# Cyclability of *id*-cycles in graphs*

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**Abstract**

Let $G$ be a graph on $n$ vertices and $C' = v_0v_1\cdots v_{p-1}v_0$ a vertex sequence of $G$ with $p \geq 3$ ($v_i \neq v_j$ for all $i, j = 0, 1, \ldots, p-1$, $i \neq j$). If for any successive vertices $v_i$, $v_{i+1}$ on $C'$, either $v_iv_{i+1} \in E(G)$ or both of the first implicit-degrees of $v_i$ and $v_{i+1}$ are at least $n/2$ (indices are taken modulo $p$), then $C'$ is called an *id*-cycle of $G$. In this paper, we prove that for every *id*-cycle $C'$, there exists a cycle $C$ in $G$ with $V(C') \subseteq V(C)$. This generalizes several early results on the Hamiltonicity and cyclability of graphs.

**Keywords:** degree, implicit-degree, Hamiltonicity, cyclability
**Mathematics Subject Classification:** 05C38, 05C45

## 1 Introduction

All graphs considered in this paper are finite, simple and undirected. For terminology and notation not defined here we refer the reader to [1].

Let $G$ be a graph. The vertex set and edge set of $G$ are denoted by $V(G)$ and $E(G)$, respectively. For a vertex $v$ and a subgraph $H$ of $G$, the *neighborhood* of $v$ in $H$ is defined as $N_H(v) = \{u : u \in V(H), uv \in E(G)\}$ and the *degree* of $v$ in $H$ is defined as $d_H(v) = |N_H(v)|$. If there is no ambiguity, we write $N(v)$ for $N_G(v)$ and $d(v)$ for $d_G(v)$.

In the study of the existence of Hamilton cycles in graphs, degree conditions play very important roles. Among the many results of this direction, the following two are well known.

**Theorem 1** (Dirac [2]). *Let $G$ be a graph on $n \geq 3$ vertices. If $d(v) \geq n/2$ for every vertex $v \in V(G)$, then $G$ is Hamiltonian.*

**Theorem 2** (Ore [4]). *Let $G$ be a graph on $n \geq 3$ vertices. If $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is Hamiltonian.*

For a vertex $v$ of a graph $G$, denote by $N_2(v)$ the vertices which are at distance of 2 from $v$ in $G$. In order to weaken the condition in Theorem 2 (Ore's condition), Zhu et al. [6] gave the definition of the first implicit-degree of the vertex $v$ based on the degrees of vertices in $N(v) \cup N_2(v) \cup \{v\}$.

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**Definition 1** (Zhu et al. [6]). Let $G$ be a graph on $n$ vertices and $v$ a vertex in $G$. If $N_2(v) \neq \emptyset$, then let $d(v) = k+1$, $M_2 = \max\{d(u)|u \in N_2(v)\}$. Denote by $d_1 \leq d_2 \leq \cdots \leq d_{k+1} \leq d_{k+2} \leq \cdots$ the nondecreasing degree sequence of vertices in $N(v) \cup N_2(v)$. Then the first implicit-degree of $v$ is defined as

$$d_1(v) = \begin{cases} \max\{d_{k+1}, k+1\}, & \text{if } d_{k+1} > M_2; \\ \max\{d_k, k+1\}, & \text{otherwise.} \end{cases} \tag{1}$$

If $N_2(v) = \emptyset$, then $d(v) = n-1$. In this case, let $d_1(v) = d(v) = n-1$.

It is clear that $d_1(v) \geq d(v)$ for every vertex $v \in V(G)$. Zhu et al. [6] obtained the following result as a generalization of Theorem 2.

**Theorem 3** (Zhu et al. [6]). *Let $G$ be a 2-connected graph on $n \geq 3$ vertices. If $d_1(u) + d_1(v) \geq n$ for every pair of nonadjacent vertices $u, v \in V(G)$, then $G$ is Hamiltonian.*

Let $G$ be a graph and $X$ a subset of of $V(G)$. If there exists a cycle $C$ in $G$ with $X \subseteq V(C)$, then we say $X$ is *cyclable* in $G$. A subgraph $H$ of $G$ is called *cyclable* if $V(H)$ is cyclable. Apparently, $G$ is Hamiltonian if and only if every spanning subgraph of $G$ is cyclable.

For a graph $G$, a vertex of degree at least $|V(G)|/2$ is called *heavy*. In 1992, Shi [5] proved the following result.

