_S DT)\to D\operatorname{Hom}_S(Y,B\otimes_S DT)\to 0.$$

By the isomorphism in Step 1 again, we have isomorphisms of complexes

$$D\operatorname{Hom}_S(\sharp,B\otimes_S DT)\simeq \operatorname{Hom}_R(B,\sharp\otimes_S T).$$

It follows that, for any $i\geq 2$,

$$\operatorname{Ext}^1_R(B,Y_i\otimes_S T)\simeq \operatorname{H}^{i-1}(\operatorname{Hom}_R(B,\sharp\otimes_S T))$$
$$\simeq \operatorname{H}^{i-1}(D\operatorname{Hom}_S(\sharp,B\otimes_S DT))=D\operatorname{Ext}^{i-1}_R(Y,B\otimes_S T)=0,$$

where $Y_i=\operatorname{Im}f_i$, since $B\in\operatorname{KerExt}^1_R(-,T)$ and $\operatorname{Hom}_R(B,-)$ is left right exact. This shows that $Y_i\otimes_S T\in \operatorname{KerExt}^1_R(\mathcal{B},-)=\mathcal{A}$, for all $i\geq 2$. As $S\otimes_S T\simeq T\in\mathcal{A}$ and $\mathcal{A}$ is coresolving, we obtain that $Y\otimes_S T\in\mathcal{A}$ too. Then we have an induced exact sequence $\operatorname{Hom}_R(T,\sharp\otimes_S T)$:

$$\cdots\to \operatorname{Hom}_R(T,S_n\otimes_S T)\to\cdots\to \operatorname{Hom}_R(T,S_1\otimes_S T)$$
$$\to \operatorname{Hom}_R(T,S_0\otimes_S T)\to \operatorname{Hom}_R(T,Y\otimes_S T)\to 0.$$

It follows that $Y\simeq \operatorname{Hom}_R(T,Y\otimes_S T)\in \operatorname{Hom}_R(T,\mathcal{A})$, since $S_i\simeq \operatorname{Hom}_R(T,S_i\otimes_S T)$ for each i. This shows that $\operatorname{KerExt}^1_S(-,\mathcal{B}\otimes_R DT)\subseteq \operatorname{Hom}_R(T,\mathcal{A})$. Together with Step 1, we obtain that $\operatorname{KerExt}^1_S(-,\mathcal{B}\otimes_R DT)=\operatorname{Hom}_R(T,\mathcal{A})$.

Step 3 $\operatorname{KerExt}^1_S(\operatorname{Hom}_R(T,\mathcal{A}),-)\subseteq \mathcal{B}\otimes_R DT$.

Note that there is an isomorphism

$$\mathrm{Hom}_S(\mathrm{Hom}_R(T,A),DS)\simeq D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,DS),A),$$

for any $A\in \mathrm{mod}R$ and that it induces an isomorphism

$$\mathrm{Hom}_S(\mathrm{Hom}_R(T,A),I_i)\simeq D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,I_i),A),$$

for any $I_i\in \mathrm{add}DS$.

Now take any $X\in \mathrm{KerExt}^1_S(\mathrm{Hom}_R(T,\mathcal{A}),-)$ and consider an injective resolution of X:

$$0\to X\xrightarrow{g_0} I_0\xrightarrow{g_1} I_1\xrightarrow{g_1}\cdots\xrightarrow{g_{n-1}} I_n\xrightarrow{g_n}\cdots. \qquad (\natural)$$

Since $DT\simeq \mathrm{Hom}_R(T,DR)\in \mathrm{Hom}_R(T,\mathcal{A})$ and $\mathrm{Hom}_R(T,\mathcal{A})$ is resolving, we obtain that

$$X\in \mathrm{KerExt}^1_S(\mathrm{Hom}_R(T,\mathcal{A}),-)=\mathrm{KerExt}^{\geq 0}_S(\mathrm{Hom}_R(T,\mathcal{A}),-)\subseteq \mathrm{KerExt}^{\geq 0}_S(DT,-).$$

Thus, for any $A\in\mathcal{A}$, applying the functor $\mathrm{Hom}_S(\mathrm{Hom}_R(T,A),-)$, we have an induce exact complex $\mathrm{Hom}_S(\mathrm{Hom}_R(T,A),\natural)$:

$$0\to \mathrm{Hom}_S(\mathrm{Hom}_R(T,A),X)\longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(T,A),I_0)$$
$$\longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(T,A),I_1)\longrightarrow\cdots\longrightarrow \mathrm{Hom}_S(\mathrm{Hom}_R(T,A),I_n)\longrightarrow\cdots.$$

On the other hand, we also have the following induced complex $D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,\natural),A)$, by applying the functor $D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,-),A)$:

$$0\to D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,X),A)\longrightarrow D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,I_0),A)$$
$$\longrightarrow D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,I_1),A)\longrightarrow\cdots\longrightarrow D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,I_n),A)\longrightarrow\cdots.$$

By the above mentioned isomorphism $(*)$ and the fact that $\mathrm{Hom}_S(DT,\natural)$ is exact, we have that, for any $i\geq 1$,

$$\mathrm{Ext}^1_R(\mathrm{Hom}_S(DT,X_{i+2}),A)\simeq \mathrm{H}^i(\mathrm{Hom}_R(\mathrm{Hom}_S(DT,\natural),A))$$
$$\simeq D\mathrm{H}^i(D\mathrm{Hom}_R(\mathrm{Hom}_S(DT,\natural),A))\simeq D\mathrm{H}^i(\mathrm{Hom}_S(\mathrm{Hom}_R(T,A),\natural))=0,$$

where $X_i=\mathrm{Img}_i$, since $A\in \mathrm{KerExt}^1_R(-,T)$ and $\mathrm{Hom}_R(-,A)$ is left exact. It follows that $\mathrm{Hom}_S(DT,X_{i+2})\in \mathrm{KerExt}^1_R(-,\mathcal{A})=\mathcal{B}$ for any $i\geq 1$. Since $\mathcal{B}$ is resolving, we also obtain each $\mathrm{Hom}_S(DT,X_i)\in\mathcal{B}$ for for each $0\leq i\leq 2$, where $X_0:=X$. Then we have an induced exact sequence $\mathrm{Hom}_S(DT,\natural)\otimes_R DT$, since $\mathcal{B}\subseteq \mathrm{KerExt}^1_R(-,T)=\mathrm{KerTor}^R_1(-,DT)$:

$$0\to \mathrm{Hom}_S(DT,X)\otimes_S DT\to \mathrm{Hom}_S(DT,I_0)\otimes_S DT$$
$$\to \mathrm{Hom}_S(DT,I_1)\otimes_S DT\to\cdots\to \mathrm{Hom}_S(DT,I_n)\otimes_S DT\to\cdots.$$

It follows that $X\simeq \mathrm{Hom}_S(DT,X)\otimes_S DT\in\mathcal{B}\otimes_S DT$, since $I_i\simeq \mathrm{Hom}_S(DT,I_i)\otimes_S DT$ for each i. This shows that $\mathrm{KerExt}^1_S(\mathrm{Hom}_R(T,\mathcal{A}),-)\subseteq \mathcal{B}\otimes_R DT$. Together with Step 1, we obtain that $\mathrm{KerExt}^1_S(\mathrm{Hom}_R(T,\mathcal{A}),-)=\mathcal{B}\otimes_R DT$.

Altogether, we obtain that $(\mathrm{Hom}_R(T,\mathcal{A}),\mathcal{B}\otimes_R DT)$ is a hereditary cotorsion pair. □

3.2.6 Corollary *Let T be a Wakamatsu-tilting R-module with $S=\mathrm{End}(T_R)$. Assume that the functor $\mathrm{Hom}_R(T,-)$ gives an equivalence between a covariantly finite coresolving subcategory $\mathcal{A}$ in $\mathrm{mod}R$ and a contravariantly finite resolving subcategory $\mathcal{G}$ in $\mathrm{mod}S$. If T is an Ext-projective generator in $\mathcal{A}$, then T is a good Wakamatsu-tilting module.*

Proof. Since $\mathcal{G}$ is a contravariantly finite resolving subcategory in $\mathrm{mod}S$, there is a cotorsion pair $(\mathcal{G},\mathrm{KerExt}^1_R(\mathcal{G},-))$ in $\mathrm{mod}S$, by [3, Proposition 1.10]. Dually, there is a cotorsion pair

$(\mathrm{KerExt}_R^1(-,\mathcal{A}),\mathcal{A})$ in mod$R$, since $\mathcal{A}$ is a covariantly finite coresolving subcategory $\mathcal{A}$ in mod$R$. Note that both cotrosion pairs are complete and hereditary. Since T is an Ext-projective generator in $\mathcal{A}$, by Proposition 3.2.5, the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the above two cotorsion pairs. Hence T is a good Wakamatsu-tilting module by the definition. □

4 The proof of the main results

The whole section will be devoted to the proof of the two results mentioned in the introduction.

Let R be an artin algebra and T be a good Wakamatsu-tilting module with $S=\mathrm{End}(T_R)$. Then ${}_ST_R$ be a good Wakamatsu-tilting bimodule. Assume that $(\mathcal{B},\mathcal{A})$ is a compete hereditary cotorsion pair in mod$R$ and $(\mathcal{G},\mathcal{K})$ is a compete hereditary cotorsion pair in mod$S$ such that the bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between these two cotorsion pairs.

The sketch of our proof on the main theorem is as follows.

Firstly we construct a functor $\mathrm{L}_T:\mathcal{RC}^{\mathrm{tr}}(R)\to\mathcal{RC}(S)$ and a functor $-\hat{\otimes}DT:\mathcal{RC}(R)\to\mathcal{RC}(S)$. Then we give a natural homomorphism

$$l_Y^X:\mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(X,Y)\to\mathrm{Hom}_{\mathcal{RC}(S)}(X\hat{\otimes}DT,\mathrm{L}_T(Y))$$

which is functional in both variables. After this, associated with an object $X\in\mathcal{RC}(R)$, we use the condition that $(\mathcal{B},\mathcal{A})$ is a compete cotorsion pair in mod$R$ to obtain an object $A_X\in\mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$ and establish a homomorphism $u_X\in\mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(X,A_X)$. We then show that the assignment that setting $X\longmapsto\mathrm{Cok}(l(u_X))$ induces our desired functor $\mathbf{S}_T:\underline{\mathcal{RC}}(R)\to\underline{\mathcal{RC}}(S)$. We use the dual method to construct another desired functor $\mathbf{Q}_{DT}:\underline{\mathcal{RC}}(S)\to\underline{\mathcal{RC}}(R)$. Then we prove that there are natural isomorphisms $\mathbf{Q}_{DT}\mathbf{S}_T\simeq 1_{\underline{\mathcal{RC}}(R)}$ and $\mathbf{S}_T\mathbf{Q}_{DT}\simeq 1_{\underline{\mathcal{RC}}(R)}$.

4.1 From $\mathcal{RC}(R)$ to $\mathcal{RC}(S)$: the functor $\mathbf{S}_T$

4.1.1 *The functor* $\mathrm{L}_T:\mathcal{RC}^{\mathrm{tr}}(R)\to\mathcal{RC}(S)$

Let $X=\{X_i\}\in\mathcal{RC}^{\mathrm{tr}}(R)$. We define $\mathrm{L}_T(X)\in\mathcal{RC}(S)$ as follows.

($l1$) The underlying module $\mathrm{L}_T(X)_i=\mathrm{Hom}_R(T,X_{i-1})\oplus X_i\otimes_R DT$ and,

($l2$) The structure map $\delta_i^{\otimes}(\mathrm{L}_T(X)):\mathrm{L}_T(X)_i\to\mathrm{L}_T(X)_{i-1}$ is given by $\begin{pmatrix}0&\delta_{L_i}\\0&0\end{pmatrix}$, where δ_{L_i} is the composition:

$$\mathrm{Hom}_R(T,X_{i-1})\otimes_S DS\xrightarrow{\simeq}\mathrm{Hom}_R(T,X_{i-1})\otimes_S T\otimes_R \mathrm{D}T\xrightarrow{\epsilon_{X_{i-1}}^T\otimes_R\mathrm{D}T}X_{i-1}\otimes_R DT.$$

From the functor property of $\mathrm{Hom}_R(T,-)$ and $-\otimes_R DT$, one can easily see that L_T is a functor from $\mathcal{RC}^{\rm tr}(R)$ to $\mathcal{RC}(S)$.

Remark (1) If $X \in \mathcal{RC}^{\rm tr}(\mathrm{add}_R T)$, i.e., $X = \{X_i\}$ with each $X_i \in \mathrm{add}_R T$, then $\mathrm{Hom}_R(T, X_{i-1}) \in \mathrm{add}_S S$ and δ_{L_i} defined above is an isomorphism for each i. It follows that $\mathrm{L}_T(X)$ is a projective object in $\mathcal{RC}(S)$ in the case.

