Classification of modules over laterally complete regular algebras

Vladimir I. Chilin, Jasurbek A. Karimov

Abstract

Let $\mathcal{A}$ be a laterally complete commutative regular algebra and X be a laterally complete $\mathcal{A}$-module. In this paper we introduce a notion of passport $\Gamma(X)$ for X, which consist of uniquely defined partition of unity in the Boolean algebra of idempotents in $\mathcal{A}$ and the set of pairwise different cardinal numbers. It is proved that $\mathcal{A}$-modules X and Y are isomorphic if and only if $\Gamma(X) = \Gamma(Y)$.

Key words: Commutative regular algebra, Homogeneous module, Finite dimensional module

Mathematics Subject Classification (2000): MSC 13C05, MSC 16D70

1 Introduction

J. Kaplansky [4] introduced a class of AW^*-algebras to describe C^*-algebras, which is close to von Neumann algebras by their algebraic and order structure. The class of AW^*-algebras became a subject of many researches in the operator theory (see review in [1]). One of the important results in this direction is the realization of an arbitrary AW^*-algebra M of type I as a $*$-algebra of all linear bounded operators, which act in a special Banach module over the center $Z(M)$ of the algebra M [5]. The Banach $Z(M)$-valued norm in this module is generated by the scalar product with values in the commutative AW^*-algebra $Z(M)$. Later, these modules were called Kaplansky-Hilbert modules (KHM). Detailed exposition of many useful properties of KHM is given, for example, in ([9], 7.4). One of the important properties is a representation of an arbitrary Kaplansky-Hilbert module as a direct sum of homogeneous KHM ([6], [9], 7.4.7).

Development of the noncommutative integration theory stimulated an interest to the different classes of algebras of unbounded operators, in particular, to the $*$-algebras $LS(M)$ of locally measurable operators, affiliated with von Neumann algebras or AW^*-algebras M. If M

is a von Neumann algebra, then the center $Z(LS(M))$ in the algebra $LS(M)$ identifies with the algebra $L^0(\Omega, \Sigma, \mu)$ of all classes of equal almost everywhere measurable complex functions, defined on some measurable space (Ω, Σ, μ) with a complete locally finite measure μ ([11], 2.1, 2.2). If M is an AW^*-algebra, then $Z(LS(M))$ is an extended f-algebra $C_\infty(Q)$, where Q is the Stone compact coresponding to the Boolean algebra of central projectors in M [1]. The problem (like the one in the work of J. Kaplansky [5] for AW^*-algebras) on possibility of realization of $*$-algebras $LS(M)$, in the case, when M has the type I, as $*$-algebras of linear $L^0(\Omega, \Sigma, \mu)$-bounded (respectively, $C_\infty(Q)$-bounded) operators, which act in corresponding KHM over the $L^0(\Omega, \Sigma, \mu)$ or over the $C_\infty(Q)$ naturally arises. In order to solve this problem it is necessary to construct corresponding theory of KHM over the algebras $L^0(\Omega, \Sigma, \mu)$ and $C_\infty(Q)$. In a particular case of KHM over the algebras $L^0(\Omega, \Sigma, \mu)$ this problem is solved in [7], where the decomposition of KHM over $L^0(\Omega, \Sigma, \mu)$ as a direct sum of homogeneous KHM is given. Similar decomposition as a direct sum of strictly γ-homogeneous modules is given in the paper [2] for arbitrary regular laterally complete modules over the algebra $C_\infty(Q)$ (the definitions see in the Section 3 below).

The algebra $C_\infty(Q)$ is an example of a commutative unital regular algebra over the field of real numbers. In this algebra the following property of lateral completeness holds: for any set $\{a_i\}_{i\in I}$ of pairwise disjoint elements in $C_\infty(Q)$ there exists an element $a \in C_\infty(Q)$ such that $as(a_i) = a_i$ for all $i \in I$, where $s(a_i)$ is a support of the element a_i (the definitions see in the Section 2 below). This property of $C_\infty(Q)$ plays a crucial role in classification of regular laterally complete $C_\infty(Q)$-mpdules [2]. Thereby, it is natural to consider the class of laterally complete commutative unital regular algebras $\mathcal{A}$ over arbitrary fields and to obtain variants of structure theorems for modules over such algebras. Current work is devoted to solving this problem. For every faithful regular laterally complete $\mathcal{A}$-module X the concept of passport $\Gamma(X)$, which consist of the uniquely defined partition of unity in the Boolean algebra of idempotents in $\mathcal{A}$ and the set of pairwise different cardinal numbers is constructed. It is proved, that the equality of passports $\Gamma(X)$ and $\Gamma(Y)$ is necessary and sufficient condition for isomorphism of $\mathcal{A}$-modules X and Y.

2 Laterally complete commutative regular algebras

Let $\mathcal{A}$ be a commutative algebra over the field K with the unity $\mathbf{1}$ and $\nabla = \{e \in \mathcal{A} : e^2 = e\}$ be a set of all idempotents in $\mathcal{A}$. For all $e, f \in \nabla$ we write $e \leq f$ if $ef = e$. It is well known (see, for example [10, Prop. 1.6]) that this binary relation is partial order in ∇ and ∇ is a Boolean algebra with respect to this order. Moreover, we have the following equalities: $e \vee f = e + f - ef$, $e \wedge f = ef$, $Ce = \mathbf{1} - e$ with respect to the lattice operations and the complement Ce in ∇.

The commutative unital algebra $\mathcal{A}$ is called regular if the following equivalent conditions hold [12, §2, item 4]:

1. For any $a \in \mathcal{A}$ there exists $b \in \mathcal{A}$ such that $a = a^2 b$;
2. For any $a \in \mathcal{A}$ there exists $e \in \nabla$ such that $a\mathcal{A} = e\mathcal{A}$.

A regular algebra $\mathcal{A}$ is a regular semigroup with respect to the multiplication operation [3, Ch. I, §1.9]. In this case all idempotents in $\mathcal{A}$ commute pairwisely. Therefore, $\mathcal{A}$ is a commutative inverse semigroup, i.e. for any $a \in \mathcal{A}$ there exists an unique element $i(a) \in \mathcal{A}$, which is an unique solution of the system: $a^2 x = a$, $ax^2 = x$ [3, Ch. I, §1.9]. The element $i(a)$ is called an inversion of the element a. Obviously, $ai(a) \in \nabla$ for any $a \in \mathcal{A}$. In this case the map $i : \mathcal{A} \to \mathcal{A}$ is a bijection and an automorphism (by multiplication) in semigroup $\mathcal{A}$. Moreover, $i(i(a)) = a$ and $i(g) = g$ for all $a \in \mathcal{A}$, $g \in \nabla$.

Let $\mathcal{A}$ be a commutative unital regular algebra and ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$. Idempotent $s(a) \in \nabla$ is called the support of an element $a \in \mathcal{A}$ if $s(a)a = a$ and $ga = a$, $g \in \nabla$ imply $s(a) \leq g$. It is clear that $s(a) = ai(a) = s(i(a))$. In particular, $s(e) = ei(e) = e$ for any $e \in \nabla$.

It is easy to show that supports of elements in a commutative regular unital algebra $\mathcal{A}$ satisfy the following properties:

Proposition 2.1. *Let $a, b \in \mathcal{A}$, then*

(i). $s(ab) = s(a)s(b)$, in particular, $ab = 0 \Leftrightarrow s(a)s(b) = 0$;

(ii). If $ab = 0$, then $i(a + b) = i(a) + i(b)$ and $s(a + b) = s(a) + s(b)$.

Two elements a and b in a commutative unital regular algebra $\mathcal{A}$ are called *disjoint elements*, if $ab = 0$, which equivalent to the equality $s(a)s(b) = 0$ (see Proposition 2.1 (i)). If the Boolean algebra ∇ of all idempotents in $\mathcal{A}$ is complete, $a \in \mathcal{A}$ and $r(a) = \sup\{e \in \nabla : ae = 0\}$, then

$$s(a)r(a) = s(a) \wedge r(a) = s(a) \wedge (\sup\{e : ae = 0\}) =$$
$$= \sup\{s(a) \wedge e : ae = 0\} = \sup\{s(a)e : ae = 0\} = 0.$$

Hence $s(a) \leq \mathbf{1} - r(a)$. If $q = (\mathbf{1} - r(a) - s(a))$, then $aq = as(a)q = 0$, thus $q \leq r(a)$. This yields that $q = 0$, i.e. $s(a) = \mathbf{1} - r(a)$. This implies the following

Proposition 2.2. *Let $\mathcal{A}$ be a commutative unital regular algebra and let ∇ be complete Boolean algebra of idempotents in $\mathcal{A}$. If $\{e_i\}_{i \in I}$ is a partition of unity in ∇, $a, b \in \mathcal{A}$ and $ae_i = be_i$ for all $i \in I$, then $a = b$.*

Proof. Since $(a-b)e_i = 0$ for any $i \in I$, then $\mathbf{1} = \sup\limits_{i\in I} e_i \leq r(a-b)$, i.e. $r(a-b) = \mathbf{1}$. Hence, $s(a-b) = 0$, i.e. $a = b$. □

Commutative unital regular algebra $\mathcal{A}$ is called *laterally complete* (*l-complete*) if the Boolean algebra of its idempotents is complete and for any set $\{a_i\}_{i\in I}$ of pairwise disjoint elements in $\mathcal{A}$ there exists an element $a \in \mathcal{A}$ such that $as(a_i) = a_i$ for all $i \in I$. The element $a \in \mathcal{A}$ such that $as(a_i) = a_i$, $i \in I$, in general, is not uniquely determined. However, by Proposition 2.2, it follows that the element a is unique in the case, when $\sup\limits_{i\in I} s(a_i) = \mathbf{1}$. In general case, due to the equality $as(a_i) = a_i = bs(a_i)$ for all $i \in I$ and $a, b \in \mathcal{A}$, it follows that $a \sup_{i\in I} s(a_i) = b \sup_{i\in I} s(a_i)$.