**Theorem 4** (Shi [5]). *Let $G$ be a 2-connected graph on $n \geq 3$ vertices and $S = \{v : d(v) \geq n/2, v \in V(G)\}$. Then $S$ is cyclable in $G$.*

It is clear that Theorem 4 implies Theorems 1 and 2.

Recently Li et al. [3] gave another generalization of Ore's condition.

**Definition 2** (Li et al. [3]). Let $G$ be a graph on $n$ vertices and $C' = v_0v_1 \cdots v_{p-1}v_0$ a vertex sequence in $G$ with $p \geq 3$ ($v_i \neq v_j$ for all $i, j = 0, 1, \ldots, p-1$, $i \neq j$). If for any successive vertices $v_i$, $v_{i+1}$ on $C'$, either $v_iv_{i+1} \in E(G)$ or $d(v_i) + d(v_{i+1}) \geq n$ (indices are taken modulo $p$), then $C'$ is called an *Ore-cycle* of $G$ or briefly, an *o-cycle* of $G$.

**Theorem 5** (Li et al. [3]). *Let $C'$ an o-cycle of a graph $G$. Then $C'$ is cyclable in $G$.*

Obviously, Theorem 5 implies Theorems 2 and 4. Our aim in this paper is to consider whether Theorem 5 can be generalized to the first implicit-degree condition.

**Definition 3.** Let $G$ be a graph on $n$ vertices and $C' = v_0v_1 \cdots v_{p-1}v_0$ a vertex sequence in $G$ with $p \geq 3$ ($v_i \neq v_j$ for all $i, j = 0, 1, \ldots, p-1$, $i \neq j$). If for any successive vertices $v_i$, $v_{i+1}$ on $C'$, either $v_iv_{i+1} \in E(G)$ or $d_1(v_i) + d_1(v_{i+1}) \geq n$ (indices are taken modulo $p$), then we call $C'$ an *implicit-Ore-cycle* of $G$ or briefly, an *io-cycle* of $G$. Specifically, if either $v_iv_{i+1} \in E(G)$ or $d_1(v_i) \geq n/2$ and $d_1(v_{i+1}) \geq n/2$ for all $i = 0, 1, \ldots, p-1$, then we call $C'$ an *implicit-Dirac-cycle* of $G$ or briefly, an *id-cycle* of $G$.

**Problem 1.** Is every *io*-cycle cyclable ?

Although we are unable to solve Problem 1, we can show that every *id*-cycle is cyclable.

**Theorem 6.** *Let $C'$ an id-cycle of a graph $G$. Then $C'$ is cyclable in $G$.*

Note that in Definition 2, if either $v_iv_{i+1} \in E(G)$ or $d(v_i) \geq n/2$ and $d(v_{i+1}) \geq n/2$ for all $i = 0, 1, \ldots, p-1$, then we can similarly call $C'$ a *Dirac-cycle* of $G$ or briefly, a *d-cycle* of $G$. It is clear that a $d$-cycle is a special $o$-cycle and we can regard an $id$-cycle as a generalization of a $d$-cycle.

**Fact 1.** *Theorem 6 implies Theorem 3.*

*Proof.* Let $A = \{a : d_1(a) < n/2, a \in V(G)\}$. For any vertices $u, v \in A$ $(u \neq v)$, we have $d_1(u) + d_1(v) < n$. So $uv \in E(G)$. This implies that $G[A] \cong K_{|A|}$. Let $B = \{b : d_1(b) \geq n/2, b \in V(G)\}$. Now, consider the size of $|A|$ and $|B|$, respectively.

If $|B| = 0$ or $|A| = 0$, then any vertex sequence of length $n$ is an $id$-cycle in $G$. By Theorem 6, $G$ is Hamiltonian.

If $|A| = 1$, then let $A = \{a\}$ and $B = \{b_1, b_2, \ldots, b_{n-1}\}$. Since $G$ is 2-connected, there are at least two neighbors of $a$ in $B$, say $b_1$ and $b_2$. Thus $b_1ab_2b_3 \cdots b_{n-1}b_1$ is an $id$-cycle of length $n$. By Theorem 6, $G$ is Hamiltonian. The proof is similar when $|B| = 1$.