(2) In particular, in case $T = R$, we obtain the functor $\mathrm{L}_R : \mathcal{RC}^{\rm tr}(R) \to \mathcal{RC}(R)$ which specially send objects in $\mathcal{RC}^{\rm tr}(\mathrm{add}_R R)$ to a projective object in $\mathcal{RC}(R)$.

4.1.2 *The functor* $-\hat{\otimes} DT : \mathcal{RC}(R) \to \mathcal{RC}(S)$

Let $Y = \{Y_i, \delta_i^{\otimes}(Y)\} \in \mathcal{RC}(R)$. We define $Y \hat{\otimes} DT \in \mathcal{RC}(S)$ by setting

$(t1)$ the underlying module is $(Y\hat{\otimes} DT)_i = Y_i \otimes_R DT$ and,

$(t2)$ the structure map $\delta_i^{\otimes}(Y\hat{\otimes} DT)$ is given by the composition

$$Y_i \otimes_R DT \otimes_S DS \xrightarrow{\simeq} Y_i \otimes_R DT \otimes_S T \otimes_R DT \xrightarrow{\simeq} Y_i \otimes_R DR \otimes_R DT \xrightarrow{\delta_i^{\otimes}(Y)\otimes_R DT} Y_{i-1} \otimes_R DT.$$

From the functor property of $- \otimes_R DT$, one can see that $-\hat{\otimes} DT$ is a functor from $\mathcal{RC}(R)$ to $\mathcal{RC}(S)$.

4.1.3 *The homomorphism* $l_Y^X : \mathrm{Hom}_{\mathcal{RC}^{\rm tr}(R)}(X, Y) \to \mathrm{Hom}_{\mathcal{RC}(S)}(X\hat{\otimes} DT, \mathrm{L}_T(Y))$

Recall that we have a forgotting functor from $\mathcal{RC}(R)$ to $\mathcal{RC}^{\rm tr}(R)$. For any $X \in \mathcal{RC}(R)$ and $Y \in \mathcal{RC}^{\rm tr}(R)$, there is a canonical homomorphism

$$l_Y^X : \mathrm{Hom}_{\mathcal{RC}^{\rm tr}(R)}(X, Y) \to \mathrm{Hom}_{\mathcal{RC}(S)}(X\hat{\otimes} DT, \mathrm{L}_T(Y))$$

which is functional in both variables, defined by

$$l_Y^X : u = \{u_i\} \quad \longmapsto \quad f = \{f_i\} \text{ with } f_i = \big(-\theta_{l_i}, \quad u_i \otimes_R DT\big),$$

where θ_{l_i} is given by the composition

$$X_i \otimes_R DT \xrightarrow{\eta^T_{X_i}} \mathrm{Hom}_R(T, X_i \otimes_R DT \otimes_S T) \xrightarrow{\simeq} \mathrm{Hom}_R(T, X_i \otimes_R DR)$$
$$\xrightarrow{\mathrm{Hom}_R(T,\delta_{X_i})} \mathrm{Hom}_R(T, X_{i-1}) \xrightarrow{\mathrm{Hom}_R(T,u_{i-1})} \mathrm{Hom}_R(T, Y_{i-1}).$$

Remark Using the fact ${}_R DR_R \simeq {}_R(DT \otimes_S T)_R$ and the adjoint isomorphism

$$\mathbf{\Gamma}^T : \mathrm{Hom}_R(X_i \otimes_R DR, Y_{i-1}) \simeq \mathrm{Hom}_S(X_i \otimes_R DT, \mathrm{Hom}_R(T, Y_{i-1})),$$

one can easily check that θ_{l_i} is just the image of the natural homomorphism $\delta_i^{\otimes}(X) \circ u_{i-1}$ under Γ^T, i.e., $\theta_{l_i} = \mathbf{\Gamma}^T(\delta_i^{\otimes}(X) \circ u_{i-1})$.

It is easy to see that, for any commutative diagram in $\mathcal{RC}(R)$

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \downarrow{\scriptstyle x} & & \downarrow{\scriptstyle y} \\ X' & \xrightarrow{u'} & Y', \end{array}$$

there is an induced commutative diagram in $\mathcal{RC}(S)$

$$\begin{array}{ccc} X\hat{\otimes}DT & \xrightarrow{l(u)} & \mathrm{L}_T(Y) \\ \big\downarrow{\scriptstyle x\hat{\otimes}DT} & & \big\downarrow{\scriptstyle y} \\ X'\hat{\otimes}DT & \xrightarrow{l(u')} & \mathrm{L}_T(Y'). \end{array}$$

4.1.4 *A monomorphism* $u_X : X \to A_X$ *with* $A_X \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$*, for* $X \in \mathcal{RC}(R)$.

Let $X = \{X_i\} \in \mathcal{RC}(R)$. Since $(\mathcal{B}, \mathcal{A})$ is a complete cotorsion pair in mod$R$, there are exact sequences $0 \to X_i \xrightarrow{u_{X_i}} A_{X_i} \to B_{X_i} \to 0$ with $A_{X_i} \in \mathcal{A}$ and $B_{X_i} \in \mathcal{B}$, for each i. This gives an exact sequence $0 \to X \xrightarrow{u_X} A_X \xrightarrow{\pi_{u_X}} B_X \to 0$ in $\mathcal{RC}^{\mathrm{tr}}(R)$ with $A_X = \{A_{X_i}\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$ and $B_X = \{B_{X_i}\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{B})$.

Now let $Y = \{Y_i\} \in \mathcal{RC}(R)$ and $h = \{h_i\} \in \mathrm{Hom}_{\mathcal{RC}(R)}(X, Y)$. Then we have an exact sequence $0 \to Y \xrightarrow{u_Y} A_Y \xrightarrow{\pi_{u_Y}} B_Y \to 0$ in $\mathcal{RC}^{\mathrm{tr}}(R)$ with $A_Y = \{A_{Y_i}\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$ and $B_Y = \{B_{Y_i}\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{B})$, as above. Note that $\mathcal{B} \subseteq \mathrm{KerExt}^1_R(-, \mathcal{A})$, it is easy to see that there is a homomorphism $h_A \in \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(A_X, A_Y)$ and further $h_B \in \mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(R)}(B_X, B_Y)$, such that the following diagram in $\mathcal{RC}^{\mathrm{tr}}(R)$ is commutative with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u_X} & A_X & \xrightarrow{\pi_X} & B_X & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle h} & & \big\downarrow{\scriptstyle h_A} & & \big\downarrow{\scriptstyle h_B} & & \\ 0 & \longrightarrow & Y & \xrightarrow{u_Y} & A_Y & \xrightarrow{\pi_Y} & B_X & \longrightarrow & 0 \end{array}$$

4.1.5 *The cokernel* $\mathrm{Cok}(l(u_X))$

Applying the functor $- \otimes_R DT$ to the exact sequences $0 \to X_i \xrightarrow{u_{X_i}} A_{X_i} \to B_{X_i} \to 0$ in the above, we obtain an induced exact sequences

$$0 \longrightarrow X_i \otimes_R DT \xrightarrow{u_{X_i}\otimes_R DT} A_{X_i} \otimes_R DT \longrightarrow B_{X_i} \otimes_R DT \longrightarrow 0,$$

since $B_{X_i} \in \mathcal{B} \subseteq \mathrm{KerExt}^1_R(-, T) = \mathrm{KerTor}^R_1(-, DT)$ for each i. It follows that, by applying the homomorphism l in 4.1.3 to the homomorphism u_X in 4.1.4, there is an induced exact sequence

$$0 \longrightarrow X\hat{\otimes}DT \xrightarrow{l(u_X)} \mathrm{L}_T(A_X) \xrightarrow{\pi_{l_X}} \mathrm{Cok}(l_X) \longrightarrow 0.$$

Remark From the definition of $\mathrm{Cok}(l(u_X))$, one see that, for each i, $\mathrm{Cok}(l(u_X))_i$ is given by the pushout

$$\begin{array}{ccc} X_i \otimes_R DT & \xrightarrow{\theta_{l_i}} & \mathrm{Hom}_R(T, A_{X_{i-1}}) \\ {\scriptstyle u_{X_i}\otimes_R DT}\big\downarrow & & \big\downarrow \\ A_{X_i} \otimes_R DT & \longrightarrow & \mathrm{Cok}(l(u_X))_i. \end{array}$$

Moreover, for $Y \in \mathcal{RC}(R)$ and $h \in \mathrm{Hom}_{\mathcal{RC}(R)}(X, Y)$, by applying the homomorphism l to the left square in the commutation diagram in 4.1.4, we obtain the following commutative diagram in $\mathcal{RC}(S)$, for some $h_{\rm Cok}$.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X\hat{\otimes}DT & \xrightarrow{l(u_X)} & \mathrm{L}_T(A_X) & \xrightarrow{\pi_{l_X}} & \mathrm{Cok}(l(u_X)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle h\hat{\otimes}DT} & & \downarrow{\scriptstyle \mathrm{L}_T(h_A)} & & \downarrow{\scriptstyle h_{\rm Cok}} & & \\
0 & \longrightarrow & Y\hat{\otimes}DT & \xrightarrow{l(u_Y)} & \mathrm{L}_T(A_Y) & \xrightarrow{\pi_{l_Y}} & \mathrm{Cok}(l(u_Y)) & \longrightarrow & 0
\end{array}$$

4.1.6 *The assignment* $\mathbf{S}_T : \mathcal{RC}(R) \to \underline{\mathcal{RC}}(S)$ *by setting* $X \longmapsto \mathrm{Cok}(l(u_X))$ *is a functor.*

By 4.1.5, it is sufficient to prove that $\mathbf{S}_T(h) := h_{\rm Cok} = 0$ in $\underline{\mathcal{RC}}(S)$ provided $h = 0$. We divide the proof into two steps.

Step 1: Consider each piece in the commutative diagram in 4.1.4. If $h = \{h_i\} = 0$, then $h_i = 0$ for each i. Thus, we have that $u_{X_i}h_{A_i} = 0$ and consequently, $h_{A_i} = \pi_{X_i}g_i$ for some $g_i : B_{X_i} \to A_{Y_i}$. Since $B_{X_i} \in \mathcal{B} \subseteq \mathcal{X}_T$ for each i, there are exact sequences $0 \to B_{X_i} \xrightarrow{b_i} T_{B_{X_i}} \to B'_i \to 0$ with $T_{B_{X_i}} \in \mathrm{add}_R T$ and $B'_i \in \mathcal{B} \subseteq \mathrm{KerExt}^1_R(-, \mathcal{A})$. It follows that there exists $t_i \in \mathrm{Hom}_R(T_{B_{X_i}}, A_{Y_i})$ such that $g_i = b_i t_i$. Altogether we obtain the following commutative diagram

$$\begin{array}{ccc}
A_{X_i} & \xrightarrow{\pi_{X_i}} & B_{X_i} \\
\downarrow{\scriptstyle h_{A_i}} & \swarrow{\scriptstyle g_i} & \downarrow{\scriptstyle b_i} \\
A_{Y_i} & \xleftarrow[t_i]{} & T_{B_{X_i}}.
\end{array}$$

This induces the following commutative diagram in $\mathcal{RC}^{\rm tr}(R)$, where $T_{B_X} = \{T_{B_{X_i}}\}$.

$$\begin{array}{ccc}
A_X & \xrightarrow{\pi_X} & B_X \\
\downarrow{\scriptstyle h_A} & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle b} \\
A_Y & \xleftarrow[t]{} & T_{B_X}.
\end{array}$$

Set $k := \pi_X b$. Then $\mathrm{L}_T(h_A) = \mathrm{L}_T(kt) = \mathrm{L}_T(k)\mathrm{L}_T(t)$.