Let us give examples of l-complete and not l-complete commutative regular algebras. Let Δ be an arbitrary set and K^Δ be a Cartesian product of Δ copies of the field K, i.e. the set of all K-valued functions on Δ. The set K^Δ is a commutative unital regular algebra with respect to pointwise algebraic operations, moreover, the Boolean algebra ∇ of all idempotents in K^Δ is an isomorphic atomic Boolean algebra of all subsets in Δ. In particular ∇ is complete Boolean algebra. If $\{a_j = (\alpha_q^{(j)})_{q\in\Delta}, j \in J\}$ is a family of pairwise disjoint elements in K^Δ, then setting $\Delta_j = \{q \in \Delta : \alpha_q^{(j)} \neq 0\}$, $j \in J$ and $a = (\alpha_q)_{q\in\Delta} \in K^\Delta$, where $\alpha_q = \alpha_q^{(j)}$ for any $q \in \Delta_j$, $j \in J$, and $\alpha_q = 0$ for $q \in \Delta \setminus \bigcup\limits_{j\in J} \Delta_j$, we obtain that $as(a_j) = a_j$ for all $j \in J$. Hence, K^Δ is a l-complete algebra.

Now let $\mathcal{A}$ be an arbitrary commutative unital regular algebra over the field K and ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$. An element $a \in \mathcal{A}$ is called a *step element* in $\mathcal{A}$ if it has the following form $a = \sum_{k=1}^{n} \lambda_k e_k$, here $\lambda_k \in K$, $e_k \in \nabla$, $k = 1, \ldots, n$. The set $K(\nabla)$ of all step elements is the smallest subalgebra in $\mathcal{A}$, which contains ∇. Any nonzero element $a = \sum_{k=1}^{n} \lambda_k e_k$ in $K(\nabla)$ can be represented as $a = \sum_{l=1}^{m} \alpha_l g_l$, here $g_l \in \nabla$, $g_l g_k = 0$ when $l \neq k$, $0 \neq \alpha_k \in K$, $l, k = 1, \ldots, m$. Setting $b = \sum_{l=1}^{m} \alpha_l^{-1} g_l \in K(\nabla)$, we obtain $a^2 b = a$. Hence, $K(\nabla)$ is a regular subalgebra in $\mathcal{A}$. Since $\nabla \subset K(\nabla)$, the Boolean algebra of idempotents in $K(\nabla)$ coincides with ∇. Assume that $\operatorname{card}(K) = \infty$ and $\operatorname{card}(\nabla) = \infty$. We choose a countable set $K_0 = \{\lambda_n\}_{n=1}^\infty$ of pairwise different nonzero elements in K and a countable set $\{e_n\}_{n=1}^\infty$ of nonzero pairwise disjoint elements in ∇. Let us consider a set $\{\lambda_n e_n\}_{n=1}^\infty$ of pairwise disjoint elements in $K(\nabla)$. Assume that there exists $b = \sum_{l=1}^{m} \alpha_l g_l \in K(\nabla)$, $0 \neq \alpha_l \in K$, $g_l \in \nabla$, $g_l g_k = 0$ and $l \neq k$, $l, k = 1, \ldots, m$, such that $be_n = bs(\lambda_n e_n) = \lambda_n e_n$. In this case for any positive integer n there exists natural number $l(n)$, such that $\alpha_{l(n)} g_{l(n)} e_n = \lambda_n g_{l(n)} e_n \neq 0$, i.e. $\alpha_{l(n)} = \lambda_n$. This implies that the set $\{\lambda_n\}_{n=1}^\infty$ is finite, which is not true. Hence, the commutative unital regular algebra $K(\nabla)$ is not l-complete.

Let ∇ be complete Boolean algebra and let $Q(\nabla)$ be a Stone compact corresponding to ∇.

An algebra $C_\infty(Q(\nabla))$ of all continuous functions $a : Q(\nabla) \to [-\infty, +\infty]$, taking the values $\pm\infty$ only on nowhere dense sets in $Q(\nabla)$ [9, 1.4.2], is an important example of a l-complete commutative regular algebra.

An element $e \in C_\infty(Q(\nabla))$ is an idempotent if and only if $e(t) = \chi_V(t)$, $t \in Q(\nabla)$, for some clopen set $V \subset Q(\nabla)$, where

$$\chi_V(t) = \begin{cases} 1, & t \in V; \\ 0, & t \notin V, \end{cases}$$

i.e. $\chi_V(t)$ is a characteristic function of the set V. In particular, the Boolean algebra ∇ can be identified with the Boolean algebra of all idempotents in algebra $C_\infty(Q(\nabla))$.

If $a \in C_\infty(Q(\nabla))$, then $G(a) = \{t \in Q(\nabla) : 0 < |a(t)| < +\infty\}$ is open set in the Stone compact set $Q(\nabla)$. Hence, the closure $V(a) = \overline{G(a)}$ in $Q(\nabla)$ of the set $G(a)$ is an clopen set, i.e. $\chi_{V(a)}$ is an idempotent in the algebra $C_\infty(Q(\nabla))$. We consider a continuous function $b(t)$, given on the dense open set $G(a) \cup (Q(\nabla) \setminus V(a))$ and defines by the following equation

$$b(t) = \begin{cases} \frac{1}{a(t)}, & t \in G(a), \\ 0, & t \in Q(\nabla) \setminus V(a). \end{cases}$$

This function uniquely extends to a continuous function defined on $Q(\nabla)$ with values in $[-\infty, +\infty]$ [14, Ch.5, §2] (we also denote this extension by $b(t)$). Since $ab = \chi_{V(a)}$, then $a^2b = a$ and $s(a) = \chi_{V(a)}$. Hence, $C_\infty(Q(\nabla))$ is a commutative unital regular algebra over the field of real numbers $\mathbf{R}$. In this case, the Boolean algebra of all idempotents in $C_\infty(Q(\nabla))$ is complete.

It is known that (see, for example [9, 1.4.2]) $C_\infty(Q(\nabla))$ is an extended complete vector lattice. In particular, for any set $\{a_j\}_{j\in J}$ of pairwise disjoint positive elements in $C_\infty(Q(\nabla))$ there exists the least upper bound $a = \sup_{j\in J} a_j$ and $as(a_j) = a_j$ for all $j \in J$. It follows that the commutative regular algebra $C_\infty(Q(\nabla))$ is laterally complete.

In the case, when ∇ is a complete atomic Boolean algebra and Δ is the set of all atoms in ∇, then $C_\infty(Q(\nabla))$ is isomorphic to the algebra $\mathbf{R}^\Delta$.

The following examples of laterally complete commutative regular algebras are variants of algebras $C_\infty(Q(\nabla))$ for any topological fields, in particular, for the field $\mathbf{Q}_p$ of p-adic numbers.

Let K be an arbitrary field and t be the Hausdorff topology on K. If operations $\alpha \to (-\alpha)$, $\alpha \to \alpha^{-1}$ and operations $(\alpha, \beta) \to \alpha + \beta$, $(\alpha, \beta) \to \alpha\beta$, $\alpha, \beta \in K$, are continuous with respect to this topology, we say that (K, t) is a *topological field* (see, for example, [13, Ch.20, §165]).

Let (K, t) be a topological field, (X, τ) be any topological space and $\nabla(X)$ be a Boolean algebra of all clopen subsets in (X, τ). A map $\varphi : (X, \tau) \to (K, t)$ is called *almost continuous* if there exists a dense open set U in (X, τ) such that the restriction $\varphi|_U : U \to (K, t)$ of the map φ on the subset U is continuous in U. The set of all almost continuous maps from (X, τ) to (K, t) we denote by $AC(X, K)$.

We define pointwise algebraic operations in $AC(X, K)$ by

$$(\varphi + \psi)(t) = \varphi(t) + \psi(t);$$

$$(\alpha\varphi)(t) = \alpha\varphi(t);$$

$$(\varphi \cdot \psi)(t) = \varphi(t)\psi(t)$$

for all $\varphi, \psi \in AC(X, K)$, $\alpha \in K$, $t \in X$.

Since an intersection of two dense open sets in (X, τ) is a dense open set in (X, τ), then $\varphi + \psi$, $\varphi \cdot \psi \in AC(X, K)$ for any $\varphi, \psi \in AC(X, K)$. Obviously, $\alpha\varphi \in AC(X, K)$ for all $\varphi \in AC(X, K)$, $\alpha \in K$. It can be easily checked that $AC(X, K)$ is a commutative algebra over K with the unit element $\mathbf{1}(t) = 1_K$ for all $t \in X$, where 1_K is the unit element of K. In this case, the algebra $C(X, K)$ of all continuous maps from (X, τ) to (K, t) is a subalgebra in $AC(X, K)$.

In the algebra $AC(X, K)$ consider the following ideal

$$I_0(X, K) = \{\varphi \in AC(X, K) : \text{ interior of preimage } \varphi^{-1}(0) \text{ is dense in } (X, \tau)\}.$$

By $C_\infty(X, K)$ denote the quotient algebra $AC(X, K)/I_0(X, K)$ and by

$$\pi : AC(X, K) \to AC(X, K)/I_0(X, K)$$

denote the corresponding canonical homomorphism.

Theorem 2.3. *The quotient algebra $C_\infty(X, K)$ is a commutative unital regular algebra over the field K. Moreover, if (X, τ) is a Stone compact set, then algebra $C_\infty(X, K)$ is laterally complete, and the Boolean algebra ∇ of all its idempotents is isomorphic to the Boolean algebra $\nabla(X)$.*

Proof. Since $AC(X, K)$ is a commutative unital algebra over K, then $C_\infty(X, K)$ is also a commutative unital algebra over K with unit element $\pi(\mathbf{1})$. Now we show that $C_\infty(X, K)$ is a regular algebra, i.e. for any $\varphi \in AC(X, K)$ there exists $\psi \in AC(X, K)$, such that $\pi^2(\varphi)\pi(\psi) = \pi(\varphi)$.

We fix an element $\varphi \in AC(X, K)$ and choose a dense open set $U \in \tau$, such that the restriction $\varphi|_U : U \to (K, t)$ is continuous. Since $K \setminus \{0\}$ is an open set in (K, t), then the set

$V = U \cap \varphi^{-1}(K \setminus \{0\})$ is open in (X, τ). Clearly, the set $W = X \setminus \overline{V}^{\tau}$ is also open in (X, τ), in this case $V \cup W$ is a dense open set in (X, τ).

We define a map $\psi : X \to K$, as follow: $\psi(x) = (\varphi(x))^{-1}$ if $x \in V$, and $\psi(x) = 0$ if $x \in X \setminus V$. It is clear that $\psi \in AC(X, K)$ and $\varphi^2\psi - \varphi \in I_0(X, K)$, i.e. $\pi^2(\varphi)\pi(\psi) = \pi(\varphi)$. Hence, the algebra $C_\infty(X, K)$ is regular.