If $|A| \geq 2$ and $|B| \geq 2$, since $G$ is 2-connected, there exist $a_1, a_2 \in A$ $(a_1 \neq a_2)$ and $b_1, b_2 \in B$ $(b_1 \neq b_2)$ such that $a_1b_1, a_2b_2 \in E(G)$. Let $A = \{a_1, a_2, \ldots, a_k\}$ and $B = \{b_1, b_2, \ldots, b_{n-k}\}$. Then $b_1a_1a_3a_4 \cdots a_ka_2b_2b_3 \cdots b_{n-k}b_1$ is an $id$-cycle of length $n$ in $G$. By Theorem 6, $G$ is Hamiltonian. □

**Fact 2.** *Let $G$ be a 2-connected graph on $n \geq 3$ vertices and $S = \{v : d_1(v) \geq n/2, v \in V(G)\}$. Then $S$ is cyclable in $G$.*

Obviously, Fact 2 is a generalization of Theorem 4 and can be directly obtained from Theorem 6.

## 2 Definitions and Lemmas

In this section, we will give some additional definitions and useful lemmas.

Let $G$ be a graph and $C' = v_0v_1 \cdots v_{p-1}v_0$ $(p \geq 3)$ an $id$-cycle in $G$ with a fixed orientation. For vertices $x, y \in V(C')$, let $xC'y$ be the segment on $C'$ from $x$ to $y$ along the direction of $C'$ and $x\overline{C'}y$ the segment on $C'$ along the reverse direction. For a vertex $v_i \in V(C')$, if $v_{i-1}v_i$ or $v_iv_{i+1} \notin E(G)$, then we call $v_i$ a *break-vertex* on $C'$. Denote by $Bre(C')$ the set of break-vertices on $C'$. Let

$$Bre^+(C') = \{v_i : v_iv_{i+1} \notin E(G)\} \text{ and } Bre^-(C') = \{v_i : v_{i-1}v_i \notin E(G)\}.$$

Then $Bre(C') = Bre^+(C') \cup Bre^-(C')$. Note that $Bre^+(C') \cap Bre^-(C')$ is not necessarily empty. For a vertex $v_i \in V(C')$, let $v_i^+ = v_{i+1}$ and $v_i^- = v_{i-1}$. Then $v_i^+$ and $v_i^-$ represent the immediate successor and predecessor of $v_i$ on $C'$, respectively. Denote by $N_{C'}(v_i)^-$ the predecessors of vertices in $N_{C'}(v_i)$. To measure the gap between $C'$ and a cycle, we define the *deficit-degree* of $C'$ as

$$def(C') = |\{i : v_iv_{i+1} \notin E(G)\}|.$$

If $def(C') \leq def(C)$ for any $id$-cycle $C$ satisfying $V(C') \subseteq V(C)$, then we say $C'$ is *def-minimal*. Let $u$ be a break-vertex on $C'$. We say $u$ is a *heavy-break-vertex* if $d(u) \geq |V(G)|/2$. Denote by $Hb(C')$ the set of heavy-break-vertices on $C'$. To measure the difference between $C'$ and a $d$-cycle, we define the *heavy-index* of $C'$ as

$$hb(C') = |Hb(C') \cap Bre^+(C')| + |Hb(C') \cap Bre^-(C')|.$$

If $hb(C') \geq hb(C)$ for any *id*-cycle $C$ satisfying $V(C') \subseteq V(C)$ and $def(C') = def(C)$, then we say $C'$ is *hb-maximal*.

Let $P = u_0u_1 \cdots u_{t-1}$ be a path in $G$. Then we call $u_0$ and $u_{t-1}$ the *end-vertices* of $P$. For vertices $a, b \in V(P)$, denote by $aPb$ the segment on $P$ from $a$ to $b$. If $a = b$, then $aPb = \{a\}$. Apparently, an *id*-cycle $C'$ in $G$ is composed of some vertex-disjoint paths and we can write $C' = x_1P_1y_1x_2P_2y_2 \cdots x_sP_sy_sx_1$, where $x_i$ and $y_i$ are the end-vertices of $P_i$ satisfying $d_1(x_i) \geq |V(G)|/2$ and $d_1(y_i) \geq |V(G)|/2$ for all $i = 1, 2, \ldots, s$. Hence, the set of break-vertices on $C'$ can be regarded as the set of end-vertices of $P_i$ $(i = 1, 2 \ldots, s)$.

Let $C' = v_0v_1 \cdots v_{p-1}v_0$ be an *id*-cycle in a graph $G$. Then we have $def(C') \geq 0$ and $hb(C') \leq 2def(C')$. If $def(C') = 0$, then $C'$ is a cycle. If $hb(C') = 2def(C')$, then $C'$ is a $d$-cycle and cyclable. In this paper, we mainly consider the case that $def(C') > 0$ and $hb(C') < 2def(C')$. In order to make the paper easy to follow, we name a specific kind of break-vertex as "strange-vertex".

