AN INFINITE PRESENTATION FOR THE MAPPING CLASS GROUP OF A NON-ORIENTABLE SURFACE

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Abstract. We give an infinite presentation for the mapping class group of a non-orientable surface. The generating set consists of all Dehn twists and all crosscap pushing maps along simple loops.

1. Introduction

Let $\Sigma_{g,n}$ be a compact connected orientable surface of genus $g \geq 0$ with $n \geq 0$ boundary components. The *mapping class group* $\mathcal{M}(\Sigma_{g,n})$ of $\Sigma_{g,n}$ is the group of isotopy classes of orientation preserving self-diffeomorphisms on $\Sigma_{g,n}$ fixing the boundary pointwise. A finite presentation for $\mathcal{M}(\Sigma_{g,n})$ was given by Hatcher-Thurston [5], Wajnryb [15] and Harer [4]. Gervais [3] obtained an infinite presentation for $\mathcal{M}(\Sigma_{g,n})$ by using the finite presentation for $\mathcal{M}(\Sigma_{g,n})$, and Luo [10] rewrote Gervais's presentation into a simpler infinite presentation (See Theorem 2.5).

Let $N_{g,n}$ be a compact connected non-orientable surface of genus $g \geq 1$ with $n \geq 0$ boundary components. The surface $N_g = N_{g,0}$ is a connected sum of g real projective planes. The mapping class group $\mathcal{M}(N_{g,n})$ of $N_{g,n}$ is the group of isotopy classes of self-diffeomorphisms on $N_{g,n}$ fixing the boundary pointwise. For $g \geq 2$ and $n \in \{0,1\}$, a finite presentation for $\mathcal{M}(N_{g,n})$ was given by Lickorish [8], Birman-Chillingworth [1], Stukow [12] and Paris-Szepietowski [11]. Note that $\mathcal{M}(N_1)$ and $\mathcal{M}(N_{1,1})$ are trivial (See [2, Theorem 3.4]) and $\mathcal{M}(N_2)$ is finite (See [8, Lemma 5]). Stukow [13] rewrote Paris-Szepietowski's presentation into a finite presentation with Dehn twists and a "Y-homeomorphism" as generators (See Theorem 2.11).

In this paper, we give a simple infinite presentation for $\mathcal{M}(N_{g,n})$ (Theorem 3.1) when $g \geq 3$ and $n \in \{0,1\}$, or $(g,n) = (2,1)$. The generating set consists of all Dehn twits and all "crosscap pushing maps" along simple loops. We review the crosscap pushing map in Section 2. We prove Theorem 3.1 by applying Gervais's argument to Stukow's finite presentation.

2. Preliminaries

2.1. **Relations among Dehn twists and Gervais's presentation.** Let S be either $N_{g,n}$ or $\Sigma_{g,n}$. We denote by $\mathcal{N}_S(A)$ a regular neighborhood of a subset A in S . For every simple closed curve c on S, we choose an orientation of c and fix it throughout this paper. However, for simple closed curves c_1, c_2 on S and $f \in \mathcal{M}(S)$, $f(c_1) = c_2$ means $f(c_1)$ is isotopic to c_2 or the inverse curve of c_2. If S is a non-orientable surface, we also fix an orientation of $\mathcal{N}_S(c)$ for each two-sided simple closed curve c. For a two-sided simple closed curve c on S, denote by t_c the right-handed Dehn twist along c on S. In particular, for a given explicit two-sided

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simple closed curve, an arrow on a side of the simple closed curve indicates the direction of the Dehn twist (See Figure 1).

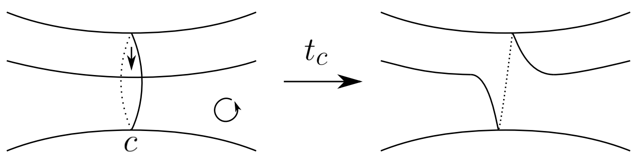


FIGURE 1. The right-handed Dehn twist t_c along a two-sided simple closed curve c on S.

Recall the following relations on $\mathcal{M}(S)$ among Dehn twists along two-sided simple closed curves on S.