For any clopen set $U \in \nabla(X)$ its characteristic function χ_U belongs to $AC(X, K)$, in this case, $\pi(\chi_U)^2 = \pi(\chi_U^2) = \pi(\chi_U)$, i.e. $\pi(\chi_U)$ is an idempotent in the algebra $C_\infty(X, K)$.

Assume that (X, τ) is a Stone compact and we show that for any idempotent $e \in C_\infty(X, K)$ there exists $U \in \nabla(X)$ such that $e = \pi(\chi_U)$.

If $e \in \nabla$, then $e = \pi(\varphi)$ for some $\varphi \in AC(X, K)$ and

$$\pi(\varphi) = e^2 = \pi(\varphi^2),$$

i.e. $(\varphi^2 - \varphi) \in I_0(X, K)$. Hence, there exists a dense open set V in X such that $\varphi^2(t) - \varphi(t) = 0$ for all $t \in V$. Denote by U a dense open set in X such that the restriction $\varphi|_U : U \to K$ is continuous. Put $U_0 = \varphi^{-1}(\{0\}) \cap (U \cap V)$, $U_1 = \varphi^{-1}(\{1_K\}) \cap (U \cap V)$. Since $U_0 \cap U_1 = \emptyset, U_0 \cup U_1 = U \cap V \in \tau$ and the sets U_0, U_1 are closed in $U \cap V$ with respect to the topology induced from (X, τ), it follows that $U_0, U_1 \in \tau$. Hence, the set $U_\varphi = \overline{U_1}$ belongs to the Boolean algebra $\nabla(X)$, besides, $U_\varphi \cap U_0 = \emptyset$.

Since $U_0 \cup U_1 = U \cap V$ is a dense open set in (X, τ) and $\varphi(t) = \chi_{U_\varphi}(t)$ for all $t \in U_0 \cup U_1$, it follows that $e = \pi(\varphi) = \pi(\chi_{U_\varphi})$. Thus, the mapping $\Phi : \nabla(X) \to \nabla$ defined by the equality $\Phi(U) = \pi(\chi_U)$, $U \in \nabla(X)$, is a surjection.

Moreover, for $U, V \in \nabla(X)$ the following equalities hold

$$\Phi(U \cap V) = \pi(\chi_{U \cap V}) = \pi(\chi_U \chi_V) = \pi(\chi_U)\pi(\chi_V) = \Phi(U)\Phi(V),$$

$$\Phi(X \setminus U) = \pi(\chi_{X \setminus U}) = \pi(\mathbf{1} - \chi_U) = \Phi(X) - \Phi(U).$$

Furthermore, the equality $\Phi(U) = \Phi(V)$ implies that the continuous mappings χ_U and χ_V coincide on a dense set in X. Therefore $\chi_U = \chi_V$, that is $U = V$.

Hence, Φ is an isomorphism from the Boolean algebra $\nabla(X)$ onto the Boolean algebra ∇ of all idempotents from $C_\infty(X, K)$, in particular, ∇ is a complete Boolean algebra.

Finally, to prove l-completeness of the algebra $C_\infty(X, K)$ we show that for any family $\{\pi(\varphi_i) : \varphi \in AC(X, K)\}_{i \in I}$ of nonzero pairwise disjoint elements in $C_\infty(X, K)$ there exists $\varphi \in AC(X, K)$ such that $\pi(\varphi)s(\pi(\varphi_i)) = \pi(\varphi_i)$ for all $i \in I$. For any $i \in I$ we choose a dense open set U_i such that the restriction $\varphi_i|_{U_i}$ is continuous and put $V_i = U_i \cap \varphi_i^{-1}(K \setminus \{0\})$, $i \in I$. It is not hard to prove that $s(\pi(\varphi_i)) = \Phi(\overline{V_i})$. In particular, $V_i \cap V_j = \emptyset$ when $i \neq j$,

$i, j \in I$. Define the mapping $\varphi : X \to K$, as follows $\varphi(t) = \varphi_i(t)$ if $t \in V_i$ and $\varphi(t) = 0$ if $t \in X \setminus \left(\bigcup\limits_{i \in I} V_i\right)$. Clearly, $\varphi \in AC(X, K)$ and $\pi(\varphi)s(\pi(\varphi_i)) = \pi(\varphi\chi_{\overline{V_i}}) = \pi(\varphi_i\chi_{\overline{V_i}}) = \pi(\varphi_i)$ for all $i \in I$.

$\square$

3 Laterally complete regular modules

Let $\mathcal{A}$ be a laterally complete commutative regular algebra and let ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$. Let X be a left $\mathcal{A}$-module with algebraic operations $x + y$ and ax, $x, y \in X$, $a \in \mathcal{A}$. Since the algebra $\mathcal{A}$ is commutative, then a left $\mathcal{A}$-module X becomes a right $\mathcal{A}$-module, if we put $xa := ax$, $x \in X$, $a \in \mathcal{A}$. Hence, we can assume, that X is a bimodule over $\mathcal{A}$, where the following equality $ax = xa$ holds for any $x \in X$, $a \in \mathcal{A}$. Next, an $\mathcal{A}$-bimodule X we shall call an $\mathcal{A}$-module.

An $\mathcal{A}$-module X is called faithful, if for any nonzero $e \in \nabla$ there exists $x \in X$ such that $ex \neq 0$. Clearly, for a faithful $\mathcal{A}$-module X the set $X_e := eX$ is a faithful $\mathcal{A}_e$-module for any $0 \neq e \in \nabla$, where $\mathcal{A}_e := e\mathcal{A}$.

An $\mathcal{A}$-module X is said to be a regular module, if for any $x \in \mathcal{A}$ the condition $ex = 0$ for all $e \in L \subset \nabla$ implies $(\sup L)x = 0$. In this case, for $x \in X$ the idempotent

$$s(x) = \mathbf{1} - \sup\{e \in \nabla : ex = 0\}$$

is called the support of an element x. In case, when $X = \mathcal{A}$, the notions of support of an element in an $\mathcal{A}$-module X and of support of an element in $\mathcal{A}$ coincide. If X is a regular $\mathcal{A}$-module, then X_e is also a regular $\mathcal{A}_e$-module for any nonzero $e \in \nabla$.

We need the following properties of supports of elements in a regular $\mathcal{A}$-module X.

Proposition 3.1. *Let X be a regular $\mathcal{A}$-module, $x, y \in X$, $a \in \mathcal{A}$. Then*

(i). $s(x)x = x$;

(ii). if $e \in \nabla$ and $ex = x$, then $e \geq s(x)$;

(iii). $s(ax) = s(a)s(x)$.

Proof. (i). If $r(x) = \sup\{e \in \nabla : ex = 0\}$, then $s(x) = \mathbf{1} - r(x)$ and $r(x)x = 0$. Hence, $x = (s(x) + r(x))x = s(x)x$.

(ii). As $ex = x$, then $(\mathbf{1} - e)x = 0$, and therefore $\mathbf{1} - e \leq r(x)$. Thus $e \geq \mathbf{1} - r(x) = s(x)$.

(iii). Since $(s(a)s(x)) \cdot (ax) = (s(a)a) \cdot (s(x)x) = ax$, then by (ii) we have $s(ax) \leq s(a)s(x)$. If $g = s(a)s(x) - s(ax) \neq 0$, then $ga \neq 0$, $g \leq s(a)$ and $gs(ax) = 0$. Hence $gax = 0$ and

$0 = i(ga)(gax) = (i(g)i(a)ga)x = (gi(a)a)x = gs(a)x = gx \neq 0$. This contradiction implies $g = 0$, i.e. $s(ax) = s(a)s(x)$. ☐

We say that a regular $\mathcal{A}$-module X is laterally complete (l-complete), if for any set $\{x_i\}_{i\in I} \subset X$ and for any partition $\{e_i\}_{i\in I}$ of unity of the Boolean algebra ∇ there exists $x \in X$ such that $e_i x = e_i x_i$ for all $i \in I$. In this case, the element x is called mixing of the set $\{x_i\}_{i\in I}$ with respect to the partition of unity $\{e_i\}_{i\in I}$ and denote by $\operatorname{mix}_{i\in I}(e_i x_i)$. Mixing $\operatorname{mix}_{i\in I}(e_i x_i)$ is defined uniquely, whereas the equalities $e_i x = e_i x_i = e_i y$, $x, y \in X$, $i \in I$, implies $e_i(x - y) = 0$ for all $i \in I$, and, by regularity of the $\mathcal{A}$-module X, we obtain $x = y$.

Let $\{x_i\}_{i\in I} \subset E \subset X$ and let $\{e_i\}_{i\in I}$ be a partition of unity in ∇. The set of all mixings $\operatorname{mix}_{i\in I}(e_i x_i)$ is called a cyclic hull of the set E in X and denotes by $\operatorname{mix}(E)$. Obviously, the inclusion $E \subset \operatorname{mix}(E)$ is always true. If $E = \operatorname{mix}(E)$, then E is called a cyclic set in X (compare with [8], 1.1.2).

Thus, a regular $\mathcal{A}$-module X is a l-complete $\mathcal{A}$-module if and only if X is a cyclic set. In particular, in any l-complete $\mathcal{A}$-module X its submodule X_e is also a l-complete $\mathcal{A}_e$-module for any nonzero idempotent e in $\mathcal{A}$.

We need the following properties of cyclic hulls of sets.

Proposition 3.2. *Let X be a l-complete $\mathcal{A}$-module and let E be a nonempty subset in X, $a \in \mathcal{A}$. Then*

(i). $\operatorname{mix}(\operatorname{mix}(E)) = \operatorname{mix}(E)$*;*

(ii). $\operatorname{mix}(aE) = a\operatorname{mix}(E)$*;*

(iii). If Y is an $\mathcal{A}$-submodule in X, then $\operatorname{mix}(Y)$ *is a l-complete $\mathcal{A}$-submodule in X;*

(iv). If U is an isomorphism from $\mathcal{A}$-module X onto $\mathcal{A}$-module Z, then Z is a l-complete $\mathcal{A}$-module and $\operatorname{mix}(U(E)) = U(\operatorname{mix}(E))$.