**Definition 4.** Let $G$ be a graph on $n \geq 3$ vertices and $C' = x_1P_1y_1x_2P_2y_2 \cdots x_sP_sy_sx_1$ an *id*-cycle in $G$. Let $R$ be the subgraph of $G$ induced by $V(G)\backslash V(C')$ and $u$ an end-vertex of $P_i$. If the following conditions hold:
$(a)$ $d(u) < n/2$;
$(b)$ $d(v) < d_1(u)$ for every vertex $v \in N_R(u)$;
$(c)$ $N(u) \cap V(P_j) = \emptyset$ $(j = 1, 2, \ldots, s, j \neq i)$;
$(d)$ $|V(P_i)| \geq 3$ and $uw \in E(G)$ ($w$ is the other end-vertex of $P_i$),
then we call $u$ a *strange-vertex* on $C'$. Denote by $Str(C')$ the set of strange-vertices on $C'$.

**Lemma 1.** *Let $G$ be a graph on $n \geq 3$ vertices and $C' = v_0v_1 \cdots v_{p-1}v_0$ an id-cycle in $G$. If $v_0v_{p-1} \notin E(G)$ and $d(v_0) + d(v_{p-1}) \geq n$, then there exists an id-cycle $C$ such that $V(C') \subseteq V(C)$ and $def(C) < def(C')$.*

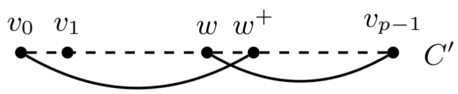


Figure 1

*Proof.* Let $R$ be the subgraph of $G$ induced by $V(G)\backslash V(C')$. If there exists a vertex $w \in N_R(v_0) \cap N_R(v_{p-1})$, then construct a new *id*-cycle as $C = wv_0v_1 \cdots v_{p-1}w$. Obviously, $def(C) = def(C') - 1 < def(C')$. If $N_R(v_0) \cap N_R(v_{p-1}) = \emptyset$, then $|N_R(v_0)| + |N_R(v_{p-1})| \leq |R|$. Since $d(v_0) + d(v_{p-1}) \geq n$, we have $|N_{C'}(v_0)| + |N_{C'}(v_{p-1})| \geq |C'|$. Note that $|N_{C'}(v_0)^-| = |N_{C'}(v_0)|$, so

$$|N_{C'}(v_0)^-| + |N_{C'}(v_{p-1})| \geq |C'|.$$

Since $v_0v_{p-1} \notin E(G)$, we have

$$|N_{C'}(v_0)^- \cup N_{C'}(v_{p-1})| \leq |C'| - 1.$$

This implies that $N_{C'}(v_0)^- \cap N_{C'}(v_{p-1}) \neq \emptyset$. Choose a vertex $w \in N_{C'}(v_0)^- \cap N_{C'}(v_{p-1})$, then $w^+ \in N_{C'}(v_0)$. Construct an *id*-cycle as $C = v_0w^+C'v_{p-1}w\overline{C'}v_0$ (see Fig. 1). Apparently, $V(C') \subseteq V(C)$ and $def(C) < def(C')$. $\square$

**Lemma 2.** *Let $G$ be a graph on $n \geq 3$ vertices and $C' = v_0v_1 \cdots v_{p-1}v_0$ a def-minimal id-cycle in $G$ with $v_0v_{p-1} \notin E(G)$. Let $R$ be the subgraph of $G$ induced by $V(G)\backslash V(C')$. If $v_0$ satisfies the following conditions:*
*(a) $N_{C'}(v_0)^- \subseteq N(v_0) \cup N_2(v_0) \cup \{v_0\}$;*
*(b) $N_2(v_0) \nsubseteq N_{C'}(v_0)^-$;*
*(c) $d(v_0) < n/2$ and $d(v) < d_1(v_0)$ for any $v \in N_R(v_0)$,*
*then there must exist a vertex $u \in N_{C'}(v_0)^-$ such that $d(u) \geq d_1(v_0)$ and $C'$ is not hb-maximal.*