Lemma 2.1. *For a two-sided simple closed curve c on S which bounds a disk or a Möbius band in S, we have $t_c = 1$ on $\mathcal{M}(S)$.*

Lemma 2.2 (The braid relation (i))**.** *For a two-sided simple closed curve c on S and $f \in \mathcal{M}(S)$, we have*

$$t_{f(c)}^{\varepsilon_{f(c)}} = f t_c f^{-1},$$

where $\varepsilon_{f(c)} = 1$ if the restriction $f|_{\mathcal{N}_S(c)} : \mathcal{N}_S(c) \to \mathcal{N}_S(f(c))$ is orientation preserving and $\varepsilon_{f(c)} = -1$ if the restriction $f|_{\mathcal{N}_S(c)} : \mathcal{N}_S(c) \to \mathcal{N}_S(f(c))$ is orientation reversing.

When f in Lemma 2.2 is a Dehn twist t_d along a two-sided simple closed curve d and the geometric intersection number $|c \cap d|$ of c and d is m, we denote by T_m the braid relation.

Let $c_1, c_2, \ldots, c_k$ be two-sided simple closed curves on S. The sequence $c_1, c_2, \ldots, c_k$ of simple closed curves on S is the *k-chain on S* if $c_1, c_2, \ldots, c_k$ satisfy $|c_i \cap c_{i+1}| = 1$ for each $i = 1, 2, \ldots, k-1$ and $|c_i \cap c_j| = 0$ for $|j - i| > 1$.

Lemma 2.3 (The k-chain relation)**.** *Let $c_1, c_2, \ldots, c_k$ be a k-chain on S and let δ_1, δ_2 (resp. δ) be distinct boundary components (resp. the boundary component) of $\mathcal{N}_S(c_1 \cup c_2 \cup \cdots \cup c_k)$ when k is odd (resp. even). Then we have*

$$\begin{aligned} (t_{c_1}^{\varepsilon_{c_1}} t_{c_2}^{\varepsilon_{c_2}} \cdots t_{c_k}^{\varepsilon_{c_k}})^{k+1} &= t_{\delta_1}^{\varepsilon_{\delta_1}} t_{\delta_2}^{\varepsilon_{\delta_2}} \quad \text{when } k \text{ is odd}, \\ (t_{c_1}^{\varepsilon_{c_1}} t_{c_2}^{\varepsilon_{c_2}} \cdots t_{c_k}^{\varepsilon_{c_k}})^{2k+2} &= t_{\delta}^{\varepsilon_{\delta}} \quad \text{when } k \text{ is even}, \end{aligned}$$

where $\varepsilon_{c_1}, \varepsilon_{c_2}, \ldots, \varepsilon_{c_k}, \varepsilon_{\delta_1}, \varepsilon_{\delta_2}$ and ε are 1 or -1, and $t_{c_1}^{\varepsilon_{c_1}}, t_{c_2}^{\varepsilon_{c_2}}, \ldots, t_{c_k}^{\varepsilon_{c_k}}, t_{\delta_1}^{\varepsilon_{\delta_1}}$ and $t_{\delta_2}^{\varepsilon_{\delta_2}}$ (resp. $t_{\delta}^{\varepsilon_{\delta}}$) are right-handed Dehn twists for some orientation of $\mathcal{N}_S(c_1 \cup c_2 \cup \cdots \cup c_k)$.

Lemma 2.4 (The lantern relation)**.** *Let Σ be a subsurface of S which is diffeomorphic to $\Sigma_{0,4}$ and let $\delta_{12}, \delta_{23}, \delta_{13}, \delta_1, \delta_2, \delta_3$ and δ_4 be simple closed curves on Σ as in Figure 2. Then we have*

$$t_{\delta_{12}}^{\varepsilon_{\delta_{12}}} t_{\delta_{23}}^{\varepsilon_{\delta_{23}}} t_{\delta_{13}}^{\varepsilon_{\delta_{13}}} = t_{\delta_1}^{\varepsilon_{\delta_1}} t_{\delta_2}^{\varepsilon_{\delta_2}} t_{\delta_3}^{\varepsilon_{\delta_3}} t_{\delta_4}^{\varepsilon_{\delta_4}},$$

where $\varepsilon_{\delta_{12}}, \varepsilon_{\delta_{23}}, \varepsilon_{\delta_{13}}, \varepsilon_{\delta_1}, \varepsilon_{\delta_2}, \varepsilon_{\delta_3}$ and ε_{δ_4} are 1 or -1, and $t_{\delta_{12}}^{\varepsilon_{\delta_{12}}}, t_{\delta_{23}}^{\varepsilon_{\delta_{23}}}, t_{\delta_{13}}^{\varepsilon_{\delta_{13}}}, t_{\delta_1}^{\varepsilon_{\delta_1}}, t_{\delta_2}^{\varepsilon_{\delta_2}}, t_{\delta_3}^{\varepsilon_{\delta_3}}$ and $t_{\delta_4}^{\varepsilon_{\delta_4}}$ are right-handed Dehn twists for some orientation of Σ.