Step 2: Consider the commutative diagram in 4.1.5. Since $u_{X_i}\pi_{X_i} = 0$, we see that $l(u_X)\mathrm{L}_T(k) = 0$ by the definitions. Hence there is some $\theta \in \mathrm{Hom}_{\mathcal{RC}(S)}(\mathrm{Coker}(l(u_X)), \mathrm{L}_T(A_Y))$ such that $\mathrm{L}_T(k) = \pi_{l_X}\theta$. Consequently, we have that $\mathrm{L}_T(h_A) = \mathrm{L}_T(k)L(t) = \pi_{l_X}\theta\mathrm{L}_T(t)$. Now we obtain that $\pi_{l_X}h_{\rm Coker} = \mathrm{L}_T(h_A)\pi_{l_Y} = \pi_{l_X}\theta\mathrm{L}_T(t)\pi_{l_Y}$. Since π_{l_X} is epic, we get that $h_{\rm Cok} = \mathrm{L}_T(h_A)\pi_{l_Y} = \theta\mathrm{L}_T(t)\pi_{l_Y}$. That is, we have the following commutative diagram.

$$\begin{array}{ccc}
\mathrm{Cok}(l(u_X)) & \xrightarrow{\theta} & \mathrm{L}_T(T_{B_X}) \\
\downarrow{\scriptstyle h_{\rm Cok}} & & \downarrow{\scriptstyle \mathrm{L}_T(t)} \\
\mathrm{Cok}(l(u_Y)) & \xleftarrow[\pi_{l_Y}]{} & L(A_Y)
\end{array}$$

Note that $\mathrm{L}_T(T_{B_X})$ is a projective-injective object in $\mathcal{RC}(S)$, so $h_{\rm Cok} = 0$ in $\underline{\mathcal{RC}}(S)$.

4.1.7 *The functor* $\mathbf{S}_T : \underline{\mathcal{RC}}(R) \to \underline{\mathcal{RC}}(S)$

We will show that the functor $\mathbf{S}_T$ factors through $\underline{\mathcal{RC}}(R)$.

To see this, it is enough to show that $\mathbf{S}_T(X)$ is a projective object in $\mathcal{RC}(S)$ whenever X is a projective object in $\mathcal{RC}(R)$.

W.l.o.g., we assume that $X = \{X_i\}$ is an indecomposable projective object in $\mathcal{RC}(R)$. Thus, X has the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow \mathrm{Hom}_R(DR, I) \overset{1}{\rightsquigarrow} I \rightsquigarrow 0 \rightsquigarrow \cdots$$

where I is on the (k-1)-th position, for some k.

Note that $X_k = \mathrm{Hom}_R(DR, I) \in \mathrm{add}_R R \subseteq \mathcal{B}$ and $X_{k-1} = I \in \mathrm{add}_R(DR) \subseteq \mathcal{A}$, so we can choose A_X as the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow T_k \rightsquigarrow I \rightsquigarrow 0 \rightsquigarrow \cdots$$

where $A_{X_k} = T_k \in \mathrm{add}_R T$. And we have that the homomorphism $u_X : X \to A_X$ is of the form

$$\begin{array}{llllllll}
X: & \cdots \rightsquigarrow 0 \rightsquigarrow 0 \rightsquigarrow & \mathrm{Hom}_R(DR, I) & \rightsquigarrow & I & \rightsquigarrow 0 \rightsquigarrow \cdots \\
\downarrow u_X & & \downarrow u_k & & \downarrow 1 & \\
A_X: & \cdots \rightsquigarrow 0 \rightsquigarrow 0 \rightsquigarrow & T_k & \rightsquigarrow & I & \rightsquigarrow 0 \rightsquigarrow \cdots
\end{array}$$

Then, from the structure of $l(u_X)$, we can see that $l(u_X)$ is of the form

$$\begin{array}{llllllll}
X\hat{\otimes}DT: & \cdots \rightsquigarrow 0 \rightsquigarrow & 0 & \rightsquigarrow & \mathrm{Hom}_R(DR, I) \otimes_R DT & \rightsquigarrow & I \otimes_R DT & \rightsquigarrow 0 \rightsquigarrow \cdots \\
\downarrow l(u_X) & & \downarrow & & \downarrow (-\theta_{l_k}, u_k \otimes_D T) & & \downarrow 1 & \\
\mathrm{L}_T(A_X): & \cdots \rightsquigarrow 0 \rightsquigarrow & \mathrm{Hom}_R(T, T_k) & \overset{(0, \delta_{L_{k+1}})}{\rightsquigarrow} & \mathrm{Hom}_R(T, I) \oplus T_k \otimes_R DT & \rightsquigarrow & I \otimes_R DT & \rightsquigarrow 0 \rightsquigarrow \cdots
\end{array}$$

where θ_{l_k} is defined as in 4.1.3 and $\delta_{L_{k+1}}$ is defined as in 4.1.1 repectively. One checks that both homomorphisms θ_{l_k} and $\delta_{L_{k+1}}$ are in fact isomorphisms. So we obtain that $S_T(X) = \mathrm{Coker}(l(u_X))$ is of the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow \mathrm{Hom}_R(T, T_k) \overset{\delta'_{k+1}}{\rightsquigarrow} T_k \otimes_R DT \rightsquigarrow 0 \rightsquigarrow 0 \rightsquigarrow \cdots$$

where δ'_{k+1} is the induced isomorphism: $\mathrm{Hom}_R(T, T_k) \otimes_S DS \to T_k \otimes_R DT$. Since $T_k \otimes_R DT \in \mathrm{add}_S(T \otimes_R DT) = \mathrm{add}_S(DS)$, we see that $T_k \otimes_R DT$ is an injective S-module and that $\mathbf{S}_T(X)$ is a projective object in $\mathcal{RC}(S)$.

It follows that the functor $\mathbf{S}_T$ factors through $\underline{\mathcal{RC}}(R)$. We still denote by $\mathbf{S}_T$ the induced functor from $\underline{\mathcal{RC}}(R)$ to $\underline{\mathcal{RC}}(S)$.

4.2 From $\underline{\mathcal{RC}}$ to $\underline{\mathcal{RC}}(R)$: The functor $\mathbf{Q}_{DT}$

The functor $\mathbf{Q}_T$ is indeed defined in a way dual to the construction of $\mathbf{S}_T$.

4.2.1 *The functor* $\mathrm{R}_{DT} : \mathcal{RC}^{\mathrm{tr}}(S) \to \mathcal{RC}(R)$

Dually to 4.1.1, for any $X = \{X_i\} \in \mathcal{RC}^{\mathrm{tr}}(S)$, we define $\mathrm{R}_{DT}(X) \in \mathcal{RC}(R)$ as follows.

($r1$) the underlying module is $\mathrm{R}_{DT}(X)_i = \mathrm{Hom}_S(DT, X_i) \oplus X_{i+1} \otimes_S T$ and,

($r2$) the structure map $\delta_i^{\mathrm{H}}(\mathrm{R}_{DT})$ is given by $\begin{pmatrix} 0 & \delta_{R_i} \\ 0 & 0 \end{pmatrix}$, where δ_{R_i} is the composition

$$\mathrm{Hom}_S(DT,X_i)\overset{\mathrm{Hom}_S(DT,\eta^T_{X_i})}{\longrightarrow}\mathrm{Hom}_S(DT,\mathrm{Hom}_R(T,X_i\otimes_S T))$$
$$\simeq \mathrm{Hom}_R(DT\otimes_S T,X_i\otimes_S T)$$
$$\simeq \mathrm{Hom}_R(DR,X_i\otimes_S T).$$

It is easy to see that R_{DT} is a functor.

Remark (1) If $X\in\mathcal{RC}^{\mathrm{tr}}(\mathrm{add}_S DT)$, i.e., $X=\{X_i\}$ with each $X_i\in\mathrm{add}_S DT$, then $\mathrm{Hom}_S(DT,X_i)\in\mathrm{add}_R R$ and δ_{R_i} defined above is an isomorphism for each i. It follows that $\mathrm{R}_{DT}(X)$ is a projective object in $\mathcal{RC}(R)$ in the case.

(2) In particular, in case $T=S$, we obtain the functor $\mathrm{R}_{DS}:\mathcal{RC}^{\mathrm{tr}}(S)\to\mathcal{RC}(S)$ which specially send objects in $\mathcal{RC}^{\mathrm{tr}}(\mathrm{add}_S DS)$ to a projective object in $\mathcal{RC}(S)$.

4.2.2 *The functor* $\hat{\mathrm{H}}\mathrm{om}(DT,-):\mathcal{RC}(S)\to\mathcal{RC}(R)$

Let $Y=(Y_i,\delta^{\mathrm{H}}_i(Y))\in\mathcal{RC}(S)$. We definie $\hat{\mathrm{H}}\mathrm{om}_S(DT,Y)\in\mathcal{RC}(R)$ by setting

($h1$) the underlying module is $\hat{\mathrm{H}}\mathrm{om}_S(DT,Y)_i=\mathrm{Hom}_S(DT,Y_i)$ and,

($h2$) the structure map $\delta^{\mathrm{H}}_i(\hat{\mathrm{H}}\mathrm{om}_S(DT,Y))$ is given by the composition:

$$\mathrm{Hom}_S(DT,\delta^{\mathrm{H}}_i(Y))\overset{\mathrm{Hom}_S(DT,-y_i)}{\longrightarrow}\mathrm{Hom}_S(DT,\mathrm{Hom}_S(DS,Y_{i-1}))\simeq\mathrm{Hom}_R(DR,\mathrm{Hom}_S(DT,Y_{i-1}))).$$

Then from the functor property of $\mathrm{Hom}_S(DT,-)$, one can see that $\hat{\mathrm{H}}\mathrm{om}(DT,-)$ is a functor from $\mathcal{RC}(S)$ to $\mathcal{RC}(R)$.

4.2.3 *The homomorphism* $r^X_Y:\mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(S)}(X,Y)\to\mathrm{Hom}_{\mathcal{RC}(R)}(\mathrm{R}_{DT}(X),\hat{\mathrm{H}}\mathrm{om}(DT,Y))$

Dually to the homomorphism l^X_Y, for any $X\in\mathcal{RC}^{\mathrm{tr}}(S)$ and $Y\in\mathcal{RC}(S)$, we have a canonical homomorphism

$$r^X_Y:\mathrm{Hom}_{\mathcal{RC}^{\mathrm{tr}}(S)}(X,Y)\to\mathrm{Hom}_{\mathcal{RC}(R)}(\mathrm{R}_{DT}(X),\hat{\mathrm{H}}\mathrm{om}(DT,Y))$$

which is functional in both variables, defined by

$$r^X_Y:u=\{u_i\}\quad\longmapsto\quad f=\{f_i\}\text{ with }f_i=\begin{pmatrix}\mathrm{Hom}_S(DT,u_i)\\ -\zeta_{r_i}\end{pmatrix},$$

where $\zeta_{r_i}:X_{i+1}\otimes_S T\to\mathrm{Hom}_S(DT,Y_i)$ equals to $(u_{i+1}\otimes_S T)\circ(\delta^{\mathrm{H}}_{i+1}(Y)\otimes_S T)\circ\epsilon^T_{\mathrm{Hom}_S(DT,Y_i)}$, i.e., the composition

$$X_{i+1}\otimes_S T\overset{u_{i+1}\otimes_S T}{\longrightarrow}Y_{i+1}\otimes_S T\overset{\delta^Y_{i+1}\otimes_S T}{\longrightarrow}\mathrm{Hom}_S(DS,Y_i)\otimes_S T$$
$$\simeq\mathrm{Hom}_S(T\otimes_R DT,Y_i)\otimes_S T$$
$$\simeq\mathrm{Hom}_R(T,\mathrm{Hom}_S(DT,Y_i))\otimes_S T\overset{\epsilon^T_{\mathrm{Hom}_S(DT,Y_i)}}{\longrightarrow}\mathrm{Hom}_S(DT,Y_i)$$

Remark Using the fact ${}_SDS_S\simeq{}_ST\otimes_R DT_S$ and the adjoint isomorphism

$$\mathbf{\Gamma}^{DT}:\mathrm{Hom}_S(X_{i+1}\otimes_S DS,Y_i)\simeq\mathrm{Hom}_S(X_{i+1}\otimes_S T,\mathrm{Hom}_S(DT,Y_i)),$$

one can easily check that ζ_{r_i} is the image of the natural homomorphism $(u_{i+1}\otimes_S DS)\circ\delta^{\otimes}_{i+1}(Y)$ under Γ^{DT}, i.e., $\zeta_{r_i}=\mathbf{\Gamma}^{DT}((u_{i+1}\otimes_S DS)\circ\delta^{\otimes}_{i+1}(Y))$.

4.2.4 *An epimorphism* $v_Y:G_Y\to Y$ *with* $G_Y\in\mathcal{RC}^{\mathrm{tr}}(\mathcal{G})$, for $Y\in\mathcal{RC}(S)$.

Since $(\mathcal{G}, \mathcal{K})$ is a complete hereditary cotorsion pair in mod$S$. It follows that, for any $Y = \{Y_i\} \in \mathcal{RC}(S)$, there is an exact sequence $0 \to K_Y \xrightarrow{k_{v_Y}} G_Y \xrightarrow{v_Y} Y \to 0$ with $K_Y \in \mathcal{RC}^{\text{tr}}(\mathcal{K})$ and $G_Y \in \mathcal{RC}^{\text{tr}}(\mathcal{G})$.