*Proof.* Suppose that $d(v_0) = k+1$. Denote by $d_1 \leq d_2 \leq \cdots \leq d_{k+1} \leq d_{k+2} \leq \cdots$ the non-decreasing degree sequence of $N(v_0) \cup N_2(v_0)$. Let $M_2 = \max\{d(u)|u \in N_2(v_0)\}$. Since $N_{C'}(v_0)^- \subseteq V(C')$ and $N_R(v_0) \subseteq V(R)$, we have $N_{C'}(v_0)^- \cap N_R(v_0) = \emptyset$. Furthermore, $|N_{C'}(v_0)^-| = |N_{C'}(v_0)|$, so $|N_{C'}(v_0)^-\backslash\{v_0\} \cup N_R(v_0)| \geq d(v_0) - 1$ (the equation holds if and only if $v_0v_1 \in E(G)$). Thus we get

$$|N_{C'}(v_0)^-\backslash\{v_0\} \cup N_R(v_0)| \geq k. \tag{2}$$

By $(a)$, we have

$$N_{C'}(v_0)^-\backslash\{v_0\} \cup N_R(v_0) \subseteq N(v_0) \cup N_2(v_0). \tag{3}$$

Since $d_1(v_0) \geq n/2 > d(v_0)$, $d_1(v_0) = d_k$ or $d_1(v_0) = d_{k+1}$.

If $d_1(v_0) = d_k$, then there are at most $k-1$ vertices in $N(v_0) \cup N_2(v_0)$ having degrees smaller than $d_1(v_0)$. By (2) and (3), there exists a vertex $u \in N_{C'}(v_0)^-\backslash\{v_0\} \cup N_R(v_0)$ such that $d(u) \geq d_1(v_0)$.

If $d_1(v_0) = d_{k+1} > d_k$, then by Definition 1, $d_{k+1} > M_2$. By $(b)$, there is a vertex $w \in N_2(v_0)$ and $w \notin N_{C'}(v_0)^-\backslash\{v_0\} \cup N_R(v_0)$ satisfying $d(w) \leq M_2 < d_{k+1}$. Similarly, by (2) and (3), there exists at least one vertex $u \in N_{C'}(v_0)^-\backslash\{v_0\} \cup N_R(v_0)$ such that $d(u) \geq d_{k+1} = d_1(v_0)$.

Recall that $d(v) < d_1(v_0)$ for any $v \in N_R(v_0)$. In all cases, there is a vertex $u \in N_{C'}(v_0)^-$ satisfying $d(u) \geq d_1(v_0)$.

Let $u = v_s$ and $C = v_sv_{s-1}\cdots v_0v_{s+1}v_{s+2}\cdots v_{p-1}$. Thus $V(C) = V(C')$ and $def(C) \leq def(C')$. Note that $C'$ is def-minimal, we have $def(C) = def(C')$.

Now, we will prove that $hb(C) > hb(C')$. Considering the construction of $C$, we know that

$$Bre^+(C) = \{v_t : v_t \in Bre^-(C'), t \leq s\} \cup \{v_t : v_t \in Bre^+(C'), t > s\},$$

and

$$Bre^-(C) = \{v_t : v_t \in Bre^+(C'), t \leq s\} \cup \{v_t : v_t \in Bre^-(C'), t > s\} \cup \{v_s\}\backslash\{v_0\}.$$

Thus we have

$$\begin{aligned} hb(C) &= |Hb(C) \cap Bre^+(C)\}| + |Hb(C) \cap Bre^-(C)\}| \\ &= |\{v_t : v_t \in Hb(C') \cap Bre^-(C'), t \leq s\}| \\ &\quad + |\{v_t : v_t \in Hb(C') \cap Bre^+(C'), t > s\}| \\ &\quad + |\{v_t : v_t \in Hb(C') \cap Bre^+(C'), t \leq s\}| \\ &\quad + |\{v_t : v_t \in Hb(C') \cap Bre^-(C'), t > s\}| + |\{v_s\}| \\ &= hb(C') + 1. \end{aligned}$$

Hence, $C'$ is not hb-maximal. The proof is complete. □

**Lemma 3.** *Let $G$ be a graph on $n \geq 3$ vertices and $C' = x_1P_1y_1x_2P_2y_2 \cdots x_sP_sy_sx_1$ a def-minimal and then hb-maximal id-cycle in $G$ with $def(C') \geq 1$. Then the following statements hold:*
*(1) $x_ix_j, x_iy_j, y_iy_j \notin E(G)$ for any $i, j = 1, 2, \ldots, s, i \neq j$;*
*(2) $Bre(C') = Hb(C') \cup Str(C')$.*

*Proof.* (1) By contradiction. Assume that $x_ix_j \in E(G)$. Combine $P_i$ and $P_j$ into a new path $P' = y_iP_ix_ix_jP_jy_j$. Note that although we change the orders or orientations of $P_i$ and $P_j$ in $C'$, it always produces an *id*-cycle. We can assume that $C$ is an arbitrary permutation of $\{P_1, P_2, \ldots, P_s\}\backslash\{P_i, P_j\} \cup \{P'\}$. Thus $def(C) \leq def(C') - 1 < def(C')$. This contradicts that $C'$ is a def-minimal. Similarly, we can prove that $x_iy_j, y_iy_j \notin E(G)$.