Luo's presentation for $\mathcal{M}(\Sigma_{g,n})$, which is an improvement of Gervais's one, is as follows.

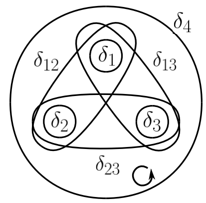


FIGURE 2. Simple closed curves δ_{12}, δ_{23}, δ_{13}, δ_1, δ_2, δ_3 and δ_4 on Σ.

Theorem 2.5 ([3], [10]). *For $g \geq 0$ and $n \geq 0$, $\mathcal{M}(\Sigma_{g,n})$ has the following presentation:*

generators: $\{t_c \mid c:$ s.c.c. on $\Sigma_{g,n}\}$.

relations:

(0′) *$t_c = 1$ when c bounds a disk in $\Sigma_{g,n}$,*

(I′) *All the braid relations T_0 and T_1,*

(II) *All the 2-chain relations,*

(III) *All the lantern relations.*

2.2. **Relations among the crosscap pushing maps and Dehn twists.** Let μ be a one-sided simple closed curve on $N_{g,n}$ and let α be a simple closed curve on $N_{g,n}$ such that μ and α intersect transversely at one point. Recall that α is oriented. For these simple closed curves μ and α, we denote by $Y_{\mu,\alpha}$ a self-diffeomorphism on $N_{g,n}$ which is described as the result of pushing the Möbius band $\mathcal{N}_{N_{g,n}}(\mu)$ once along α. We call $Y_{\mu,\alpha}$ a *crosscap pushing map*. In particular, if α is two-sided, we call $Y_{\mu,\alpha}$ a *Y-homeomorphism* (or *crosscap slide*), where a *crosscap* means a Möbius band in the interior of a surface. We have the following fundamental relation on $\mathcal{M}(N_{g,n})$ and we also call the relation the *braid relation*.

Lemma 2.6 (The braid relation (ii)). *Let μ be a one-sided simple closed curve on $N_{g,n}$ and let α be a simple closed curve on $N_{g,n}$ such that μ and α intersect transversely at one point. For $f \in \mathcal{M}(N_{g,n})$, we have*

$$Y_{f(\mu),f(\alpha)}^{\varepsilon_{f(\alpha)}} = f Y_{\mu,\alpha} f^{-1},$$

where $\varepsilon_{f(\alpha)} = 1$ if the given orientation of $f(\alpha)$ coincides with that of $f(\alpha)$ induced by the orientation of α, and $\varepsilon_{f(\alpha)} = -1$ if the given orientation of $f(\alpha)$ does not coincide with that of $f(\alpha)$ induced by the orientation of α.

We describe crosscap pushing maps as a different view. Let $e : D' \hookrightarrow \mathrm{int}S$ be a smooth embedding of the unit disk $D' \subset \mathbb{C}$. Put $D := e(D')$. Let S' be the surface obtained from $S - \mathrm{int}D$ by the identification of antipodal points of ∂D. We call the manipulation that gives S' from S the *blowup of S on D*. Note that the image $M \subset S'$ of $\mathcal{N}_{S-\mathrm{int}D}(\partial D) \subset S - \mathrm{int}D$ with respect to the blowup of S on D is a crosscap. Conversely, the *blowdown of S' on M* is the following manipulation that gives S from S'. We paste a disk on the boundary obtained by cutting S along the center line μ of M. The blowdown of S' on M is the inverse manipulation of the blowup of S on D.

Let μ be a one-sided simple closed curve on $N_{g,n}$. Note that we obtain $N_{g-1,n}$ from $N_{g,n}$ by the blowdown of $N_{g,n}$ on $\mathcal{N}_{N_{g,n}}(\mu)$. Denote by x_μ the center point of a disk D_μ that is pasted on the boundary obtained by cutting S along μ. Let