Moreover, for any $h \in \text{Hom}_{\mathcal{RC}(S)}(X, Y)$, there is an induced commutative diagram as follows, since $\text{Ext}^1_S(\mathcal{G}, \mathcal{K}) = 0$.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_X & \xrightarrow{k_{v_X}} & G_X & \xrightarrow{v_X} & X & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_K} & & \downarrow{\scriptstyle h_G} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & K_Y & \xrightarrow{k_{v_Y}} & G_Y & \xrightarrow{v_Y} & Y & \longrightarrow & 0 \end{array}$$

4.2.5 *The kernel* $\text{Ker}(r(v_Y))$

Applying the functor $\text{Hom}_S(DT, -)$ to the bottom exact sequence in the above diagram, we obtain an induced exact sequence

$$0 \to \text{Hom}_S(DT, K_Y) \to \text{Hom}_S(DT, G_Y) \to \text{Hom}_S(DT, Y) \to 0,$$

since $K_{Y_i} \in \mathcal{K} \subseteq \text{KerExt}^1_S(DT, -)$ for each i. Thus, after applying the homomorphism r in 4.2.3 to the homomorphism v_X in 4.2.4, we obtain the following exact sequence in $\mathcal{RC}(R)$.

$$0 \to \text{Ker}(r(v_Y)) \xrightarrow{\lambda_{r_Y}} \text{R}_{DT}(G_Y) \xrightarrow{r(v_Y)} \hat{\text{H}}\text{om}(DT, Y) \to 0$$

Moreover, for any $h \in \text{Hom}_{\mathcal{RC}(S)}(X, Y)$, by applying the homomorphism r to the right part of the commutative diagram in 4.2.4, we obtain the following commutative diagram in $\mathcal{RC}(R)$, for some h_{Ker}.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ker}(r(v_X)) & \xrightarrow{\lambda_{r_X}} & \text{R}_{DT}(G_X) & \xrightarrow{r_{v_X}} & \hat{\text{H}}\text{om}(DT, X) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle h_{\text{Ker}}} & & \downarrow{\scriptstyle \text{R}_{DT}(h_G)} & & \downarrow{\scriptstyle \hat{\text{H}}\text{om}(DT,h)} & & \\ 0 & \longrightarrow & \text{Ker}(r(v_Y)) & \xrightarrow{\lambda_{r_Y}} & \text{R}_{DT}(G_Y) & \xrightarrow{r_{v_Y}} & \hat{\text{H}}\text{om}(DT, Y) & \longrightarrow & 0 \end{array}$$

4.2.6 *The assignment* $\mathbf{Q}_{DT} : \mathcal{RC}(S) \to \underline{\mathcal{RC}}(R)$ *by setting* $Y \longmapsto \text{Ker}(r(v_Y))$ *is a functor.*

By 4.2.5, it is sufficient to prove that $\mathbf{Q}_{DT}(h) := h_{\text{Ker}} = 0$ in $\underline{\mathcal{RC}}(R)$ provided $h = 0$. This is also divided into two steps.

Step 1: Consider each piece in the commutative diagram in 4.2.4. If $h = \{h_i\} = 0$, then $h_i = 0$ for each i. Thus, we have that $h_{G_i} v_{Y_i} = 0$ and consequently, $h_{G_i} = g_i k_{v_{Y_i}}$ for some $g_i : G_{X_i} \to K_{Y_i}$. Since $K_{Y_i} \in \mathcal{K} \subseteq {}_{DT}\mathcal{X}$ for all i, there are exact sequences $0 \to K'_{Y_i} \to DT_{K_{Y_i}} \xrightarrow{b_i} K_{Y_i} \to 0$ with $DT_{K_{Y_i}} \in \text{add}_S DT$ and $K'_{Y_i} \in \mathcal{K} \subseteq \text{KerExt}^1_S(\mathcal{G}, -)$. It follows that there exists $t_i \in \text{Hom}_R(G_{Y_i}, DT_{K_{X_i}})$ such that $g_i = t_i b_i$. Altogether we obtain the following commutative diagram.

$$\begin{array}{ccc} G_{X_i} & \xrightarrow{t_i} & DT_{K_{Y_i}} \\ {\scriptstyle h_{G_i}}\downarrow & \overset{g_i}{\searrow} & \downarrow{\scriptstyle b_i} \\ G_{Y_i} & \xleftarrow[k_{v_{Y_i}}]{} & K_{Y_i} \end{array}$$

It follows that there is a commutative diagram in $\mathcal{RC}^{\mathrm{tr}}(S)$, where $DT_{K_Y} := \{DT_{K_{Y_i}}\}$,

$$\begin{array}{ccc} G_X & \xrightarrow{t} & DT_{K_Y} \\ {\scriptstyle h_G}\downarrow & \overset{g}{\searrow} & \downarrow{\scriptstyle b} \\ G_Y & \xleftarrow[k_{v_Y}]{} & K_Y. \end{array}$$

Set $\beta := bk_{v_Y}$. Then $\mathrm{R}_{DT}(h_G) = \mathrm{R}_{DT}(t\beta) = \mathrm{R}_{DT}(t)\mathrm{R}_{DT}(\beta)$.

Step 2: Consider the commutative diagram in 4.2.5. Since $k_{v_{Y_i}} v_{Y_i} = 0$, we see that $\mathrm{R}_{DT}(\beta)r(v_Y) = 0$ by the definitions. Hence there is some $\theta \in \mathrm{Hom}_{\mathcal{RC}(R)}(\mathrm{R}_{DT}(G_X), \mathrm{Ker}(r(v_Y)))$ such that $\mathrm{R}_{DT}(\beta) = \theta\lambda_{r_Y}$. Consequently, we have that $\mathrm{R}_{DT}(h_G) = \mathrm{R}_{DT}(t)\mathrm{R}_{DT}(\beta) = \mathrm{R}_{DT}(t)\theta\lambda_{r_Y}$. Now we obtain that $h_{\mathrm{Ker}}\lambda_{r_Y} = \lambda_{r_X}\mathrm{R}_{DT}(h_G) = \lambda_{r_X}\mathrm{R}_{DT}(t)\theta\lambda_{r_Y}$. Since λ_{r_Y} is monomorphic, we get that $h_{\mathrm{Ker}} = \lambda_{r_X}\mathrm{R}_{DT}(t)\theta$. That is, we have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Ker}(r(v_X)) & \xrightarrow{\lambda_{r_X}} & \mathrm{R}_{DT}(G_Y) \\ {\scriptstyle h_{\mathrm{Ker}}}\downarrow & & \downarrow{\scriptstyle \mathrm{R}_{DT}(t)} \\ \mathrm{Ker}(r(v_Y)) & \xleftarrow[\theta]{} & \mathrm{R}_{DT}(DT_{K_Y}) \end{array}$$

Note that $\mathrm{R}_{DT}(DT_{K_Y})$ is a projective-injective object in $\mathcal{RC}(R)$, so $h_{\mathrm{Ker}} = 0$ in $\underline{\mathcal{RC}}(R)$.

4.2.7 *The functor* $\mathbf{Q}_{DT} : \underline{\mathcal{RC}}(S) \to \underline{\mathcal{RC}}(R)$

We will show that the functor $\mathbf{Q}_{DT}$ factors through $\underline{\mathcal{RC}}(S)$.

To see this, it is enough to show that $\mathbf{Q}_{DT}(X)$ is a projective object in $\mathcal{RC}(R)$, whenever X is a projective object in $\mathcal{RC}(S)$.

W.l.o.g., we assume that $X = \{X_i\}$ is an indecomposable projective object in $\mathcal{RC}(S)$. Thus, we have that X has the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow P \overset{1}{\rightsquigarrow} P \otimes_S DS \rightsquigarrow 0 \rightsquigarrow \cdots$$

where P is on the $(k{+}1)$-th position, for some k.

Note that $X_{k+1} = P \in \mathrm{add}_R R \subseteq \mathcal{G}$ and $X_k = P \otimes_S DS \in \mathrm{add}_R(DR) \subseteq \mathcal{K}$, so we can choose G_X as the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow P \rightsquigarrow DT_k \rightsquigarrow 0 \rightsquigarrow \cdots$$

where $G_{X_k} = DT_k \in \mathrm{add}_R DT$. And we have that the homomorphism $v_X : G_X \to X$ is of the form

$$
\begin{array}{lcccccccccc}
G_X: & \cdots & \rightsquigarrow 0 \rightsquigarrow & 0 \rightsquigarrow & P & \rightsquigarrow & DT_k & \rightsquigarrow 0 \rightsquigarrow & \cdots \\
\downarrow v_X & & & & \downarrow 1 & & \downarrow v_k & & \\
X: & \cdots & \rightsquigarrow 0 \rightsquigarrow & 0 \rightsquigarrow & P & \rightsquigarrow & P \otimes_S DS & \rightsquigarrow 0 \rightsquigarrow & \cdots
\end{array}
$$

Then, from the structure of $r(v_X)$ in 4.2.3, we can see that $r(v_X)$ is of the form

$$
\begin{array}{lcccccc}
\mathrm{R}_{DT}(X): & \cdots \rightsquigarrow 0 \rightsquigarrow & \mathrm{Hom}_S(DT,P) \rightsquigarrow & \mathrm{Hom}_S(DT,DT_k) \oplus P \otimes_S T & \overset{(0,\delta_{R_k})^{\mathrm{T}}}{\rightsquigarrow} & DT_k \otimes_S T & \rightsquigarrow 0 \rightsquigarrow \cdots \\
\downarrow r(v_X) & & \downarrow 1 & \downarrow (\mathrm{Hom}_S(DT,v_k),\zeta_{r_k})^{\mathrm{T}} & & \downarrow & \\
\hat{\mathrm{H}}\mathrm{om}(DT,X): & \cdots \rightsquigarrow 0 \rightsquigarrow & \mathrm{Hom}_S(DT,P) \rightsquigarrow & \mathrm{Hom}_S(DT, P \otimes_S DS) & \rightsquigarrow & 0 & \rightsquigarrow 0 \rightsquigarrow \cdots
\end{array}
$$

where ζ_{r_k} is defined as in 4.2.3 and δ_{R_k} is defined as in 4.2.1 repectively. One checks that both homomorphisms ζ_{r_k} and δ_{R_k} are in fact isomorphisms. So we obtain that $\mathbf{Q}_{DT}(X) = \mathrm{Ker}(r(v_X))$ is of the form

$$\cdots \rightsquigarrow 0 \rightsquigarrow 0 \rightsquigarrow \mathrm{Hom}_S(DT,DT_k) \overset{\delta'_k}{\rightsquigarrow} DT_k \otimes_S T \rightsquigarrow 0 \rightsquigarrow \cdots$$

where δ'_k is an induced isomorphism: $\mathrm{Hom}_S(DT,DT_k) \to \mathrm{Hom}_S(DS, DT_k \otimes_S T)$. Since $\mathrm{Hom}_S(DT,DT_k) \in \mathrm{add}_R(\mathrm{Hom}_S(DT,DT)) = \mathrm{add}_R R$, we see that $\mathrm{Hom}_S(DT,DT_k)$ is projective and that $\mathbf{Q}_{DT}(X)$ is a projective object in $\mathcal{RC}(R)$.

It follows that the functor $\mathbf{Q}_{DT}$ factors through $\underline{\mathcal{RC}}(S)$. We still denote by $\mathbf{Q}_{DT}$ the induced functor from $\underline{\mathcal{RC}}(S)$ to $\underline{\mathcal{RC}}(R)$.