Proof. (i). It is sufficient to show that $\operatorname{mix}(\operatorname{mix}(E)) \subset \operatorname{mix}(E)$. If $x \in \operatorname{mix}(\operatorname{mix}(E))$, then $x = \operatorname{mix}_{i\in I}(e_i x_i)$, where $x_i \in \operatorname{mix}(E)$, $i \in I$. Since $x_i \in \operatorname{mix}(E)$, then $x_i = \operatorname{mix}_{j\in J(i)}(e_j^{(i)} x_j^{(i)})$, where $x_j^{(i)} \in E$, $j \in J(i)$ and $\{e_j^{(i)}\}_{j\in J(i)}$ is a partition of unity in the Boolean algebra ∇ for all $i \in I$. Fix $i \in I$ and put $g_j^{(i)} := e_i e_j^{(i)}$. It is clear that $\{g_j^{(i)}\}_{j\in J(i)}$ is a partition of the idempotent e_i. Hence, $\{g_j^{(i)}\}_{j\in J(i), i\in I}$ is a partition of unity $\mathbf{1}$. Besides,

$$g_j^{(i)} x = g_j^{(i)} e_i x = g_j^{(i)} e_i x_i = e_i e_j^{(i)} x_i = e_i e_j^{(i)} x_j^{(i)} = g_j^{(i)} x_j^{(i)}.$$

This yields that $x = \operatorname{mix}_{j\in J(i), i\in I}(g_j^{(i)} x_j^{(i)}) \in \operatorname{mix}(E)$.

(ii). If $x \in \operatorname{mix}(aE)$, then $x = \operatorname{mix}_{i\in I}(e_i a y_i)$, where $y_i \in E, i \in I$. Since X is a l-complete $\mathcal{A}$-module, then there exists $y = \operatorname{mix}_{i\in I}(e_i y_i) \in \operatorname{mix}(E)$ and $e_i x = a e_i y_i = e_i(ay)$ for all $i \in I$.

Hence, $e_i(x - ay) = 0$, and regularity of the $\mathcal{A}$-module X implies the equality $x = ay$. Thus, $\mathrm{mix}(aE) \subset a\mathrm{mix}(E)$.

Conversely, if $x \in a\mathrm{mix}(E)$, then $x = az$, where $z = \underset{i \in I}{\mathrm{mix}}(e_i z_i), z_i \in E, i \in I$. Since $az_i \in aE$ and $e_i x = e_i(az) = e_i a e_i z = e_i(az_i)$ for all $i \in I$, we have that $x = \underset{i \in I}{\mathrm{mix}}(e_i(az_i)) \in \mathrm{mix}(aE)$. Hence, $a\mathrm{mix}(E) \subset \mathrm{mix}(aE)$.

(iii). Let $x, y \in \mathrm{mix}(Y)$, $x = \underset{i \in I}{\mathrm{mix}}(e_i x_i)$, $y = \underset{j \in J}{\mathrm{mix}}(g_j y_j)$, where $x_i, y_j \in Y$, $i \in I$, $j \in J$, $\{e_i\}_{i \in I}$, $\{g_j\}_{j \in J}$ are partitions of unity in ∇. Clearly, that $p_{ij} = e_i g_j$, $i \in I$, $j \in J$, is also a partition of unity in ∇ and $p_{ij}(x + y) = p_{ij}(x_i + y_j)$, where $x_i + y_j \in Y$ for all $i \in I$, $j \in J$. This means that $(x + y) \in \mathrm{mix}(Y)$.

Since $aY \subset Y$, then by (ii) we have that $ax \in a\mathrm{mix}(Y) = \mathrm{mix}(aY) \subset \mathrm{mix}(Y)$. Hence, $\mathrm{mix}(Y)$ is an $\mathcal{A}$-submodule in X, and by regularity of the $\mathcal{A}$-module X, it is a regular $\mathcal{A}$-module. The equality $\mathrm{mix}(Y) = \mathrm{mix}(\mathrm{mix}Y)$ (see (i)) implies that $\mathrm{mix}Y$ is a l-complete $\mathcal{A}$-module.

(iv). If $U(x) = y \in Z$, $x \in X$, $\emptyset \neq L \subset \nabla$ and $ey = 0$ for all $e \in L$, then $U(ex) = eU(x) = ey = 0$. Since U is a bijection, then $ex = 0$ for any $e \in L$. By regularity of the $\mathcal{A}$-module X, we have that $(\sup L)x = 0$, and, therefore, $(\sup L)y = U((\sup L)x) = 0$. Hence, Z is a regular $\mathcal{A}$-module. In the same way we show that Z is a l-complete $\mathcal{A}$-module and the equality $\mathrm{mix}(U(E)) = U(\mathrm{mix}(E))$ holds.

□

Let ∇ be an arbitrary complete Boolean algebra. For any nonzero element $e \in \nabla$ we put $\nabla_e = \{q \in \nabla : q \leq e\}$. The set ∇_e is a Boolean algebra with the unity e with respect to partial order, induced from ∇.

We say that a set B in ∇ is a minorant subset for nonempty set $E \subset \nabla$, if for any nonzero $e \in E$ there exists nonzero $q \in B$ such that $q \leq e$. We need the following property of complete Boolean algebras.

Theorem 3.3. ([9], 1.1.6) *If ∇ is a complete Boolean algebra, e is a nonzero element in ∇ and B is a minorant subset for ∇_e, then there exists a disjoint subset $L \subset B$ such that* $\sup L = e$.

We say that a Boolean algebra ∇ has *a countable type* or is *σ-finite*, if any nonfinite family of nonzero pairwise disjoint elements in ∇ is a countable set. A complete Boolean algebra ∇ is called *multi-σ-finite*, if for any nonzero element $g \in \nabla$ there exists $0 \neq e \in \nabla$ such that $e \leq g$ and the Boolean algebra ∇_e has a countable type. By theorem 3.3, a multi-σ-finite Boolean algebra ∇ always has a partition $\{e_i\}_{i \in I}$ of unity $\mathbf{1}$ such that the Boolean algebra ∇_{e_i} has a countable type for all $i \in I$.

By theorem 3.3 we set the following useful properties of l-complete $\mathcal{A}$-modules.

Proposition 3.4. *Let X be an arbitrary l-complete $\mathcal{A}$-module and ∇ be a complete Boolean algebra of all idempotents in $\mathcal{A}$. Then*

(i). If X is a faithful $\mathcal{A}$-module, then there exists an element $x \in X$ such that $s(x) = \mathbf{1}$;

(ii). If Y is a l-complete $\mathcal{A}$-submodule in a regular $\mathcal{A}$-module X and for any nonzero $e \in \nabla$ there exists a nonzero $g_e \in \nabla$ such that $g_e \leq e$ and $g_eY = g_eX$, then $Y = X$.

Proof is in the same way as the proof of Proposition 2.4 in [2].

We need a representation of a faithful l-complete $\mathcal{A}$-module X as the Cartesian product of a faithful l-complete $\mathcal{A}_{e_i}$-modules family, where $\{e_i\}_{i\in I}$ is a partition of unity in the Boolean algebra ∇ of all idempotents in $\mathcal{A}$. In the Cartesian product

$$\prod_{i\in I} e_iX = \{\{y_i\}_{i\in I} : y_i \in e_iX\}$$

of $\mathcal{A}$-submodules e_iX we consider coordinate-wise algebraic operations. It is clear that $\prod\limits_{i\in I} e_iX$ is a faithful l-complete $\mathcal{A}$-module. We define a map $U : X \to \prod\limits_{i\in I} e_iX$ given by $U(x) = \{e_ix\}_{i\in I}$. Obviously, U is a homomorphizm from X onto $\prod\limits_{i\in I} e_iX$. If $U(x) = U(y)$, then $e_ix = e_iy$ for all $i \in I$, and by regularity of the $\mathcal{A}$-module X, it follows that $x = y$.

If $z = \{x_i\}_{i\in I} \in \prod\limits_{i\in I} e_iX$, where $x_i \in e_iX \subset X$, $i \in I$, then l-completeness of the $\mathcal{A}$-module X implies that there exists an element $x \in X$ such that $e_ix = e_ix_i = x_i$ for all $i \in I$. Hence, $U(x) = z$, i.e. U is a surjection.

Thus, the following proposition holds.

Proposition 3.5. *If X is a faithful l-complete $\mathcal{A}$-module, $\{e_i\}_{i\in I}$ is a partition of unity of the Boolean algebra ∇ of all idempotents in $\mathcal{A}$, then $\prod\limits_{i\in I} e_iX$ is also a faithful l-complete $\mathcal{A}$-module and U is an isomorphism from X onto $\prod\limits_{i\in I} e_iX$.*

4 Homogenous $\mathcal{A}$-modules

Let $\mathcal{A}$ be a laterally complete commutative regular algebra, let ∇ be a complete Boolean algebra of all idempotents in $\mathcal{A}$, let X be a faithful $\mathcal{A}$-module. The following $\mathcal{A}$-submodule in X is called $\mathcal{A}$-linear hull of a nonempty subset $Y \subset \mathcal{A}$

$$\mathrm{Lin}(Y, \mathcal{A}) = \left\{ \sum_{i=1}^{n} a_iy_i : a_i \in \mathcal{A}, y_i \in Y, i = 1, \ldots, n, n \in \mathcal{N} \right\},$$

where $\mathcal{N}$ is the set of all natural numbers. If X is a l-complete $\mathcal{A}$-module, then by proposition 3.2 (*iii*), $\mathrm{mix}(\mathrm{Lin}(Y, \mathcal{A}))$ is also a l-complete $\mathcal{A}$-submodule in X.

A set $\{x_i\}_{i\in I}$ in an $\mathcal{A}$-module X is called $\mathcal{A}$-*linearly independent*, if for any $a_1,\dots,a_n \in \mathcal{A}$, $x_{i_1},\dots,x_{i_n} \in \{x_i\}_{i\in I}$, $n \in \mathcal{N}$, the equality $\sum\limits_{k=1}^{n} a_k x_{i_k} = 0$ implies equalities $a_1 = \ldots = a_n = 0$.

Proposition 4.1. *If $Y = \{x_1,\dots,x_k\}$ is a finite $\mathcal{A}$-linearly independent subset in a l-complete $\mathcal{A}$-module X, then* $\mathrm{mix}(\mathrm{Lin}(Y,\mathcal{A})) = \mathrm{Lin}(Y,\mathcal{A})$.