(2) By contradiction. Assume that there is a break-vertex $x_i$ which is neither a strange-vertex nor a heavy-break-vertex. Then $d(x_i) < n/2$. Let $R$ be the subgraph of $G$ induced by $V(G)\backslash V(C')$. By Definition 4, at least one of the following statements fails:
$(a)$ $d(v) < d_1(x_i)$ for every vertex $v \in N_R(x_i)$;
$(b)$ $N(x_i) \cap V(P_j) = \emptyset$ for any $j = 1, 2, \ldots, s$, $j \neq i$;
$(c)$ $|V(P_i)| \geq 3$ and $x_iy_i \in E(G)$.
Without loss of generality, let $C' = x_iP_iy_ix_{i+1}P_{i+1}y_{i+1} \cdots x_{i-1}P_{i-1}y_{i-1}x_i$. Denote by $v_0v_1 \cdots v_{p-1}v_0$ the vertex sequence of $C'$ with $v_0 = x_i$. Let $l(x_i) = \max\{t | v_tv_0 \in E(G)\}$. By (1), we have $N_{C'}(x_i)^- \subseteq N(x_i) \cup N_2(x_i) \cup \{x_i\}$.

Now, we will discuss the following three cases.

**Case 1.** $(a)$ fails.

In this case, there is a vertex $v \in N_R(x_i)$ such that $d(v) \geq d_1(x_i) \geq n/2$. Let $C = vv_0v_1 \cdots v_{p-1}v$. Then $V(C') \subseteq V(C)$ and $def(C) \leq def(C')$. Since $C'$ is a def-minimal *id*-cycle, we have $def(C) = def(C')$ and $vv_{p-1} \notin E(G)$. Thus $Bre^+(C) = Bre^+(C')$, $Bre^-(C) = Bre^-(C') \cup \{v\}\backslash\{x_i\}$ and $hb(C) = hb(C') + 1$. This contradicts that $C'$ is hb-maximal.

**Case 2.** $(a)$ holds and $(b)$ fails.

In this case, there exists a path $P_j$ ($j = 1, 2, \ldots, s$, $j \neq i$) such that $N(x_i) \cap V(P_j) \neq \emptyset$. By (1), $x_j, y_j \notin N(x_i)$. Thus $v_{l(x_i)}v_{l(x_i)+1} \in E(G)$. This implies that $v_{l(x_i)+1} \in N_2(x_i)$. Since $v_{l(x_i)+1} \notin N_{C'}(x_i)^-$, we have $N_2(x_i) \nsubseteq N_{C'}(x_i)^-$. Thus, the vertex $x_i$ on the *id*-cycle $C'$ suffices the conditions in Lemma 2. Hence, $C'$ is not hb-maximal, a contradiction.

**Case 3.** $(a)$, $(b)$ hold and $(c)$ fails.

In this case, $|V(P_i)| \leq 2$ or $|V(P_i)| \geq 3$ and $x_iy_i \notin E(G)$. If $|V(P_i)| \leq 2$, then $N_{C'}(x_i)^- \subseteq \{x_i\}$. So $N_2(x_i) \cap N_{C'}(x_i)^- = \emptyset$. Since $N_2(x_i) \neq \emptyset$, we have $N_2(x_i) \nsubseteq N_{C'}(x_i)^-$. If $|V(P_i)| \geq 3$ and $x_iy_i \notin E(G)$, then $N_{C'}(x_i) \neq \emptyset$ and $v_{l(x_i)+1} \in N_2(x_i)$. Furthermore, we know that $v_{l(x_i)+1} \notin N_{C'}(x_i)^-$, so $N_2(x_i) \nsubseteq N_{C'}(x_i)^-$. No matter $|V(P_i)| \leq 2$ or $|V(P_i)| \geq 3$, the vertex $x_i$ on the *id*-cycle $C'$ suffices the conditions in Lemma 2. Thus, $C'$ is not hb-maximal, a contradiction.

Now, each break-vertex $x_i$ on $C'$ is either is strange-vertex or a heavy-break-vertex. Similarly, we can prove this conclusion for every break-vertex $y_i$ by analyzing the reversion of $C'$.