4.3 The isomorphism $\mathbf{Q}_{DT}\mathbf{S}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$

4.3.1 *Computing the composition* $\mathbf{Q}_{DT}\mathbf{S}_T$

Take any $X = \{X_i, \delta_i^{\otimes}\} \in \mathcal{RC}(R)$. From the chosen exact sequence $0 \to X \xrightarrow{u_X} A_X \xrightarrow{\pi_{u_X}} B_X \to 0$ in $\mathcal{RC}^{\mathrm{tr}}(R)$ with $A_X = \{A_{X_i}\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{A})$ and $B_X = \{B_{X_i}\} \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{B})$, as in 4.1.4, we obtain an exact sequence

$$0 \to X \hat{\otimes} DT \xrightarrow{l(u_X)} \mathrm{L}_T(A_X) \xrightarrow{s} \mathbf{S}_T(X) \to 0$$

by the construction of the functor $\mathbf{S}_T$ in 4.1.5. Note that, for each i, $\mathbf{S}_T(X)_i$ is given by the pushout diagram

$$
\begin{array}{ccc}
X_i \otimes_R DT & \xrightarrow{\theta_{l_i}} & \mathrm{Hom}_R(T, A_{X_{i-1}}) \\
{\scriptstyle u_{X_i} \otimes_R DT} \downarrow & & \downarrow {\scriptstyle s_i^1} \\
A_{X_i} \otimes_R DT & \xrightarrow{s_i^2} & \mathbf{S}_T(X)_i.
\end{array}
$$

Now we take a projective R-module $P_{A_{X_i}}$ such that $P_{A_{X_i}} \xrightarrow{p_i} A_{X_i} \to 0$ is exact. Then we have a pullback diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{X}_i & \xrightarrow{\overline{u}_{X_i}} & P_{A_{X_i}} & \longrightarrow & B_{X_i} & \longrightarrow & 0 \\
 & & {\scriptstyle q_i} \downarrow & & \downarrow {\scriptstyle p_i} & & \| & & \\
0 & \longrightarrow & X_i & \xrightarrow{u_{X_i}} & A_{X_i} & \longrightarrow & B_{X_i} & \longrightarrow & 0.
\end{array}
$$

Since $\mathcal{B}$ is closed under kernels of epimorphisms, we see that $\overline{X}_i \in \mathcal{B}$. By applying the functor $-\otimes_R DT$, the diagram above induces the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{X}_i \otimes_R DT & \xrightarrow{\overline{u}_{X_i}\otimes_R DT} & P_{A_{X_i}} \otimes_R DT & \longrightarrow & B_{X_i} \otimes_R DT & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q_i\otimes_R DT} & & \downarrow{\scriptstyle p_i\otimes_R DT} & & \| & & \\
0 & \longrightarrow & X_i \otimes_R DT & \xrightarrow{u_{X_i}\otimes_R DT} & A_{X_i} \otimes_R DT & \longrightarrow & B_{X_i} \otimes_R DT & \longrightarrow & 0
\end{array}$$

Now, one can check that the following diagram is commutative with exact rows, for each i, where the lower row is obtained from the above exact sequence.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \overline{X}_i \otimes_R DT & \xrightarrow{(-(q_i\otimes_R DT)\circ\theta_{l_i},\, \overline{u}_{X_i}\otimes_R DT)} & \mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT & \xrightarrow{s_i^P = \begin{pmatrix} s_i^1 \\ (p_i\otimes_R DT)\circ s_i^2 \end{pmatrix}} & \mathbf{S}_T(X)_i & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle q_i\otimes_R DT} & & \downarrow{\scriptstyle \begin{pmatrix} 1 & 0 \\ 0 & p_i\otimes_R DT \end{pmatrix}} & & \| & & \\
0 & \longrightarrow & X_i \otimes_R DT & \xrightarrow{(-\theta_{l_i},\, u_{X_i}\otimes_R DT)} & \mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus A_{X_i} \otimes_R DT & \xrightarrow{s_i = \begin{pmatrix} s_i^1 \\ s_i^2 \end{pmatrix}} & \mathbf{S}_T(X)_i & \longrightarrow & 0
\end{array}$$

Denote $\mathfrak{L}_{A_X}^P := \{\mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT\} \in \mathcal{RC}^{\mathrm{tr}}(S)$. Note that $\overline{X}_i \otimes_R DT \in \mathcal{K}$ and $\mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT \in \mathcal{G}$, so we have an exact sequence from the first row in the commutative diagram

$$0 \to \overline{X} \otimes_R DT \longrightarrow \mathfrak{L}_{A_X}^P \xrightarrow{s^P} \mathbf{S}_T(X) \to 0$$

with $\overline{X} \otimes_R DT \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{K})$ and $\mathfrak{L}_{A_X}^P \in \mathcal{RC}^{\mathrm{tr}}(\mathcal{G})$, as in 4.2.4. By applying the homomorphism r in 4.2.3 to the homomorphism $s^P : \mathfrak{L}_{A_X}^P \to \mathbf{S}_T(X)$, we have an exact sequence in $\mathcal{RC}(R)$ by the construction of the functor $\mathbf{Q}_{DT}$ in 4.2.5

$$0 \to \mathbf{Q}_{DT}\mathbf{S}_T(X) \xrightarrow{\lambda} \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P) \xrightarrow{r(s^P)} \hat{\mathrm{H}}\mathrm{om}(DT, \mathbf{S}_T(X)) \to 0,$$

where $r(s^P)$ is defined as in 4.2.3.

4.3.2 *The object $X \oplus \mathrm{L}_R(P_{A_X}^+)$ in $\mathcal{RC}(R)$*

Denote that $P_{A_X}^+ := \{P_{A_{X_{i+1}}}\}$, then $P_{A_X}^+ \in \mathcal{RC}^{\mathrm{tr}}(\mathrm{add}_R R)$. Applying the functor L_R in the remark in 4.1.1, we obtain that $\mathrm{L}_R(P_{A_X}^+)$ is a projective object in $\mathcal{RC}(R)$. Hence, the object $X \oplus \mathrm{L}_R(P_{A_X}^+)$ is isomorphic to X in $\underline{\mathcal{RC}}(R)$.

We will prove that $\mathbf{Q}_{DT}\mathbf{S}_T(X) \simeq X \oplus \mathrm{L}_R(P_{A_X}^+)$ naturally. And then, $\mathbf{Q}_{DT}\mathbf{S}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$.

The general strategy is as follows. Firstly, we construct a natural homomorphism $\xi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P)$. Secondly, we show that $\xi \circ r(s^P) = 0$, i.e., the composition of ξ and the homomorphism $r(s^P) : \mathrm{R}_T(\mathfrak{L}_{A_X}^P) \to \hat{\mathrm{H}}\mathrm{om}(DT, \mathbf{S}_T(X))$ in the exact sequence above is 0. Thus, we obtain a homomorphism $\phi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$. Finally, we prove that ϕ is indeed a natural isomorphism.

4.3.3 *The homomorphism $\xi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathrm{R}_{DT}(\mathfrak{L}_{A_X}^P)$*

Recall from the construction in 4.3.1 that

$X = \{X_i\}$ and,

$\mathrm{L}_R(P^+_{A_X}) = \{\mathrm{Hom}_R(R, P_{A_{X_i}}) \oplus P_{A_{X_{i+1}}} \otimes_R DR\} = \{P_{A_{X_i}} \oplus P_{A_{X_{i+1}}} \otimes_R DR\}$

and that

$\mathrm{R}_{DT}(\mathfrak{L}^P_{A_X}) = \{\mathrm{Hom}_S(DT, (\mathfrak{L}^P_{A_X})_i) \oplus (\mathfrak{L}^P_{A_X})_{i+1} \otimes_S T\}$

$= \{\mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT) \oplus (\mathrm{Hom}_R(T, A_{X_i}) \oplus P_{A_{X_{i+1}}} \otimes_R DT) \otimes_S T\}$

Let $\xi = \{\xi_i\} : X \oplus \mathrm{L}_R(P^+_{A_X}) \to \mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})$ be a homomorphism. We may assume that $\xi_i = (\xi^a_i, \xi^b_i)$, where

$\xi^a_i : X_i \oplus P_{A_{X_i}} \oplus P_{A_{X_{i+1}}} \otimes_R DR \to \mathrm{Hom}_S(DT, (\mathfrak{L}^P_{A_X})_i)$ and

$\xi^b_i : X_i \oplus P_{A_{X_i}} \oplus P_{A_{X_{i+1}}} \otimes_R DR \to (\mathfrak{L}^P_{A_X})_{i+1} \otimes_S T$.

4.3.3 (i) *The homomorphism ξ^a_i in* mod$R$

We set $\xi^a_i = \begin{pmatrix} \xi^a_{11} & \xi^a_{12} \\ \xi^a_{21} & \xi^a_{22} \\ \xi^a_{31} & \xi^a_{32} \end{pmatrix}$:

$$X_i \oplus P_{A_{X_i}} \oplus P_{A_{X_{i+1}}} \otimes_R DR \to \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT).$$

Using the isomorphism ${}_SDS_S \simeq {}_ST \otimes_R DT_S$ and the adjoint isomorphism

$$\mathbf{\Gamma}^{DT} : \mathrm{Hom}_S(- \otimes_S DT, -) \simeq \mathrm{Hom}_S(-, \mathrm{Hom}_S(DT, -)),$$

we define the components as follows.

- We set

$$\xi^a_{11} = \mathbf{\Gamma}^{DT}(\theta_{l_i}) : X_i \to \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, A_{X_{i-1}})),$$

where $\theta_{l_i} : X_i \otimes_S DT \to \mathrm{Hom}_R(T, A_{X_{i-1}})$ is defined in 4.1.3.

In the other words, the morphism ξ^a_{11} is given by the composition : $\eta^{DR}_{X_i} \circ \mathrm{Hom}_R(DR, \delta_{X_i}) \circ \mathrm{Hom}_R(DR,\ u_{X_{i-1}})$ and some natural isomorphisms

$$\begin{aligned} X_i &\xrightarrow{\eta^{DR}_{X_i}} \mathrm{Hom}_R(DR, X_i \otimes_R DR) \xrightarrow{\mathrm{Hom}_R(DR, \delta_{X_i})} \mathrm{Hom}_R(DR, X_{i-1}) \\ &\xrightarrow{\mathrm{Hom}_R(DR,\ u_{X_{i-1}})} \mathrm{Hom}_R(DR, A_{X_{i-1}}) \simeq \mathrm{Hom}_R(DT \otimes_S T, A_{X_{i-1}}) \\ &\simeq \mathrm{Hom}_R(DT, \mathrm{Hom}_R(T, A_{X_{i-1}})). \end{aligned}$$

- The morphism $\xi^a_{22} = \mathbf{\Gamma}^{DT}(1_{(P_{A_{X_i}} \otimes_R DT)}) = \eta^{DT}_{P_{A_{X_i}}} : P_{A_{X_i}} \to \mathrm{Hom}_S(DT, P_{A_{X_i}} \otimes_R DT)$.
- The remained morphisms $\xi^a_{12}, \xi^a_{21}, \xi^a_{31}, \xi^a_{32}$ are all 0.

So we have that $\xi^a_i = \begin{pmatrix} \xi^a_{11} & 0 \\ 0 & \xi^a_{22} \\ 0 & 0 \end{pmatrix}$, where $\xi^a_{11} = \mathbf{\Gamma}^{DT}(\theta_{l_i})$, and $\xi^a_{22} = \mathbf{\Gamma}^{DT}(1_{(P_{A_{X_i}} \otimes_R DT)})$.

4.3.3 (ii) *The homomorphism ξ^b_i in* mod$R$

We set $\xi_i^b = \begin{pmatrix} \xi_{11}^b & \xi_{12}^b \\ \xi_{21}^b & \xi_{22}^b \\ \xi_{31}^b & \xi_{32}^b \end{pmatrix}$:

$$X_i \oplus P_{A_{X_i}} \oplus P_{A_{X_{i+1}}} \otimes_R DR \to (\mathrm{Hom}_R(T, A_{X_i}) \oplus P_{A_{X_{i+1}}} \otimes_R DT) \otimes_S T,$$

where the components are defined naturally as follows.

- The morphism $\xi_{11}^b = u_{X_i} \circ (\epsilon^T_{A_{X_i}})^{-1} : X_i \to \mathrm{Hom}_R(T, A_{X_i}) \otimes_S T$ (note that $\epsilon^T_{A_{X_i}}$ is an isomorphism since $A_{X_i} \in \mathcal{A}$), i.e., is given by the composition
$$X_i \xrightarrow{u_{X_i}} A_{X_i} \xrightarrow{(\epsilon^T_{A_{X_i}})^{-1}} \mathrm{Hom}_R(T, A_{X_i}) \otimes_S T.$$
- The morphism $\xi_{21}^b = p_i \circ (\epsilon^T_{A_{X_i}})^{-1} : P_{A_{X_i}} \to \mathrm{Hom}_R(T, A_{X_i}) \otimes_S T$, i.e., is given by the composition
$$P_{A_{X_i}} \xrightarrow{p_i} A_{X_i} \xrightarrow{(\epsilon^T_{A_{X_i}})^{-1}} \mathrm{Hom}_R(T, A_{X_i}) \otimes_S T.$$
- The morphism $\xi_{32}^b : P_{A_{X_{i+1}}} \otimes_R DR \to P_{A_{X_{i+1}}} \otimes_R DT \otimes_S T$ is the natural isomorphism given by ${}_R(DT \otimes_S T)_R \simeq {}_R R_R$.
- The remained morphisms $\xi_{12}^b, \xi_{22}^b, \xi_{31}^b$ are all 0.