Proof. It is sufficient to show the following inclusion $\mathrm{mix}(\mathrm{Lin}(Y,\mathcal{A})) \subset \mathrm{Lin}(Y,\mathcal{A})$. Let $x \in \mathrm{mix}(\mathrm{Lin}(Y,\mathcal{A}))$, $\{e_i\}_{i\in I}$ be a partition of unity in the Boolean algebra ∇ and let $\{y_i\}_{i\in I} \subset \mathrm{Lin}(Y,\mathcal{A})$ be such that $e_i x = e_i y_i$ for all $i \in I$. Since $e_i x = e_i y_i \in \mathrm{Lin}(Y,\mathcal{A})$, then $e_i x = \sum\limits_{j=1}^{k} a_j^{(i)} x_j$ for some $a_j^{(i)} \in \mathcal{A}$, $j = 1,\dots,k$. Hence, $e_i x = e_i(e_i x) = \sum\limits_{j=1}^{k} e_i a_j^{(i)} x_j$. Since $\mathcal{A}$ is a l-complete commutative regular algebra and $\{e_i\}_{i\in I}$ is a partition of unity in ∇, then there exists a unique element $\beta_j \in \mathcal{A}$ such that $e_i\beta_j = e_i a_j^{(i)}$ for all $i \in I$, where $j \in \{1,\dots,k\}$. Thus, $e_i x = \sum\limits_{j=1}^{k} e_i\beta_j x_j = e_i\left(\sum\limits_{j=1}^{k}\beta_j x_j\right)$ for any $i \in I$, and this implies the equality $x = \sum\limits_{j=1}^{k}\beta_j x_j \in \mathrm{Lin}(Y,\mathcal{A})$.

□

We say that an $\mathcal{A}$-linearly independent system $\{x_i\}_{i\in I}$ from a l-complete $\mathcal{A}$-module X is $\mathcal{A}$-*Hamel basis*, if

$$\mathrm{mix}(\mathrm{Lin}(\{x_i\}_{i\in I},\mathcal{A})) = X.$$

In the case when an $\mathcal{A}$-Hamel basis is a finite set, we say that it is an $\mathcal{A}$-basis in X.

Theorem 4.2. *If $\{x_i\}_{i=1}^{n}$, $\{y_j\}_{j=1}^{k}$ are $\mathcal{A}$-basises in an $\mathcal{A}$-module X, then $n = k$.*

Proof. First we shall show the following $\mathcal{A}$-variant of one known fact from the linear algebra.

Lemma 4.3. *Let $\{z_i\}_{i=1}^{n} \subset X, \{y_j\}_{j=1}^{k} \subset X, \{ey_j\}_{j=1}^{k} \subset \mathrm{Lin}(\{ez_i\}_{i=1}^{n},\mathcal{A}_e)$ for nonzero $e \in \nabla$. If the set $\{ey_1,\dots,ey_k\}$ is $\mathcal{A}_e$-linearly independent, then $k \le n$.*

Proof. We use the mathematical induction. Let us suppose that for $n = 1$, $k > 1$ the equalities $ey_1 = a_1 ez_1,\dots,ey_k = a_k ez_1$ hold, where $a_i \in \mathcal{A}_e$, $i = 1,\dots,k$. Since $a_2 ey_1 + (-a_1)ey_2 = 0$, then $ea_1 = ea_2 = 0$, i.e. $ey_1 = ey_2 = 0$, this contradicts to $\mathcal{A}_e$-linear independence of the elements ey_1 and ey_2. Hence, $k = 1$.

Now assume that the lemma holds for $n = l-1$. Let $\{z_i\}_{i=1}^{l} \subset X$ and the following equalities hold

$$ey_j = \sum_{i=1}^{l} a_{ji} ez_i, a_{ji} \in \mathcal{A}_e, \quad j = 1,\dots,k, i = 1,\dots,l. \tag{1}$$

Let $a_{j_0 l}ex_l \neq 0$ for some $j_0 \in \{1, \dots, k\}$. By reindexing $\{y_j\}_{j=1}^k$, we can assume that $a_{kl}ex_l \neq 0$, in particular $p = s(a_{kl}e) \neq 0$, wherein $p \leq e$. Since the set $\{ey_j\}_{j=1}^k$ is $\mathcal{A}_e$-linearly independent, then the set $\{py_j\}_{j=1}^k$ is $\mathcal{A}_p$-linearly independent, wherein, by (1), we have

$$py_j = \sum_{i=1}^{l} a_{ji}pz_i, \quad j = 1, \dots, k. \tag{2}$$

Since $\mathcal{A}$ is a regular algebra, then for the inversion $h = i(a_{kl}) \in \mathcal{A}$ the equality $ha_{kl} = s(a_{kl})$ holds. Therefore the following equality

$$py_k = \sum_{i=1}^{l-1} a_{ki}pz_i + a_{kl}pz_l$$

implies

$$pz_l = hpy_k - \sum_{i=1}^{l-1} a_{ki}hpz_i. \tag{3}$$

Substitute pz_l from (3) in the first $(k-1)$ equalities from (2) and collect similar terms, we obtain

$$py_j - ha_{jl}py_k = \sum_{i=1}^{l-1} \beta_{ji}pz_i \in \mathrm{Lin}(\{pz_i\}_{i=1}^{l-1}, \mathcal{A}_p)$$

for some $\beta_{ji} \in \mathcal{A}_p$, $i = 1, \dots, l-1$, $j = 1, \dots, k-1$.

Let us show that the elements $u_j = py_j - ha_{jl}py_k$, $j = 1, \dots, k-1$ are $\mathcal{A}_p$-linearly independent. Let

$$\sum_{j=1}^{k-1} \gamma_j py_j - \left(\sum_{j=1}^{k-1} \gamma_j ha_{jl}\right) py_k = \sum_{j=1}^{k-1} \gamma_j u_j = 0,$$

where $\gamma_j \in \mathcal{A}_p$, $j = 1, \dots, k-1$. Since $\{py_j\}_{j=1}^k$ is $\mathcal{A}_p$-linearly independent, then $p\gamma_1 = p\gamma_2 = \dots = p\gamma_{k-1} = 0$, i.e. the set $\{u_j\}_{j=1}^{k-1}$ is $\mathcal{A}_p$-linearly independent in pX. By the assumption of the mathematical induction we have that $k-1 \leq l-1$, and thus $k \leq l$. The Lemma 4.3 is proved. □

Return to the proof of Theorem 4.2. As $\{x_i\}_{i=1}^n$ is an $\mathcal{A}$-basis in X, then by Proposition 4.1 we obtain that $X = \mathrm{Lin}(\{x_i\}_{i=1}^n, \mathcal{A})$. On the other hand, $\{y_j\}_{j=1}^k \subset X$ and $\{y_j\}_{j=1}^k$ is an $\mathcal{A}$-linearly independent set. Therefore, by Lemma 4.3 it follows that $k \leq n$.

Similarly, we show that $n \leq k$, and thus $n = k$.

□

Next we need the following characterization of $\mathcal{A}$-Hamel basises.

Proposition 4.4. *For an $\mathcal{A}$-linearly independent set $\{x_i\}_{i \in I}$ in a l-complete $\mathcal{A}$-module X the following conditions are equivalent:*

(i). $\{x_i\}_{i\in I}$ is an $\mathcal{A}$-Hamel basis;

(ii). For any $x \in X$ and any nonzero idempotent $e \in \mathcal{A}$ there exists a nonzero idempotent $g \le e$, such that $gx \in g\mathrm{Lin}(\{x_i\}_{i\in I}, \mathcal{A})$.

Proof. $(i) \Rightarrow (ii)$. If $X = \mathrm{mix}(\mathrm{Lin}(\{x_i\}_{i\in I}, \mathcal{A}))$, then for $x \in X$ there exists a partition $\{e_j\}_{j\in J}$ of unity, such that $e_j x \in e_j\mathrm{Lin}(\{x_i\}_{i\in I}, \mathcal{A})$. Since $\sup\limits_{j\in J} e_j = \mathbf{1}$, then for $0 \ne e \in \nabla$ there exists an element $j_0 \in J$ such that $g = e_{j_0}e \ne 0$, wherein $gx \in g\mathrm{Lin}(\{x_i\}_{i\in I}, \mathcal{A})$.

$(ii) \Rightarrow (i)$. Fix $0 \ne x \in X$ and for any nonzero idempotent $e \in \nabla$ choose a nonzero idempotent $g(e,x) \le e$ such that $g(e,x)x \in g(e,x)\mathrm{Lin}(\{x_i\}_{i\in I}, \mathcal{A})$. By Theorem 3.3, there exists a set $\{q_j\}_{j\in J}$ of pairwise disjoint idempotents in $\mathcal{A}$ such that $\sup\limits_{j\in J} q_j = \mathbf{1}$ and $q_j x \in q_j\mathrm{Lin}(\{x_i\}_{i\in I}, \mathcal{A})$ for all $j \in J$. This means that $x \in \mathrm{mix}(\mathrm{Lin}(\{x_i\}_{i\in I}, \mathcal{A}))$, which implies the equality $X = \mathrm{mix}(\mathrm{Lin}(\{x_i\}_{i\in I}, \mathcal{A}))$.

□

Fix some cardinal number γ. A faithful l-complete $\mathcal{A}$-module X is called *γ-homogeneous*, if there exists an $\mathcal{A}$-Hamel basis $\{x_i\}_{i\in I}$ in X with $\mathrm{card}\, I = \gamma$. We say that $\mathcal{A}$-module X *homogeneous*, if it is a γ-homogeneous $\mathcal{A}$-module for some cardinal number γ.

If X is a γ-homogeneous $\mathcal{A}$-module, then obviously, eX is also γ-homogeneous $\mathcal{A}_e$-module for any nonzero idempotent $e \in \mathcal{A}$. Besides, by Proposition 3.2 (iv) it follows that, if $\mathcal{A}$-module Y is isomorphic to a γ-homogeneous $\mathcal{A}$-module X, then Y is also a γ-homogeneous module.

By repeating the proof of Theorem 3.8 from [2], we establish the following proposition on isomorphisms of γ-homogeneous $\mathcal{A}$-modules.

Proposition 4.5. *If X and Y are γ-homogeneous $\mathcal{A}$-modules, then X and Y are isomorphic.*

Let us give examples of γ-homogeneous $\mathcal{A}$-modules for an arbitrary cardinal number γ and for any l-complete commutative regular untaly algebra $\mathcal{A}$. Consider an arbitrary set of indexes I with $\mathrm{card}\, I = \gamma$. Since the algebra $\mathcal{A}$ is l-complete, then the Cartesian product

$$Y = \prod_{i\in I} \mathcal{A} = \{\hat{\alpha} = \{\alpha_i\}_{i\in I} : \alpha_i \in \mathcal{A}, i \in I\}$$

is a l-complete $\mathcal{A}$-module with coordinate-wise algebraic operations.