The proof is complete. □

**Lemma 4.** *Let $G$ be a graph on $n \geq 3$ vertices and $C' = x_1P_1y_1x_2P_2y_2 \cdots x_sP_sy_sx_1$ a def-minimal and then hb-maximal id-cycle in $G$ with $def(C') \geq 1$. If $x_i \in Str(C')$, then $N_2(x_i) \subseteq N_{C'}(x_i)^- \subseteq V(P_i)$.*

*Proof.* Without loss of generality, assume $C' = v_0v_1 \cdots v_{p-1}v_0$ starts at $v_0 = x_i$ with $v_0v_{p-1} \notin E(G)$. First, we will prove that $N_2(x_i) \subseteq N_{C'}(x_i)^-$.

By contradiction. Assume that $N_2(x_i) \nsubseteq N_{C'}(x_i)^-$. Recall the definition of strange-vertex. We know that the vertex $x_i$ on the *id*-cycle $C'$ suffices the conditions of Lemma 2. Thus, $C'$ is not hb-maximal, a contradiction.

Furthermore, by the definition of strange-vertex, we have $N_{C'}(x_i)^- \subseteq V(P_i)$. So $N_2(x_i) \subseteq N_{C'}(x_i)^- \subseteq V(P_i)$. □

## 3 Proof of Theorem 6

By contradiction. Assume that $|V(G)| = n$. Let $C_1$ be a def-minimal and then hb-maximal counterexample with $def(C_1) \geq 1$. By Lemma 3, we have $Bre(C_1) = Str(C_1) \cup Hb(C_1)$.

**Claim 1.** $def(C_1) \geq 2$.

*Proof.* Assume that $def(C_1) = 1$. Then $C_1$ is a path in $G$. Let $C_1 = v_0v_1 \cdots v_{p-1}$ and $v_0v_{p-1} \notin E(G)$. By the definition of strange-vertex, we have $v_0, v_{p-1} \notin Str(C_1)$. This implies that $d(v_0) \geq n/2$ and $d(v_{p-1}) \geq n/2$. By Lemma 1, there must exist an *id*-cycle $C_2$ in $G$ such that $V(C_1) \subseteq V(C_2)$ and $def(C_2) < def(C_1)$, a contradiction. □

Now, let $C_1 = x_1P_1y_1x_2P_2y_2 \cdots x_sP_sy_sx_1$. By Claim 1, we have $s \geq 2$. Let $i, j$ be arbitrary integers satisfying $1 \leq i < j \leq s$.

**Claim 2.** $x_i \in Str(C_1)$ or $x_j \in Str(C_1)$.

*Proof.* By contradiction. Assume that $x_i \in Hb(C_1)$ and $x_j \in Hb(C_1)$. Then by changing the orders and orientations of the paths in $C_1$ appropriately we can construct a new *id*-cycle $C_2$ such that $x_i$ and $x_j$ are successive on $C_2$. Since $d(x_i) + d(x_j) \geq n$, by Lemma 1, there exists an *id*-cycle $C_3$ satisfying $V(C_1) = V(C_2) \subseteq V(C_3)$ and $def(C_3) < def(C_2) = def(C_1)$, a contradiction. □

**Claim 3.** $x_i \in Hb(C_1)$ or $x_j \in Hb(C_1)$.

*Proof.* By contradiction. Assume that $x_i \in Str(C_1)$ and $x_j \in Str(C_1)$. By the definitions of strange-vertex and implicit-degree, there must exist vertices $u_i \in V(P_i) \cap N(x_i)$ and $u_j \in V(P_j) \cap N(x_j)$ satisfying $d(u_i) \geq d_1(x_i) \geq n/2$ and $d(u_j) \geq d_1(x_j) \geq n/2$, respectively. Thus we have $d(u_i) + d(u_j) \geq n$. So either $u_iu_j \in E(G)$ or $N(u_i) \cap N(u_j) \neq \emptyset$.

**Case 1.** $u_iu_j \in E(G)$

In this case, $x_iu_iu_j$ is a shortest path from $x_i$ to $u_j$ in $G$. So $u_j \in N_2(x_i)$ and $N_2(x_i) \nsubseteq V(P_i)$. This contradicts to Lemma 4.

**Case 2.** $u_iu_j \notin E(G)$

In this case, there is a vertex $w \in N(u_i) \cap N(u_j)$.

If $w \in V(P_i)$ (or $V(P_j)$), then it follows from the definition of strange-vertex that $w \in N_2(x_j)$ (or $N_2(x_i)$). So $N_2(x_j) \nsubseteq V(P_j)$ (or $N_2(x_i) \nsubseteq V(P_i)$). This contradicts to Lemma 4.