So we have that $\xi_i^b = \begin{pmatrix} \xi_{11}^b & 0 \\ \xi_{21}^b & 0 \\ 0 & \xi_{32}^b \end{pmatrix}$, where $\xi_{11}^b = u_{X_i} \circ (\epsilon^T_{A_{X_i}})^{-1}$, $\xi_{21}^b = p_i \circ (\epsilon^T_{A_{X_i}})^{-1}$, and ξ_{32}^b is the natural isomorphism.

4.3.3 (iii) *ξ is a homomorphism in $\mathcal{RC}(R)$*

It is not difficult to prove that the above-defined morphism ξ is in fact a homomorphism in $\mathcal{RC}(R)$ by the involved definitions.

4.3.4 *The composition $\xi \circ r(s^P) = 0$, and so ξ factors through a homomorphism $\phi : X \oplus \mathrm{L}_R(P^+_{A_X}) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$.*

4.3.4 (i) *The analysis of the homomorphism $s : \mathrm{L}_T(A_X) \to \mathbf{S}_T(X)$ in 4.3.1.*

Recall from 4.3.1 that $s = \{s_i\} : \mathrm{L}_T(A_X) \to \mathbf{S}_T(X)$ is a homomorphism in $\mathcal{RC}(S)$ which is the cokernel of the homomorphism $l(u_X)$. Note that $\mathrm{L}_T(A_X) = \mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus A_{X_i} \otimes DT$, so we write that $s_i = \begin{pmatrix} s_i^1 \\ s_i^2 \end{pmatrix}$ as we have done in the last commutative diagram in 4.3.1. The fact that s is a homomorphism in $\mathcal{RC}(S)$ implies that there is the following commutative diagram, for each i.

$$\begin{array}{ccc} [\mathrm{Hom}_R(T,A_{X_{i-1}}) \oplus A_{X_i} \otimes_R DT] \otimes_S DS & \xrightarrow{\begin{pmatrix} s_i^1 \\ s_i^2 \end{pmatrix} \otimes_S DS} & \mathbf{S}_T(X)_i \otimes_S DS \\ \downarrow \delta_i^{\otimes}(\mathrm{L}_T(A_X)) & & \downarrow \delta_i^{\otimes}(\mathbf{S}_T(X)) \\ \mathrm{Hom}_R(T,A_{X_{i-2}}) \oplus A_{X_{i-1}} \otimes_R DT & \xrightarrow{\begin{pmatrix} s_{i-1}^1 \\ s_{i-1}^2 \end{pmatrix}} & \mathbf{S}_T(X)_{i-1} \end{array}$$

By the definition of $\delta_i^{\otimes}(\mathrm{L}_T(A_X))$ (see 4.1.1) and the above commutative diagram, we obtain that $(s_i^2 \otimes DS) \circ \delta_i^{\otimes}(\mathbf{S}_T(X)) = 0$ and that $(s_i^1 \otimes DS) \circ \delta_i^{\otimes}(\mathbf{S}_T(X)) = (\epsilon^T_{A_{X_{i-1}}} \otimes DT) \circ s^2_{i-1}$.

4.3.4 (ii) *The analysis of the homomorphism* $r(s^P) : \mathrm{R}_T(\mathfrak{L}^P_{A_X}) \to \hat{\mathrm{H}}\mathrm{om}(DT, \mathbf{S}_T(X))$

Recall from 4.3.1 that $s^P = \{s_i^P\} = \{\begin{pmatrix} s_i^1 \\ (p_i \otimes_R DT) \circ s_i^2 \end{pmatrix}\} : \mathfrak{L}^P_{A_X} \to \mathbf{S}_T(X)$. By 4.2.3, we know that $\mathrm{R}_T(\mathfrak{L}^P_{A_X}) = \{\mathrm{Hom}_S(DT, (\mathfrak{L}^P_{A_X})_i) \oplus (\mathfrak{L}^P_{A_X})_{i+1} \otimes_S T\}$ and that $r(s^P)_i = \begin{pmatrix} \mathrm{Hom}_S(DT, s_i^P) \\ -\zeta_{r_i} \end{pmatrix}$, where $\zeta_{r_i} = \mathbf{\Gamma}^{DT}((s^P_{i+1} \otimes_S DS) \circ \delta^{\otimes}_{i+1}(\mathbf{S}_T(X)))$.

For convenience, we set $r(s^P)_i = \begin{pmatrix} r_i^1 \\ r_i^2 \end{pmatrix}$, where

$$r_i^1 = \mathrm{Hom}_S(DT, s_i^P) : \mathrm{Hom}_S(DT, (\mathfrak{L}^P_{A_X})_i) \to \mathbf{S}_T(X)_i \text{ and}$$

$$r_i^2 = -\zeta_{r_i} : (\mathfrak{L}^P_{A_X})_{i+1} \otimes_S T \to \mathbf{S}_T(X)_i.$$

4.3.4 (iii) *Checking* $\xi \circ r(s^P) = 0$

To check $\xi \circ r(s^P) = 0$, we need only to check that $\xi_i^a r_i^1 + \xi_i^b r_i^2 = 0$ for each i, since $\xi_i = (\xi_i^a, \xi_i^b)$ and $r(s^P)_i = \begin{pmatrix} r_i^1 \\ r_i^2 \end{pmatrix}$. Note that $r_i^1 = \mathrm{Hom}_S(DT, s_i^P)$ and $r_i^2 = -\zeta_{r_i}$, so it is enough to check that $\xi_i^a \circ \mathrm{Hom}_S(DT, s_i^P) = \xi_i^b \circ \zeta_{r_i}$.

Since $\xi_i^a = \begin{pmatrix} \xi_{11}^a & 0 \\ 0 & \xi_{22}^a \\ 0 & 0 \end{pmatrix}$, $\xi_i^b = \begin{pmatrix} \xi_{11}^b & 0 \\ \xi_{21}^b & 0 \\ 0 & \xi_{32}^b \end{pmatrix}$, $\zeta_{r_i} = \mathbf{\Gamma}^{DT}((s^P_{i+1} \otimes_S DS) \circ \delta^{\otimes}_{i+1}(\mathbf{S}_T(X)))$ and $s_i^P = \begin{pmatrix} s_i^1 \\ (p_i \otimes_R DT) \circ s_i^2 \end{pmatrix}$, we just check the following.

(1) $\xi_{11}^a \circ \mathrm{Hom}_S(DT, s_i^1) = \xi_{11}^b \circ \mathbf{\Gamma}^{DT}((s^1_{i+1} \otimes_S DS) \circ \delta^{\otimes}_{i+1}(\mathbf{S}_T(X)))$.

By 4.3.3 (i), we have that

$$\xi_{11}^a \circ \mathrm{Hom}_S(DT, s_i^1) = \mathbf{\Gamma}^{DT}(\theta_{l_i}) \circ \mathrm{Hom}_S(DT, s_i^1) = \mathbf{\Gamma}^{DT}(\theta_{l_i} \circ s_i^1),$$

where later equality uses the naturality of $\mathbf{\Gamma}^{DT}$. On the other hand, by 4.3.3 (ii) and 4.3.4 (i), we obtain that

$$\begin{aligned} &\xi_{11}^b \circ \mathbf{\Gamma}^{DT}((s^1_{i+1} \otimes_S DS) \circ \delta^{\otimes}_{i+1}(\mathbf{S}_T(X))) \\ &= (u_{X_i} \circ (\epsilon^T_{A_{X_i}})^{-1}) \circ \mathbf{\Gamma}^{DT}((\epsilon^T_{A_{X_i}} \otimes_R DT) \circ s_i^2) \\ &= \mathbf{\Gamma}^{DT}(((u_{X_i} \circ (\epsilon^T_{A_{X_i}})^{-1}) \otimes_R DT) \circ (\epsilon^T_{A_{X_i}} \otimes_R DT) \circ s_i^2) \\ &= \mathbf{\Gamma}^{DT}((u_{X_i} \otimes_R DT) \circ s_i^2). \end{aligned}$$

But $(u_{X_i} \otimes_R DT) \circ s_i^2 = \theta_{l_i} \circ s_i^1$ by the pushout diagram on $\mathbf{S}_T(X)_i$ in 4.3.1. Hence, the equality (1) holds.

(2) $\xi_{22}^a \circ \mathrm{Hom}_S(DT, (p_i \otimes_R DT) \circ s_i^2) = \xi_{21}^b \circ \mathbf{\Gamma}^{DT}((s^1_{i+1} \otimes_S DS) \circ \delta^{\otimes}_{i+1}(\mathbf{S}_T(X)))$.

By 4.3.3 (i) and the naturality of $\mathbf{\Gamma}^{DT}$,

$$\begin{aligned}
&\xi_{22}^{a} \circ \operatorname{Hom}_S(DT, (p_i \otimes_R DT) \circ s_i^2) \\
&= \mathbf{\Gamma}^{DT}(1_{P_{A_{X_i}} \otimes_R DT}) \circ \operatorname{Hom}_S(DT, (p_i \otimes_R DT) \circ s_i^2) \\
&= \mathbf{\Gamma}^{DT}(1_{P_{A_{X_i}} \otimes_R DT} \circ ((p_i \otimes_R DT) \circ s_i^2)) \\
&= \mathbf{\Gamma}^{DT}((p_i \otimes_R DT) \circ s_i^2).
\end{aligned}$$

On the other hand, by 4.3.3 (ii) and 4.3.4 (i) and the naturality of $\mathbf{\Gamma}^{DT}$,

$$\begin{aligned}
&\xi_{21}^{b} \circ \mathbf{\Gamma}^{DT}((s_{i+1}^1 \otimes_S DS) \circ \delta_{i+1}^{\otimes}(\mathbf{S}_T(X))) \\
&= (p_i \circ (\epsilon_{A_{X_i}}^T)^{-1}) \circ \mathbf{\Gamma}^{DT}((\epsilon_{A_{X_i}}^T \otimes_R DT) \circ s_i^2) \\
&= \mathbf{\Gamma}^{DT}(((p_i \circ (\epsilon_{A_{X_i}}^T)^{-1}) \otimes_R DT) \circ (\epsilon_{A_{X_i}}^T \otimes_R DT) \circ s_i^2) \\
&= \mathbf{\Gamma}^{DT}((p_i \otimes_R DT) \circ s_i^2).
\end{aligned}$$

Hence, the equality (2) holds.

(3) $0 = \xi_{32}^{b} \circ \mathbf{\Gamma}^{DT}((((p_{i+1} \otimes_R DT) \circ s_{i+1}^2) \otimes_S DS) \circ \delta_{i+1}^{\otimes}(\mathbf{S}_T(X)))$.

In fact, the equality holds by observing that

$$\begin{aligned}
&\mathbf{\Gamma}^{DT}((((p_{i+1} \otimes_R DT) \circ s_{i+1}^2) \otimes_S DS) \circ \delta_{i+1}^{\otimes}(\mathbf{S}_T(X))) \\
&= \mathbf{\Gamma}^{DT}((p_{i+1} \otimes_R DT \otimes_S DS) \circ (s_{i+1}^2 \otimes_S DS) \circ \delta_{i+1}^{\otimes}(\mathbf{S}_T(X))) \\
&= 0,
\end{aligned}$$

since $(s_{i+1}^2 \otimes_S DS) \circ \delta_{i+1}^{\otimes}(\mathbf{S}_T(X))) = 0$ by 4.3.4 (i).

All together, we prove that $\xi \circ r(s^P) = 0$ and therefore, ξ factors through $\mathbf{Q}_{DT}\mathbf{S}_T(X) = \operatorname{Ker}(r(s^P))$ by a homomorphism $\phi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$ in $\mathcal{RC}(R)$, i.e., $\xi = \phi \circ \lambda$.

4.3.5 *The induced homomorphism $\phi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$ is an isomorphism*

We now prove that the induced homomorphism $\phi : X \oplus \mathrm{L}_R(P_{A_X}^+) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$ is an isomorphism. Clearly it is equivalent to show that $\phi_i : (X \oplus \mathrm{L}_R(P_{A_X}^+))_i \to \mathbf{Q}_{DT}\mathbf{S}_T(X)_i$ is an isomorphism, for each i.