For any $j \in I$ consider an element $\hat{g}_j = \{g_i^{(j)}\}_{i\in I}$ from Y, where $g_i^{(j)} = 0$, $i \ne j$ and $g_i^{(i)} = \mathbf{1}$, $i \in I$. Clearly, that the set $\{\hat{g}_j\}_{j\in I}$ is $\mathcal{A}$-linearly independent, and, therefore, the $\mathcal{A}$-submodule $X = \mathrm{mix}\,(\mathrm{Lin}\,(\{\hat{g}_j\}_{j\in I}, \mathcal{A}))$ in Y is a γ-homogeneous $\mathcal{A}$-module.

If γ is a positive integer n, then for the faithful l-complete $\mathcal{A}$-module $Y = \prod\limits_{i=1}^{n} \mathcal{A} = \mathcal{A}^n$ and for $\hat{g}_j = \{g_i^{(j)}\}_{i=1}^n$, $j = 1, \ldots, n$ we have that $\mathrm{Lin}\,(\{\hat{g}_j\}_{j=1}^n, \mathcal{A}) = Y$, i.e. the set $\{\hat{g}_j\}_{j=1}^n$ is an $\mathcal{A}$-Hamel basis in Y. Thus, Proposition 4.5 implies the following

Corollary 4.6. *For any positive integer n there exists a unique, up to isomorphism, n-homogeneous $\mathcal{A}$-module, which is isomorphic to $\mathcal{A}^n$.*

Let X be a faithful l-complete $\mathcal{A}$-module, which is γ-homogeneous and λ-homogeneous simultaneously. There is a natural question, whether in this case the equality $\gamma = \lambda$ holds. Similar question was studied in classification of Kaplansky-Hilbert modules (KHM) X over a commutative AW^*-algebra $\mathcal{A}$ with the Boolean algebra of projections ∇ (see [6]). In the case, when ∇ is a multi-σ-finite Boolean algebra in [6] it is proved that for a KHM X the equality $\lambda = \gamma$ is always true. However, for an arbitrary complete Boolean algebra ∇ this equality cannot be established. Thereby, in ([9], 7.4.6) the notion of *strictly γ-homogeneous* KHM X is defined, and this gave an opportunity to classify KHM X over an arbitrary commutative AW^*-algebra $\mathcal{A}$. For the same reason, below we introduce the notion of strictly γ-homogeneous faithful l-complete modules over laterally complete algebras $\mathcal{A}$. With this notion we obtain necessary and sufficient conditions for l-complete $\mathcal{A}$-modules to be isomorphic.

Let X be a faithful l-complete $\mathcal{A}$-module, $0 \neq e \in \nabla$. By $\varkappa(e) = \varkappa_X(e)$ we denote the smallest cardinal number γ such that the $\mathcal{A}_e$-module X_e is γ-homogeneous. If the $\mathcal{A}$-module X is homogeneous, then the cardinal number $\varkappa(e)$ is defined for all nonzero $e \in \nabla$. Further, by ([9], 7.4.7), we assume that $\varkappa(0) = 0$.

We say that an $\mathcal{A}$-module X is *strictly γ-homogeneous* (compare with [9], 7.4.6), if X is γ-homogeneous and $\gamma = \varkappa(e)$ for all nonzero $e \in \nabla$. If an $\mathcal{A}$-module X is strictly γ-homogeneous for some cardinal number γ, then such $\mathcal{A}$-module X is called *strictly homogeneous.*

Clearly, any strictly γ-homogeneous $\mathcal{A}$-module is a γ-homogeneous $\mathcal{A}$-module. By Lemma 4.3 it follows that every n-homogeneous $\mathcal{A}$-module X is a strictly n-homogeneous module. By Proposition 3.2 (iv) every $\mathcal{A}$-module Y, which is isomorphic to a strictly γ-homogeneous $\mathcal{A}$-module X, is also strictly γ-homogeneous.

The following theorem holds.

Theorem 4.7. *Let λ and γ be infinite cardinal numbers and let the Boolean algebra ∇ of all idempotents in a l-complete commutative regular algebra $\mathcal{A}$ has countable type. If a faithful l-complete $\mathcal{A}$-module X is λ-homogeneous and γ-homogeneous simultaneously, then $\gamma = \lambda$.*

Proof of Theorem 4.7 is similar to that of Theorem 3.4 in [2].

Using Theorem 4.7 to the $\mathcal{A}_e$-module X_e, we have, that Theorem 4.7 holds in the case, when in the Boolean algebra ∇ of idempotents in $\mathcal{A}$ there exists nonzero element e, which has a countable type. Thus, repeating the proof of Corollary 3.7 in [2], we obtain the following necessary and sufficient conditions for coincidence of strictly γ-homogeneous and γ-homogeneous notions for $\mathcal{A}$-modules.

Proposition 4.8. *Let a Boolean algebra ∇ of all idempotents on a l-complete commutative regular algebra $\mathcal{A}$ be multi-σ-finite. If γ is an infinite cardinal number and X is a γ-homogeneous $\mathcal{A}$-module, then the module X is strictly γ-homogeneous.*

The following proposition enables to "glue" γ-homogeneous (strictly γ-homogeneous) $\mathcal{A}$-modules.

Proposition 4.9. *Let $\mathcal{A}$ be a l-complete commutative regular algebra, let X be a l-complete $\mathcal{A}$-module and let $\{e_i\}_{i\in I}$ be a set of pairwise disjoint nonzero idempotents in $\mathcal{A}$ and $e = \sup\limits_{i\in I} e_i$. If X_{e_i} is a γ-homogeneous (respectively, strictly γ-homogeneous) $\mathcal{A}_{e_i}$-module for all $i \in I$, then the $\mathcal{A}_e$-module X_e is also γ-homogeneous (respectively, strictly γ-homogeneous).*

Proof is similar to that of Proposition 3.10 in [2].

5 Classification of faithful l-complete $\mathcal{A}$-modules

In this section it is proved that every faithful laterally complete $\mathcal{A}$-module is isomorphic to a Cartesian product of strictly homogeneous $\mathcal{A}$-modules. The important step in obtaining such an isomorphism is the following theorem.

Theorem 5.1. *Let $\mathcal{A}$ be a l-complete commutative regular algebra, let ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$ and let X be a faithful l-complete $\mathcal{A}$-module. Then there exists a nonzero idempotent $p \in \nabla$ such that X_p is a strictly homogeneous $\mathcal{A}_p$-module.*

Proof. Using Proposition 3.4 (i), we choose $x_0 \in X$ such that $s(x_0) = \mathbf{1}$. If $X = \mathrm{Lin}(x_0, \mathcal{A})$, then X is a strictly 1-homogeneous module and Theorem 5.1 is proved.

Assume that $X \neq \mathrm{mix}\,(\{x_0\})$. We consider in X the following nonempty family of subsets

$$\mathscr{E} = \{B \subset X : x_0 \in B, B - \mathcal{A}\text{-linearly independent set}\}.$$

We introduce in $\mathscr{E}$ a partial order by $B \leq C \Leftrightarrow B \subset C$. By Zorn's lemma there exists maximal element D in $\mathscr{E}$. If D is an $\mathcal{A}$-Hamel basis in X, then X is $(\mathrm{card}\, D)$-homogeneous $\mathcal{A}$-module.

Assume that $X \neq \mathrm{mix}\,(\mathrm{Lin}\,(D, \mathcal{A}))$. If for any nonzero $e \in \nabla$ there exists $0 \neq q_e \in \nabla$ such that $q_e\, \mathrm{mix}\,(\mathrm{Lin}\,(D, \mathcal{A})) = q_e X$, then from Proposition 3.2 (iii) and Proposition 3.4 (ii)

it follows that $X = \operatorname{mix}(\operatorname{Lin}(D, \mathcal{A}))$, which contradicts our assumption. Hence, there exists nonzero $e \in \nabla$ such that the following condition holds:

$$g \operatorname{mix}(\operatorname{Lin}(D, \mathcal{A})) \neq gX \text{ for all non zero } g \in \nabla_e. \tag{1}$$

Denote by $\mathscr{L}$ a set of all nonzero $e \in \nabla$ with property (1). Put $e_0 = \sup \mathscr{L}$ and show that the equality $e_0 = \mathbf{1}$ fails.

Assume that $e_0 = \mathbf{1}$. In this case for every nonzero $q \in \nabla$ there exists $e \in \mathscr{L}$ such that $g = qe \neq 0$. Hence, $gX \neq g\operatorname{mix}(\operatorname{Lin}(D, \mathcal{A}))$ (see (1)), which implies

$$qX \neq q \operatorname{mix}(\operatorname{Lin}(D, \mathcal{A})). \tag{2}$$

Show that for any nonzero $q \in \nabla$ there exists a nonzero idempotent $r \leq q$ such that for any $0 \neq g \in \nabla_r$ the following property holds:

$$\text{There exists } x_g \in gX \text{ such that } s(x_g) = g \text{ and } \ lx_g \notin \operatorname{Lin}(D, \mathcal{A}) \text{ for all } 0 \neq l \in \nabla_g. \tag{3}$$

If this is not true, then there exists a nonzero $q \in \nabla$ such that for every $0 \neq r \in \nabla_q$ there exists a nonzero idempotent $g_r \in \nabla_r$ without property (3), i.e. for any $x \in g_rX$ with $s(x) = g_r$ there exists a nonzero idempotent $e(x_g, r) \leq g_r \leq q$ such that

$$e(x_g, r)x \in e(x_g, r)\operatorname{Lin}(D, \mathcal{A}) \subset \operatorname{Lin}(D, \mathcal{A}).$$

Show that, in this case, $g_qX = g_q\operatorname{mix}(\operatorname{Lin}(D, \mathcal{A}))$. Let x be a nonzero element in g_qX, in particular, $0 \neq s(x) \leq g_q$. For any nonzero idempotent $a \leq s(x)$ there exists a nonzero idempotent $e(ax, a) \leq a$ such that $e(ax, a)x \in \operatorname{Lin}(D, \mathcal{A})$. By Theorem 3.3, there exists a partition $\{e_i\}_{i\in I}$ of support $s(x)$ such that $e_ix \in s(x)\operatorname{Lin}(D, \mathcal{A})$ for all $i \in I$. This means that $x \in \operatorname{mix}(s(x)\operatorname{Lin}(D, \mathcal{A})) = s(x)\operatorname{mix}(\operatorname{Lin}(D, \mathcal{A}))$ (see Proposition 3.2 (ii)). Since $s(x) \leq g_q$, we have that $x \in g_q\operatorname{mix}(\operatorname{Lin}(D, \mathcal{A}))$, which implies the inclusion $g_qX \subset g_q\operatorname{mix}(\operatorname{Lin}(D, \mathcal{A}))$. On the other hand, by l-completeness of an $\mathcal{A}_{g_q}$-module g_qX we have that

$$g_q\operatorname{mix}(\operatorname{Lin}(D, \mathcal{A})) \subset g_q\operatorname{mix}(X) = \operatorname{mix}(g_qX) = g_qX.$$

Hence, $g_qX = g_q\operatorname{mix}(\operatorname{Lin}(D, \mathcal{A}))$, which contradicts to (2).