If $w \in V(P_k)$ and $k \neq i, j$, then it follows from the definition of strange-vertex that $w \in N_2(x_i)$ and $N_2(x_i) \nsubseteq V(P_i)$. This contradicts to Lemma 4.

If $w \in V(G) \backslash V(C_1)$, then consider the relation between $w$ and $x_i$. If $wx_i \in E(G)$, then $x_i w u_j$ is a shortest path from $x_i$ to $u_j$. Thus $u_j \in N_2(x_i)$. If $wx_i \notin E(G)$, then $w \in N_2(x_i)$. So, in all cases, we have $N_2(x_i) \nsubseteq V(P_i)$. This contradicts to Lemma 4. □

**Claim 4.** $def(C_1) = 2$.

*Proof.* By contradiction. Assume that $def(C_1) \neq 2$. By Claim 1, $def(C_1) \geq 3$. For any integers $i, j, k$ satisfying $1 \leq i < j < k \leq s$, we have $|\{x_i, x_j, x_k\} \cap Str(C_1)| \geq 2$ or $|\{x_i, x_j, x_k\} \cap Hb(C_1)| \geq 2$. This contradicts to Claim 2 or Claim 3. □

Now, we can assume that $C_1 = x_1 P_1 y_1 x_2 P_2 y_2 x_1$. Without loss of generality, let $x_1 \in Str(C_1)$ and $x_2 \in Hb(C_1)$. By the definitions of strange-vertex and implicit-degree, there must exist an vertex $u \in V(P_1) \cap N(x_1)$ such that $d(u) \geq d_1(x_1) \geq n/2$. Since $d(x_2) \geq n/2$, we have $ux_2 \in E(G)$ or $N(u) \cap N(x_2) \neq \emptyset$.

If $ux_2 \in E(G)$, then $x_2 \in N_2(x_1)$ and $N_2(x_1) \nsubseteq V(P_1)$. This contradicts to Lemma 4. So there exists a vertex $w \in N(u) \cap N(x_2)$.

If $w \in V(P_2)$, then $w \in N_2(x_1)$ and $N_2(x_1) \nsubseteq V(P_1)$, a contradiction. If $w \in V(G) \backslash V(C_1)$, then consider the relation between $w$ and $x_1$. If $wx_1 \notin E(G)$, then $w \in N_2(x_1)$ and $N_2(x_1) \nsubseteq V(P_1)$, a contradiction. If $wx_1 \in E(G)$, then $x_2 \in N_2(x_1)$ and $N_2(x_1) \nsubseteq V(P_1)$, a contradiction. So the only possible situation is that $w \in V(P_1)$ and $wx_1 \notin E(G)$. Thus $w \in N_2(x_1)$. Furthermore, by Lemma 4, we have $N_2(x_1) \subseteq N_{C_1}(x_1)^-$ and $w \in N_{C_1}(x_1)^-$. So $w^+ \in N(x_1)$ (see Fig. 2).

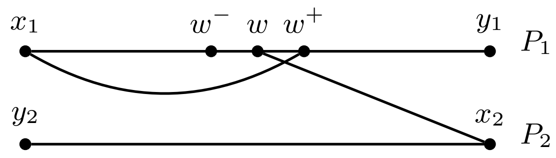


Figure 2

Let $C_2 = y_2 P_2 x_2 w w^- \overleftarrow{P_1} x_1 w^+ P_1 y_1 y_2$. Apparently, $C_2$ is an *id*-cycle, $Bre(C_2) = \{y_1, y_2\}$, $V(C_1) = V(C_2)$ and $def(C_2) = 1 < def(C_1)$. This contradicts that $C_1$ is a def-minimal *id*-cycle.

The proof is complete. □

## References


[1] J.A. Bondy, U.S.R. Murty, *Graph Theory with Applications*, Macmillan and Elsevier, London, New York, (1976).

[2] G.A. Dirac, Some theorems on abstract graphs, *Proc. Lond. Math. Soc.* **3** (1952) 69–81.

[3] B. Li, Z. Ryjáček, Y. Wang, S. Zhang, Pairs of heavy subgraphs for Hamiltonicity of 2-connected graphs, *SIAM J. Discrete Math.* **26** (2012) 1088–1103.

[4] O. Ore, Note on Hamilton circuits, *Amer. Math. Monthly* (1960) 55–55.

[5] R. Shi, 2-Neighborhoods and Hamiltonian conditions, *J. Graph Theory* **16** (3) (1992) 267–271.

[6] Y. Zhu, H. Li, X. Deng, Implicit-degree and circumference, *Graphs Combin.* **5** (1989) 283–290.