We will show that there is the following commutative diagram $(*)$ with exact rows, for each i, where $\overline{X}_i \in \mathcal{B}$ is obtained in 4.3.1. Note that $\mathrm{L}_R(P_{A_X}^+)_i = \operatorname{Hom}_R(T, A_{X_i}) \otimes_S T \oplus P_{A_{X_{i+1}}} \otimes_R DT \otimes_S T \simeq A_{X_i} \oplus P_{A_{X_{i+1}}} \otimes_R DR$.

$$(*): \quad \begin{array}{ccccccccc}
0 \to & \overline{X}_i & \xrightarrow{(-q_i, \overline{u}_{X_i}, 0)} & X_i \oplus P_{A_{X_i}} \oplus P_{A_{X_{i+1}}} \otimes_R DR & \xrightarrow{\begin{pmatrix} u_{X_i} & 0 \\ p_i & 0 \\ 0 & 1 \end{pmatrix}} & A_{X_i} \oplus P_{A_{X_{i+1}}} \otimes DR & \to 0 \\
& \downarrow \eta_{\overline{X}_i}^{DT} & & \downarrow \phi_i & & \| & \\
0 \to & \operatorname{Hom}_R(DT, \overline{X}_i \otimes_R DT) & \xrightarrow{a_i} & \mathbf{Q}_{DT}\mathbf{S}_T(X)_i & \xrightarrow{\lambda_i^2} & \operatorname{Hom}_R(T, A_{X_i}) \otimes_S T \oplus P_{A_{X_{i+1}}} \otimes_R DT \otimes_S T & \to 0
\end{array}$$

Then, since $\overline{X}_i \in \mathcal{B}$ implies that $\eta_{\overline{X}_i}^{DT}$ is an isomorphism, we obtain that ϕ_i is also an isomorphism from the above commutative diagram.

4.3.5 (i) *The upper row in the diagram (∗) is exact*

In fact, the pullback of $p_i : P_{A_{X_i}} \to A_{X_i}$ and $u_{X_i} : X_i \to A_{X_i}$ in 4.3.1 gives an exact sequence

$$0 \to \overline{X}_i \xrightarrow{(-q_i, \overline{u}_{X_i})} X_i \oplus P_{A_{X_i}} \xrightarrow{\binom{u_{X_i}}{p_i}} A_{X_i} \to 0,$$

since p_i is surjective. The direct sum of the above exact sequence and the trivial exact sequence $0 \to 0 \to P_{A_{X_{i+1}}} \xrightarrow{1} P_{A_{X_{i+1}}} \to 0$ gives us the exact sequence in the upper row in the diagram (∗).

4.3.5 (ii) *The bottom row in the diagram (∗)*

Note that we have the following exact sequence in 4.3.1

$$0 \to \mathbf{Q}_{DT}\mathbf{S}_T(X) \xrightarrow{\lambda} \mathrm{R}_{DT}(\mathfrak{L}^P_{A_X}) \xrightarrow{r(s^P)} \hat{\mathrm{H}}\mathrm{om}(DT, \mathbf{S}_T(X)) \to 0,$$

and that

$$\begin{aligned} &\mathrm{R}_{DT}(\mathfrak{L}^P_{A_X})_i = \mathrm{Hom}_S(DT, (\mathfrak{L}^P_{A_X})_i) \oplus (\mathfrak{L}^P_{A_X})_{i+1} \otimes_S T \\ &= \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT) \oplus (\mathrm{Hom}_R(T, A_{X_i}) \oplus P_{A_{X_{i+1}}} \otimes_R DT) \otimes_S T. \end{aligned}$$

So we have the following pullback diagram, for some homomorphisms a_i,

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}_S(DT, \overline{X}_i \otimes_R DT) & \xrightarrow{a_i} & \mathbf{Q}_{DT}\mathbf{S}_T(X)_i & \xrightarrow{\lambda_i^2} & (\mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT) \otimes_S T & \to & 0 \\ & & \| & & \downarrow \lambda_i^1 & & \downarrow -r_i^2 & & \\ 0 & \to & \mathrm{Hom}_R(DT, \overline{X}_i \otimes_R DT) & \xrightarrow{b_i} & \mathrm{Hom}_S(DT, \mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT) & \xrightarrow{r_i^1} & \mathrm{Hom}_S(DT, \mathbf{S}_T(X)_i) & \to & 0 \end{array}$$

where r_i^1, r_i^2 and λ_i^1, λ_i^2 are the components of the homomorphisms $r(s^P)_i$ and λ_i respectively, and $b_i = \mathrm{Hom}_S(DT, t_i)$ with

$$t_i = (-(q_i \otimes DT) \circ \theta_{l_i}, \overline{u}_{X_i} \otimes_R DT) : \overline{X}_i \otimes_R DT \to \mathrm{Hom}_R(T, A_{X_{i-1}}) \oplus P_{A_{X_i}} \otimes_R DT$$

is given in 4.3.1.

Note that r_i^1 is surjective as we indicate in 4.2.5 in the general case, so the upper row is exact. Thus we get the bottom exact sequence in the diagram (∗).

4.3.5 (iii) *The diagram (∗) is commutative*

At first, it is easy to see that the right part of the diagram (∗) is commutative from the construction of the morphism ϕ in 3.4.1.

As to the left part of the diagram (∗), we first show that the following equality of compositions

$$\eta^{DT}_{\overline{X}_i} \circ a_i \circ \lambda^1_i = (-q_i, \overline{u}_{X_i}, 0) \circ \phi_i \circ \lambda^1_i. \qquad (\dagger_1)$$

Indeed, we have that

$$\begin{aligned}
&\eta^{DT}_{\overline{X}_i} \circ a_i \circ \lambda^1_i \\
&= \eta^{DT}_{\overline{X}_i} \circ b_i && \text{(by the commutative diagram in 4.3.5 (ii))} \\
&= \eta^{DT}_{\overline{X}_i} \circ \mathrm{Hom}_S(DT, t_i) && \text{(since } b_i = \mathrm{Hom}_S(DT, t_i)\text{)} \\
&= \mathbf{\Gamma}^{DT}(1_{\overline{X}_i \otimes_R DT}) \circ \mathrm{Hom}_S(DT, t) \\
&= \mathbf{\Gamma}^{DT}(1_{\overline{X}_i \otimes_R DT} \circ t) && \text{(by the naturality of } \mathbf{\Gamma}^{DT}\text{)} \\
&= \mathbf{\Gamma}^{DT}(t) \\
&= \mathbf{\Gamma}^{DT}((-(q_i \otimes_R DT) \circ \theta_{l_i}, \overline{u}_{X_i} \otimes_R DT)) && \text{(since } t_i = (-(q_i \otimes DT) \circ \theta_{l_i}, \overline{u}_{X_i} \otimes_R DT)\text{)}
\end{aligned}$$

and we also have that

$$\begin{aligned}
&(-q_i, \overline{u}_{X_i}, 0) \circ \phi_i \circ \lambda^1_i \\
&= (-q_i, \overline{u}_{X_i}, 0) \circ \xi^a_i && \text{(since } \xi_i = (\xi^a_i, \xi^b_i) = \phi \circ \lambda\text{)} \\
&= (-q_i, \overline{u}_{X_i}, 0) \circ \begin{pmatrix} \xi^a_{11} & 0 \\ 0 & \xi^a_{22} \\ 0 & 0 \end{pmatrix} && \text{(by 4.3.3 (i))} \\
&= (-q_i \circ \xi^a_{11}, \overline{u}_{X_i} \circ \xi^a_{22}).
\end{aligned}$$

But $\xi^a_{11} = \mathbf{\Gamma}^{DT}(\theta_{l_i})$ and $\xi^a_{22} = \mathbf{\Gamma}^{DT}(1_{(P_{A_{X_i}} \otimes_R DT)})$ by the construction in 4.3.3 (i), we obtain that

$$q_i \circ \xi^a_{11} = q_i \circ \mathbf{\Gamma}^{DT}(\theta_{l_i}) = \mathbf{\Gamma}^{DT}(q_i \otimes_R DT \circ \theta_{l_i})$$

and that

$$\overline{u}_{X_i} \circ \xi^a_{22} = \overline{u}_{X_i} \circ \mathbf{\Gamma}^{DT}(1_{(P_{A_{X_i}} \otimes_R DT)}) = \mathbf{\Gamma}^{DT}(\overline{u}_{X_i} \otimes_R DT \circ 1_{(P_{A_{X_i}} \otimes_R DT)}) = \mathbf{\Gamma}^{DT}(\overline{u}_{X_i} \otimes_R DT)$$

Hence, we see that the equality $(\dagger_1)$ holds.

Since that $a_i \circ \lambda^2_i = 0$ and that

$$(-q_i, \overline{u}_{X_i}, 0) \circ \phi_i \circ \lambda^2_i = (-q_i, \overline{u}_{X_i}, 0) \circ \xi^b_i = 0,$$

we also get that

$$\eta^{DT}_{\overline{X}_i} \circ a_i \circ \lambda^2_i = (-q_i, \overline{u}_{X_i}, 0) \circ \phi_i \circ \lambda^2_i. \qquad (\dagger_2)$$

Now, from the property of the pullback in 4.3.5 (ii), we know that the two equalities $(\dagger_1)$ and $(\dagger_2)$ together imply that

$$\eta^{DT}_{\overline{X}_i} \circ a_i = (-q_i, \overline{u}_{X_i}, 0) \circ \phi_i.$$

Thus, the left part of the diagram is also commutative.

4.3.6 *The isomorphism $\phi : X \oplus \mathrm{L}_R(P^+_{A_X}) \to \mathbf{Q}_{DT}\mathbf{S}_T(X)$ is natural on X.*

For any $X, Y \in \mathcal{RC}(R)$ and $f \in \mathrm{Hom}_{\mathcal{RC}(R)}(X, Y)$, it is regular to show that the following diagram is commutative, for some natural homomorphisms $f \oplus \mathrm{L}_R(P^+_{A_f})$ and $\mathbf{Q}_{DT}\mathbf{S}_T(f)$.

$$\begin{array}{ccc} X\oplus \mathrm{L}_R(P^+_{A_X}) & \xrightarrow{\phi_X} & \mathbf{Q}_{DT}\mathbf{S}_T(X) \\ {\scriptstyle f\oplus \mathrm{L}_R(P^+_{A_f})}\downarrow & & \downarrow{\scriptstyle \mathbf{Q}_{DT}\mathbf{S}_T(f)} \\ Y\oplus \mathrm{L}_R(P^+_{A_Y}) & \xrightarrow{\phi_Y} & \mathbf{Q}_{DT}\mathbf{S}_T(X). \end{array}$$

Thus, the isomorphism ϕ is natural on X. This means that $\mathbf{Q}_{DT}\mathbf{S}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$ naturally.

4.4 The isomorphism $\mathbf{S}_T\mathbf{Q}_{DT} \simeq 1_{\underline{\mathcal{RC}}(S)}$

Dually to the proof of 4.3, one can show that $\mathbf{S}_T\mathbf{Q}_{DT} \simeq 1_{\underline{\mathcal{RC}}(S)}$ naturally.

Namely, for an object $Y \in \mathcal{RC}(S)$, one uses that $(\mathcal{G},\mathcal{K})$ is a complete hereditary cotorsion pair in mod$S$ to obtain exact sequences $0 \to K_{Y_i} \to G_{Y_i} \to Y_i \to 0$, for each i. Then take an injective S-module $I_{G_{Y_i}}$ and a monomorphism $j : G_{Y_i} \to I_{G_{Y_i}}$, one can show that there is a natural isomorphism $\mathbf{S}_T\mathbf{Q}_{DT}(Y) \to Y \oplus \mathrm{R}_{DS}(I^-_Y)$, where $I^-_Y = \{I_{G_{Y_{i-1}}}\}$ and $\mathrm{R}_{DS}(I^-_Y)$ is a projective object in $\mathcal{RC}(R)$ by Remark (2) in 4.2.1. And then one gets that $\mathbf{S}_T\mathbf{Q}_{DT} \simeq 1_{\underline{\mathcal{RC}}(S)}$ naturally.

4.5 The last proof of main theorem

Recall that [1] is an automorphism of repetitive categories, where $(X[1])_i = X_{i-1}$ for an object in a repetitive category.

Define $\mathbf{F}_T := [-1]\mathbf{S}_T : \underline{\mathcal{RC}}(R) \to \underline{\mathcal{RC}}(S)$ and $\mathbf{G}_T := \mathbf{Q}_{DT}[1] : \underline{\mathcal{RC}}(S) \to \underline{\mathcal{RC}}(R)$. Then we have that $\mathbf{F}_T\mathbf{G}_T \simeq 1_{\underline{\mathcal{RC}}(S)}$ naturally and that $\mathbf{G}_T\mathbf{F}_T \simeq 1_{\underline{\mathcal{RC}}(R)}$ naturally. So that $\mathbf{F}_T$ and $\mathbf{G}_T$ gives a repetitive equivalence between R and S.

It is easy to check that $\mathbf{F}_T|_{\mathcal{A}} \simeq \mathrm{Hom}_R(T,-)$ and that $\mathbf{G}_T|_{\mathcal{G}} \simeq -\otimes_S T$ from the definitions of two functors. Now the proof of the theorem is completed.