Thus, for every nonzero $q \in \nabla$ there exists a nonzero idempotent $r \leq q$ such that for any $0 \neq g \in \nabla_r$ property (3) holds.

Again by Theorem 3.3, we choose a partition $\{g_j\}_{j\in J}$ of the idempotent r and a set $\{x_{g_j}\}_{j\in J}$ in rX, such that $s(x_{g_j}) = g_j$ and $lx_{g_j} \notin \operatorname{Lin}(D, \mathcal{A})$ for all $0 \neq l \in \nabla_{g_i}$.

Since rX is a l-complete $\mathcal{A}_r$-module, then there exists $x \in rX$ such that $g_j x = x_{g_j}$. In particular, $s(x) = r$, wherein $lx \notin \mathrm{Lin}\,(D, \mathcal{A})$ for all $0 \neq l \in \nabla_r$.

Again by Theorem 3.3 we choose a partition $\{r_k\}_{k \in K}$ of the unity $\mathbf{1}$ in the Boolean algebra ∇ and a set $\{x_k\}_{k \in K}$ in X, such that $s(x_k) = r_k$ and $lx_k \notin \mathrm{Lin}\,(D, \mathcal{A})$ for any $0 \neq l \in \nabla_{r_k}$. By l-completeness of the $\mathcal{A}$-module X there exists $\hat{x} \in X$ such that $r_k \hat{x} = x_k$ for all $k \in K$. In this case $s(\hat{x}) = \mathbf{1}$ and $l\hat{x} \notin \mathrm{Lin}\,(D, \mathcal{A})$ for any $0 \neq l \in \nabla$.

Show that the set $D \cup \{\hat{x}\}$ is $\mathcal{A}$-linearly independent. Let $a_0 \hat{x} + \sum\limits_{i=1}^{n} a_i x_i = 0$, where $a_0, a_i \in \mathcal{A}$, $x_i \in D$, $i = 1, \ldots, n$. If $a_0 = 0$, then $\sum\limits_{i=1}^{n} a_i x_i = 0$ and by $\mathcal{A}$-linear independence of the set D it follows that $a_i = 0$ for all $i = 1, \ldots, n$. If $a_0 \neq 0$, then $s(a_0) \neq 0$ and for $i(a_0) = h \in \mathcal{A}$ we have that $ha_0 = s(a_0)$ and $s(a_0)\hat{x} = -\sum\limits_{i=1}^{n} a_i h x_i \in \mathrm{Lin}\,(D, \mathcal{A})$, which is not true. Hence, the set $D \cup \{\hat{x}\}$ is $\mathcal{A}$-linearly independent in X, which contradicts to maximality of the set D.

Thus the equality $e_0 = \mathbf{1}$ is impossible. This means that $e = \mathbf{1} - e_0 \neq 0$. By construction of the idempotent e_0, every nonzero idempotent $r \leq e$ does not have property (1). Hence, for any $0 \neq r \in \nabla_e$ there exists a nonzero idempotent $p_r \leq r$ such that

$$p_r X = p_r \mathrm{mix}\,(\mathrm{Lin}\,(D, \mathcal{A})) = \mathrm{mix}\,(\mathrm{Lin}\,(p_r D, \mathcal{A}_{p_r})) = p_r \mathrm{mix}\,(\mathrm{Lin}\,(eD, \mathcal{A}_e)).$$

From Propositions 3.2 (*iii*) and 3.4 (*ii*) it follows that

$$eX = \mathrm{mix}\,(\mathrm{Lin}\,(eD, \mathcal{A}_e)).$$

Since eD is an $\mathcal{A}_e$-linearly independent subset in the $\mathcal{A}_e$-module eX, then eD is an $\mathcal{A}_e$-basis in eX, i.e. eX is a γ-homogeneous $\mathcal{A}_e$-module, where $\gamma = \mathrm{card}\,(eD)$. In particular, a cardinal number $\varkappa(p)$ is defined for all nonzero $p \in \nabla_e$. Let γ_e be the smallest cardinal number in the set of cardinal numbers $\{\varkappa(p) : 0 \neq p \leq e\}$, i.e. $\gamma_e = \varkappa(p)$ for some nonzero $p \leq e$. By the choice of the idempotent p it follows that $\gamma_e = \varkappa(p) = \varkappa(q)$ for all $0 \neq q \in \nabla_p$. This means that the $\mathcal{A}_p$-module X_p is strictly homogeneous.

□

Now everything is ready to obtain the isomorphism from the faithful laterally complete $\mathcal{A}$-module to the Cartesian product of strictly homogeneous $\mathcal{A}$-modules.

Theorem 5.2. *Let $\mathcal{A}$ be a l-complete commutative regular algebra, let ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$ and let X be a faithful l-complete $\mathcal{A}$-module. Then there exist a uniquely defined set of pairwise disjoint nonzero idempotents $\{e_i\}_{i \in I} \subset \nabla$ and a set of pairwise different cardinal numbers $\{\gamma_i\}_{i \in I}$ such that $\sup\limits_{i \in I} e_i = \mathbf{1}$ and X_{e_i} is a strictly γ_i-homogeneous $\mathcal{A}_{e_i}$-module for all $i \in I$. In this case, the $\mathcal{A}$-modules X and $\prod\limits_{i \in I} X_{e_i}$ are isomorphic.*

Proof. By Theorem 5.1 for every nonzero idempotent $e \in \mathcal{A}$ there exists a nonzero idempotent $g \leq e$ such that X_g is a strictly homogeneous $\mathcal{A}_g$-module. By Theorem 3.3, choose a set of pairwise disjoint nonzero idempotents $\{q_j\}_{j\in J}$ such that $\sup\limits_{j\in J} q_j = \mathbf{1}$ and q_jX is a strictly λ_j-homogeneous $\mathcal{A}_{q_j}$-module for all $j \in J$. We decompose the set of cardinal numbers $A = \{\lambda_j\}_{j\in J}$ as a union of disjoint subsets A_i in such a way that every A_i consists of equal cardinal numbers from A. By γ_i denote an element in A_i. By Proposition 4.9, for $e_i = \sup\{q_j : \lambda_j \in A_i\}$ we have that the $\mathcal{A}_{e_i}$-module X_{e_i} is strictly γ_i-homogeneous. Moreover, by Proposition 3.5, the $\mathcal{A}$-module X and $\prod\limits_{i\in I} e_iX$ are isomorphic.

Assume, that there exist other sets of pairwise disjoint nonzero idempotents $\{g_j\}_{j\in J}$ and pairwise different cardinal numbers $\{\mu_j\}_{j\in J}$, such that $\sup\limits_{j\in J} g_j = \mathbf{1}$ and X_{g_j} is a strictly μ_j-homogeneous $\mathcal{A}_{g_j}$-module for all $j \in J$. For any fixed $j \in J$, by the equality $\sup\limits_{i\in I} e_i = \mathbf{1}$, we have that $g_j = \sup\limits_{i\in I} e_ig_j$. If there exist two different indexes $i_1, i_2 \in I$ such that $e_{i_1}g_j \neq 0$ and $e_{i_2}p_j \neq 0$, then

$$\mu_j = \varkappa(g_j) = \varkappa(e_{i_1}g_j) = \varkappa(e_{i_1}) = \gamma_{i_1} \neq \gamma_{i_2} = \varkappa(e_{i_2}) = \varkappa(e_{i_2}g_j) = \mu_j.$$

By this contradiction, it follows that $e_ig_j = 0$ for all $i \in I$ except one index, which we denote by $i(j)$. Since $e_{i(j)}g_j \neq 0$, we have that

$$\mu_j = \varkappa(g_j) = \varkappa(e_{i(j)}g_j) = \varkappa(e_{i(j)}) = \gamma_{i(j)}.$$

If $g_j \neq e_{i(j)}$, then by the equality $\sup\limits_{j\in J} g_j = \mathbf{1}$, there exists index $j_1 \in J$, $j_1 \neq j$ such that $e_{i(j)}g_{j_1} \neq 0$. Hence,

$$\mu_j = \gamma_{i(j)} = \varkappa(e_{i(j)}) = \varkappa(e_{i(j)}g_{j_1}) = \varkappa(g_{j_1}) = \mu_{j_1},$$

which is not true. Thus, $g_j = e_{i(j)}$ and $\mu_j = \gamma_{i(j)}$.

For the same reason, for any $i \in I$ there exists the unique index $j(i)$ such that $e_i = g_{j(i)}$ and $\gamma_i = \mu_{j(i)}$.

□

The partition $\{e_i\}_{i\in I}$ of unity in a Boolean algebra of idempotents in $\mathcal{A}$ and the set of cardinal numbers $\{\gamma_i\}_{i\in I}$ in Theorem 5.2 are called a passport for a faithful laterally complete $\mathcal{A}$-module X and denoted by $\Gamma(X) = \{(e_i(X), \gamma_i(X))\}_{i\in I(X)}$.

Thus, a passport $\Gamma(X) = \{(e_i(X), \gamma_i(X))\}_{i\in I(X)}$ for a faithful l-complete $\mathcal{A}$-module X means that $X = \prod\limits_{i\in I(X)} e_i(X)X$ (up to an isomorphism), where $e_i(X)X$ is a strictly $\gamma_i(X)$-homogeneous $\mathcal{A}_{e_i}$-module for all $i \in I(X)$, $e_i(X) \neq 0$, $e_i(X)e_j(X) = 0$, $\gamma_i(X) \neq \gamma_j(X)$, $i \neq j$, $i, j \in I(X)$, $\sup\limits_{i\in I(X)} e_i(X) = \mathbf{1}$.

The following theorem gives a criterion for isomorphism between faithful l-complete $\mathcal{A}$-modules, by using the notion of passport for these $\mathcal{A}$-modules.

Theorem 5.3. *Let $\mathcal{A}$ be a l-complete commutative regular algebra, X and Y be a faithful l-complete $\mathcal{A}$-modules. The following conditions are equivalent:*

(i) $\Gamma(X) = \Gamma(Y)$;

(ii) $\mathcal{A}$-modules X and Y are isomorphic.