4.6 The proof of the proposition in the introduction

Assume that the equivalence is given by the functor $F : \underline{\mathcal{RC}}(R) \to \underline{\mathcal{RC}}(S)$. By assumptions, F restricts to an equivalence $\mathcal{A} \to \mathcal{G}$. Note that $\mathcal{G}$ is resolving and $S \in \mathcal{G}$. Let $T = F^{-1}(S)$. Then $T \in \mathcal{A}$. By the triangle equivalence, we have that, for any $A \in \mathcal{A}$,

$$\mathrm{Ext}^i_R(T,A) \simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(R)}(T,\Sigma^i A) \simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(S)}(S,\Sigma^i F(A)) \simeq \mathrm{Ext}^i_S(S,F(A)),$$

where Σ is the translator funtor in repetitive categories. In particular, we obtain that $\mathrm{Hom}_R(T,A) \simeq \mathrm{Hom}_S(S,F(A)) \simeq F(A)$ and that $\mathrm{Ext}^i_R(T,A) = 0$ for all $i > 0$. It follows that $S \simeq \mathrm{End}(T_R)$ and $\mathrm{Ext}^i_R(T,T) = 0$ for all $i > 0$. Note that $\mathcal{A}$ is coresolving and $DR \in \mathcal{A}$, so we also have that $F(DR) \simeq \mathrm{Hom}_R(T,DR) \simeq DT$. Thus, we get that

$$\begin{aligned}\mathrm{Ext}^i_S({}_ST,{}_ST) &\simeq \mathrm{Ext}^i_S(DT,DT) \simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(S)}(DT,\Sigma^i DT)\\ &\simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(S)}(F(DR),\Sigma^i F(DR)) \simeq \mathrm{Hom}_{\underline{\mathcal{RC}}(R)}(DR,\Sigma^i DR) \simeq \mathrm{Ext}^i_R(DR,DR).\end{aligned}$$

It follows that $\mathrm{End}({}_ST) \simeq R$ and that $\mathrm{Ext}^i_S({}_ST,{}_ST) = 0$ for all $i > 0$. Thus, T is a Wakamatsu-tilting module.

By assumption, $F|_{\mathcal{A}} \simeq \mathrm{Hom}_R(T,-)$ gives the equivalence $\mathcal{A} \to \mathcal{G}$. It follows that $F^{-1}|_{\mathcal{G}} \simeq - \otimes_S T$ by the unique of the adjoint. Note that $\mathrm{Hom}_R(T,-)$ and $- \otimes_S T$ are exact functors respectively in $\mathcal{A}$ and $\mathcal{G}$, since F is a triangle functor. As $\mathcal{A}$ is coresolving, for any $A \in \mathcal{A}$, the exact sequence $0 \to A \to I \to A' \to 0$ with $I \in \mathrm{inj}R$ is a sequence in $\mathcal{A}$. Applying the exact functor $\mathrm{Hom}_R(T,-)$, we obtain that $\mathrm{Ext}^1_R(T, A) = 0$. It follows that $T \in \mathcal{A} \bigcap \mathrm{KerExt}^1_R(-, \mathcal{A})$, i.e, T is Ext-projective in $\mathcal{A}$. Dually, we have also that $\mathrm{Tor}^S_1(X, T) = 0$ for any $X \in \mathcal{G}$. In particular, for any $A \in \mathcal{A}$, suppose that $A = X \otimes_S T$ for some $X \in \mathcal{G}$ and take an exact sequence $0 \to X' \to P \to X \to 0$ with $P \in \mathrm{proj}R$, then the sequence is in $\mathcal{G}$ since $\mathcal{G}$ is resolving, and hence there is an induced exact sequence $0 \to X' \otimes_S T \to P \otimes_S T \to X \otimes_S T \to 0$, since $- \otimes_S T$ is exact in $\mathcal{G}$. The last sequence gives an exact sequence $0 \to A'' \to T_A \to A \to 0$ with $T_A = P \otimes_S T \in \mathrm{add}_R T$ and $A'' = X' \to P \in \mathcal{A}$. It follows that T is an Ext-projective generator in $\mathcal{A}$. Now applying Corollary 3.2.6, we conclude that T is a good Wakamatsu-tilting module.

5 Examples

5.1 Tilting modules and cotilting modules

Let R ba an artin algebra. Recall that an R-module T is tilting provided the following three conditions are satisfied:

(1) The projective dimension of T is finite;

(2) $\mathrm{Ext}^i_R(T, T) = 0$ for all $i > 0$;

(3) There is an exact sequence $0 \to R \to T_0 \to \cdots \to T_n \to 0$ for some integer n, where each $T_i \in \mathrm{add}_R T$.

Dually, an R-module T is cotilting provided the following three conditions are satisfied:

(1) The injective dimension of T is finite;

(2) $\mathrm{Ext}^i_R(T, T) = 0$ for all $i > 0$;

(3) There is an exact sequence $0 \to T_n \to \cdots \to T_0 \to DR \to 0$ for some integer n, where each $T_i \in \mathrm{add}_R T$.

It is known that an R-module T is a tilting module if and only if DT is a cotilting left R-module if and only if DT is a cotilting S-module. Note also that both tilting modules and cotilting modules are Wakamatsu-tilting modules.

We need the following well-known results on tilting modules and cotilting modules.

Proposition (1) *If T is a tilting module, then the cortorsion pair* $(\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}), {}_T\mathcal{X})$ *is complete.*

(2) *If T is a cotilting module, then the cortorsion pair* $(\mathcal{X}_T, \mathrm{KerExt}^1_R(\mathcal{X}_T, -))$ *is complete.*

Proof. (2) follows from [3, Section 5], and (1) is just the dual of (2). □

5.1.1 *Tilting modules are good Wakamatsu-tilting*

Assume T_R is a tilting module of finite projective dimension. Let $S = \text{End}(T_R)$. Then ${}_ST_R$ is a good Wakamatsu-tilting module. Hence there is an equivalence between repetitive categories $\underline{\mathcal{RC}}(R)$ and $\underline{\mathcal{RC}}(S)$.

Indeed, if ${}_ST_R$ is a tilting module of finite projective dimension, then T is Wakamatsu-tilting and ${}_RDT_S$ is a cotilting module of finite injective dimension. By Proposition 3.1, we obtain that bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the compete hereditary cotorsion pair $(\text{KerExt}^1_R(-, {}_T\mathcal{X}), {}_T\mathcal{X})$ in mod$R$ and the compete hereditary cotorsion pair $(\mathcal{X}_{DT}, \text{KerExt}^1_S(\mathcal{X}_{DT}, -))$ in mod$S$. It follows from the definition that ${}_ST_R$ is a good Wakamatsu-tilting bimodule.

5.1.2 *Cotilting modules are good Wakamatsu-tilting*

Assume now T_R is a cotilting module of finite injective dimension with $S = \text{End}(T_R)$. Then ${}_ST_R$ is also a good Wakamatsu-tilting module. Hence there is an equivalence between repetitive categories $\underline{\mathcal{RC}}(R)$ and $\underline{\mathcal{RC}}(S)$.

Indeed, dually to 5.1.2, if ${}_ST_R$ is a cotilting module of finite injective dimension, then ${}_RDT_S$ is a tilting module of finite projective dimension. By Proposition 3.1 again, we obtain that bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the compete hereditary cotorsion pair $(\mathcal{X}_T, \text{KerExt}^1_S(\mathcal{X}_T, -))$ in mod$R$ and the compete hereditary cotorsion pair $(\text{KerExt}^1_R(-, {}_{DT}\mathcal{X}), {}_{DT}\mathcal{X})$ in mod$S$. It follows from the definition that ${}_ST_R$ is also a good Wakamatsu-tilting bimodule.

5.2 Wakamatsu-tilting modules of finite type

5.2.1 We say that a Wakamatsu-tilting R-module T is of finite type provided that either the subcategory $\text{KerExt}^1_R(-, {}_T\mathcal{X})$ or the subcategory $\text{KerExt}^1_R(\mathcal{X}_T, -)$ is of finite representation type. In particular, if R is an algebra of finite representation type, then each subcategory of mod$R$ is of finite representation type, and hence every Wakamatsu-tilting module in mod$R$ is of finite type.

We note that, if T is a Wakamatsu-tilting R-module of finite type with $S = \text{End}(T_R)$, then DT is a Wakamatsu-tilting S-module of finite type. This is just followed from the equivalences in Proposition 3.1.

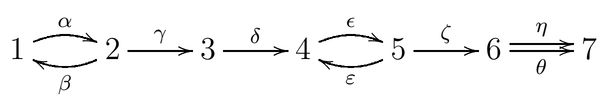
Proposition *A Wakamatsu-tilting module of finite type is always a good Wakamatsu-tilting module. In particular, every Wakamatsu-tilting module over an algebra of finite representation type is good.*

Proof. Let T be a Wakamatsu-tilting R-module of finite type with $S = \text{End}(T_R)$. Assume first that the subcategory $\text{KerExt}^1_R(\mathcal{X}_T, -)$ is of finite representation type. Then the hereditary cotorsion pair $(\mathcal{X}_T, \mathrm{KerExt}^1_S(\mathcal{X}_T, -))$ in mod$R$ is complete. Moreover, by the equivalence in Proposition 3.1 (3), the subcategory $\mathrm{KerExt}^1_R(-, {}_{DT}\mathcal{X})$ is also of finite representation type. Thus, the hereditary cotorsion pair $(\mathrm{KerExt}^1_R(-, {}_{DT}\mathcal{X}), {}_{DT}\mathcal{X})$ in mod$S$ is also complete. It follows that bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the compete hereditary cotorsion pair $(\mathcal{X}_T, \mathrm{KerExt}^1_S(\mathcal{X}_T, -))$ in mod$R$ and the compete hereditary cotorsion pair $(\mathrm{KerExt}^1_R(-, {}_{DT}\mathcal{X}), {}_{DT}\mathcal{X})$ in mod$S$. Similarly, in case that $\mathrm{KerExt}^1_R(-, {}_T\mathcal{X})$ is of finite representation type, we have that bimodules ${}_ST_R$ and ${}_RDT_S$ represent a cotorsion pair counter equivalence between the compete hereditary cotorsion pair $(\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}), {}_T\mathcal{X})$ in mod$R$ and the compete hereditary cotorsion pair $(\mathcal{X}_{DT}, \mathrm{KerExt}^1_S(\mathcal{X}_{DT}, -))$ in mod$S$. Altogether, we see that T is a good Wakamatsu-tilting module in either case. □

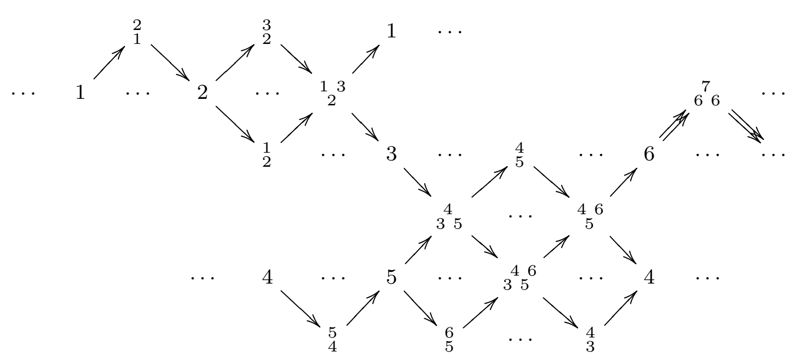
5.2.2 Two trivial examples of Wakamatsu-tilting modules of finite type over an algebra R is the module R and the module DR. In the first case, the subcategory $\mathrm{KerExt}^1_R(-, {}_T\mathcal{X}) =$ proj$R$ is of finite representation type, while the subcategory $\mathrm{KerExt}^1_R(\mathcal{X}_T, -) =$ inj$R$ is of finite representation type in the second case.

The following is an example of Wakamatsu-tilting modules of finite type over an algebra of infinite representation type.

Example Let R be the bounded quiver algebra given by the following quiver over a field with the relation given by $\mathrm{rad}^2 R = 0$.



The following is the AR-quiver of the algebra.



The algebra is of infinite representation type. Over this algebra, we have a Wakamatsu-tilting module of finite type (and hence, a good Wakamatsu-tilting module)

$$T = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix} \oplus \begin{smallmatrix} 1\;3 \\ 2 \end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix} 4 \\ 3\;5 \end{smallmatrix} \oplus \begin{smallmatrix} 5 \\ 4 \end{smallmatrix} \oplus \begin{smallmatrix} 6 \\ 5 \end{smallmatrix} \oplus \begin{smallmatrix} 7 \\ 6\;6 \end{smallmatrix}.$$

Indeed, one can check that the subcategory $\mathrm{KerExt}^1_R(-, {}_T\mathcal{X})$ is of finite representation type, while the subcategory ${}_T\mathcal{X}$ is of infinite representation type.

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