Proof. $(i) \Rightarrow (ii)$. Let $\{(e_i(X), \gamma_i(X))\}_{i\in I(X)} = \Gamma(X) = \Gamma(Y) = \{(e_i(Y), \gamma_i(Y))\}_{i\in I(Y)}$, i.e. $I(X) = I(Y) := I$, $e_i(X) = e_i(Y) := e_i$ and $\gamma_i(X) = \gamma_i(Y) := \gamma_i$ for all $i \in I$. By Theorem 5.2, there exists an isomorphism U from $\mathcal{A}$-module X onto $\mathcal{A}$-module $\prod\limits_{i\in I} e_i X$ (respectively an isomorphism V from $\mathcal{A}$-module Y onto $\mathcal{A}$-module $\prod\limits_{i\in I} e_i Y$), where $U(x) = \{e_i x\}_{i\in I}$ (respectively, $V(y) = \{e_i y\}_{i\in I}$) for every $x \in X$ (respectively, for every $y \in Y$).

Since $e_i X$ (respectively, $e_i Y$) is a strictly γ_i-homogeneous $\mathcal{A}_{e_i}$-module, then by Proposition 4.5, for all $i \in I$ there exists an isomorphism U_i from the $\mathcal{A}_{e_i}$-module $e_i X$ onto the $\mathcal{A}_{e_i}$-module $e_i Y$. It is clear that a map $\Phi : X \to Y$, defined by the equality

$$\Phi(x) = V^{-1}(\{U_i(e_i x)\}_{i\in I}).$$

is an isomorphism from the $\mathcal{A}$-module X onto the $\mathcal{A}$-module Y.

$(ii) \Rightarrow (i)$. Let Ψ be an isomorphism from X onto Y and $\Gamma(X) = \{(e_i(X), \gamma_i(X))\}_{i\in I(X)}$ be a passport for a faithful l-complete $\mathcal{A}$-module X. By Proposition 3.2 (iv), the following $\mathcal{A}_{e_i(X)}$-module

$$Y_i = \Psi(e_i(X)X) = e_i(X)\Psi(X) = e_i(X)Y$$

is strictly $\gamma_i(X)$-homogeneous. This means that $\{(e_i(X), \gamma_i(X))\}_{i\in I}$ is a passport for the faithful l-complete $\mathcal{A}$-module Y, i.e. $\Gamma(X) = \Gamma(Y)$.

□

Let $\mathcal{A}$ be a l-complete commutative regular algebra, let ∇ be a Boolean algebra of all idempotents in $\mathcal{A}$. A faithful l-complete $\mathcal{A}$-module X is called finitely-dimensional, if there exist a finite partition $\{e_i\}_{i=1}^k$ of unity in the Boolean algebra ∇ ($e_i \neq 0, i = 1, \ldots, k$) and a finite set $\{n_i\}_{i=1}^k$ of natural numbers ($n_1 < n_2 < \ldots < n_k$) such that X_{e_i} is an n_i-homogeneous $\mathcal{A}_{e_i}$-module for all $i = 1, \ldots, k$.

This means that any finitely-dimensional $\mathcal{A}$-module X has a passport of the following form

$$\Gamma(X) = \{(e_i(X), n_i(X))\}_{i=1}^k,$$

where

$$e_1(X) + \ldots + e_k(X) = \mathbf{1}, n_1(X) < \ldots < n_k(X) < \infty.$$

Theorem 5.4. *For a faithful l-complete $\mathcal{A}$-module X the following conditions are equivalent:*

(i). X is a finitely-dimensional module;

(ii). X is a finitely-generated module, i.e. there exists a finite set $\{x_i\}_{i=1}^m$ of elements in X such that $X = \mathrm{Lin}(\{x_i\}_{i=1}^m, \mathcal{A})$;

(iii). There exists a positive integer m such that for any nonzero idempotent $e \in \mathcal{A}$ any $\mathcal{A}_e$-linearly independent set in X_e consists of not more than m elements.

Proof. $(i) \Rightarrow (ii)$. Let $\Gamma(X) = \{(e_i(X), n_i(X))\}_{i=1}^k$ be a passport for the $\mathcal{A}$-module X. For every $i = 1, \ldots, k$ we choose the $\mathcal{A}_{e_i}$-basis $\{x_j^{(i)}\}_{j=1}^{n_i}$ in X_{e_i}. If $x \in X$, then $e_i x = \sum\limits_{j=1}^{n_i} a_j^{(i)} x_j^{(i)}$, where $a_j^{(i)} \in \mathcal{A}_{e_i}$. Hence,

$$x = \sum_{i=1}^{k} e_i x = \sum_{i=1}^{k} \sum_{j=1}^{n_i} a_j^{(i)} g_j^{(i)} \in \mathrm{Lin}\ (\{x_j^{(i)}\}_{j=\overline{1,n_i}, i=\overline{1,k}}, \mathcal{A}).$$

This means that $\mathcal{A}$-module X is finitely-generated.

$(ii) \Rightarrow (iii)$. If $X = \mathrm{Lin}(\{x_i\}_{i=1}^m, \mathcal{A})$, e is a nonzero idempotent in $\mathcal{A}$ and $\{y_j\}_{j=1}^l$ is an $\mathcal{A}_e$-linearly independent set in X_e, then by Lemma 4.3, it follows that $l \leq m$.

$(iii) \Rightarrow (i)$. By Theorem 5.2, there exist a set of pairwise disjoint nonzero idempotents $\{e_i\}_{i \in I}$ and a set of pairwise different cardinal numbers $\{\gamma_i\}_{i \in I}$ such that $\sup\limits_{i \in I} e_i = \mathbf{1}$ and X_{e_i} is a strictly γ_i-homogeneous $\mathcal{A}_{e_i}$-module for all $i \in I$. If $\gamma_i > m$, then in X_{e_i} there exists a finite set $\{x_i\}_{i=1}^l$, which consist of $\mathcal{A}_e$-linearly independent elements, and besides $l > m$, which contradicts to condition (iii). Hence, $\gamma_i \leq m$ for all $i \in I$. Since natural numbers $\{\gamma_i\}_{i \in I}$ are pairwise different, then I is a finite set, i.e. $\{\gamma_i\}_{i \in I} = \{n_i\}_{i=1}^k$, where $n_1 < n_2 < \ldots n_k$. Hence, the $\mathcal{A}$-module X is finitely-dimensional.

□

The following description of finitely-dimensional $\mathcal{A}$-modules follows directly from Theorem 5.2 and Corollary 4.6.

Corollary 5.5. *If X is a finitely-dimensional $\mathcal{A}$-module, then there exist an uniquely defined finite partition $\{e_i\}_{i=1}^k$ of unity in the Boolean algebra of all idempotents in $\mathcal{A}$ and a finite set of positive integers $n_1 < \ldots < n_k$ such that the $\mathcal{A}$-module X is isomorphic to the $\mathcal{A}$-module $\prod\limits_{i=1}^k \mathcal{A}_{e_i}^{n_i}$ (here $e_i \neq 0$ for all $i = 1, \ldots, k$).*

A faithful l-complete $\mathcal{A}$-module X is called σ-finitely-dimensional, if there exist a countable partition $\{e_i\}_{i=1}^{\infty}$ of unity in the Boolean algebra of all idempotents in $\mathcal{A}$ ($e_i \neq 0, i = 1, 2, \ldots$) and a countable set $\{n_i\}_{i=1}^{\infty}$ of positive integers ($n_1 < n_2 < \ldots$) such that X_{e_i} is an n_i-homogeneous $\mathcal{A}_{e_i}$-module for all $i = 1, 2, \ldots$

By Theorem 5.2 and Corollary 4.6 we obtain the following description of σ-finitely-dimensional $\mathcal{A}$-modules.

Corollary 5.6. *If X is a σ-finitely-dimensional $\mathcal{A}$-module, then there exist a uniquely defined countable partition $\{e_i\}_{i=1}^{\infty}$ of unity in the Boolean algebra of all idempotents in $\mathcal{A}$ and a countable set of positive integers $n_1 < n_2 < \ldots$ such that the $\mathcal{A}$-module X is isomorphic to the $\mathcal{A}$-module $\prod\limits_{i=1}^{\infty} \mathcal{A}_{e_i}^{n_i}$ (here $e_i \neq 0$ for all $i = 1, 2, \ldots$).*

References

[1] Chilin V.I. Partially ordered Baire's involutive algebras. Modern problems of Mathematics: The latest trends, VINITI, Moscow, **27** (1985), 99-128. (in Russian)

[2] Chilin V.I., Karimov J.A. Laterally complete $C_\infty(Q)$-modules. Vladikavkaz math J., **16(2)** (2014), 69-78. (in Russian)

[3] Clifford A.N., Preston G.B. The algebraic theory of semigroup. Amer. Math. Soc., Mathematical Surveys, Number 7, Vol.I (1964).

[4] Kaplansky J. Projections in Banach algebras. Ann. Math, **53** (1951), 235-249.

[5] Kaplansky J. Algebras of type I. Ann. Math, **56** (1952), 450-472.

[6] Kaplansky J. Modules over operator algebras. Amer. J. Math, **75(4)** (1953), 839-858.

[7] Karimov, J.A. Kaplansky-Hilbert modules over the algebra of measurable functions. Uzbek math. J., 4 (2010), 74-81. (in Russian)

[8] Kusraev, A.G. Vector duality and its applications. Nauka, Novosibirsk (1985). (in Russian)

[9] Kusraev, A.G. Dominated operators. Springer, Netherlands (2000).

[10] Maeda F. Kontinuierliche Geometrien. Berlin (1958).

[11] Muratov M.A., Chilin V.I. Algebra of measurable and locally measurbale operators. Proceedings of Mathematics institute of NAS Ukraine, **69** (2007). (in Russian)

[12] Skornyakov L.A. Dedekind's lattices with complements and regular rings. Fizmatgiz, Moscow (1961). (in Russian)

[13] Van Der Waerden B.L. Algebra. Volueme II. Springer-Verlag New York (1991).

[14] Vulikh B.Z. Introduction to the theory of partially ordered spaces. Wolters-Noordhoff Sci. Publ., Groningen (1967).

National University of Uzbekistan, 100174, Vuzgorodok, Tashkent, Uzbekistan;
e-mail: chilin@ucd.uz, vladimirchil@gmail.com;
karimovja@mail.ru