$e : D' \hookrightarrow D_\mu \subset N_{g-1,n}$ be a smooth embedding of the unit disk $D' \subset \mathbb{C}$ to $N_{g-1,n}$ such that $D_\mu = e(D')$ and $e(0) = x_\mu$. Let $\mathcal{M}(N_{g-1,n}, x_\mu)$ be the group of isotopy classes of self-diffeomorphisms on $N_{g-1,n}$ fixing the boundary $\partial N_{g-1,n}$ and the point x_μ, where isotopies also fix the boundary $\partial N_{g-1,n}$ and x_μ. Then we have the *blowup homomorphism*

$$\varphi_\mu : \mathcal{M}(N_{g-1,n}, x_\mu) \to \mathcal{M}(N_{g,n})$$

that is defined as follows. For $h \in \mathcal{M}(N_{g-1,n}, x_\mu)$, we take a representative h' of h which satisfies either of the following conditions: (a) $h'|_{D_\mu}$ is the identity map on D_μ, (b) $h'(x) = e(\overline{e^{-1}(x)})$ for $x \in D_\mu$. Such h' is compatible with the blowup of $N_{g-1,n}$ on D_μ, thus $\varphi_\mu(h) \in \mathcal{M}(N_{g,n})$ is induced and well defined (c.f. [14, Subsection 2.3]).

The *point pushing map*

$$j_{x_\mu} : \pi_1(N_{g-1,n}, x_\mu) \to \mathcal{M}(N_{g-1,n}, x_\mu)$$

is a homomorphism that is defined as follows. For $\gamma \in \pi_1(N_{g-1,n}, x_\mu)$, $j_{x_\mu}(\gamma) \in \mathcal{M}(N_{g-1,n}, x_\mu)$ is described as the result of pushing the point x_μ once along γ. Note that for γ_1, $\gamma_2 \in \pi_1(N_{g-1,n})$, $\gamma_1\gamma_2$ means $\gamma_1\gamma_2(t) = \gamma_2(2t)$ for $0 \leq t \leq \frac{1}{2}$ and $\gamma_1\gamma_2(t) = \gamma_1(2t-1)$ for $\frac{1}{2} \leq t \leq 1$.

We define the composition of the homomorphisms:

$$\psi_{x_\mu} := \varphi_\mu \circ j_{x_\mu} : \pi_1(N_{g-1,n}, x_\mu) \to \mathcal{M}(N_{g,n}).$$

For each closed curve α on $N_{g,n}$ which transversely intersects with μ at one point, we take a loop $\overline{\alpha}$ on $N_{g-1,n}$ based at x_μ such that $\overline{\alpha}$ has no self-intersection points on D_μ and α is the image of $\overline{\alpha}$ with respect to the blowup of $N_{g-1,n}$ on D_μ. If α is simple, we take $\overline{\alpha}$ as a simple loop. The next two lemmas follow from the description of the point pushing map (See [7, Lemma 2.2, Lemma 2.3]).

Lemma 2.7. *For a simple closed curve α on $N_{g,n}$ which transversely intersects with a one-sided simple closed curve μ on $N_{g,n}$ at one point, we have*

$$\psi_{x_\mu}(\overline{\alpha}) = Y_{\mu,\alpha}.$$

Lemma 2.8. *For a one-sided simple closed curve α on $N_{g,n}$ which transversely intersects with a one-sided simple closed curve μ on $N_{g,n}$ at one point, we take $\mathcal{N}_{N_{g-1,n}}(\overline{\alpha})$ such that the interior of $\mathcal{N}_{N_{g-1,n}}(\overline{\alpha})$ contains D_μ. Suppose that $\overline{\delta_1}$ and $\overline{\delta_2}$ are distinct boundary components of $\mathcal{N}_{N_{g-1,n}}(\overline{\alpha})$, and δ_1 and δ_2 are two-sided simple closed curves on $N_{g,n}$ which are image of $\overline{\delta_1}$, $\overline{\delta_2}$ with respect to the blowup of $N_{g-1,n}$ on D_μ, respectively. Then we have*

$$Y_{\mu,\alpha} = t_{\delta_1}^{\varepsilon_{\delta_1}} t_{\delta_2}^{\varepsilon_{\delta_2}},$$

where ε_{δ_1} and ε_{δ_2} are 1 or -1, and ε_{δ_1} and ε_{δ_2} depend on the orientations of α, $\mathcal{N}_{N_{g,n}}(\delta_1)$ and $\mathcal{N}_{N_{g,n}}(\delta_2)$ (See Figure 3).

By the definition of the homomorphism ψ_{x_μ} and Lemma 2.7, we have the following lemma.

Lemma 2.9. *Let α and β be simple closed curves on $N_{g,n}$ which transversely intersect with a one-sided simple closed curve μ on $N_{g,n}$ at one point. Suppose the product $\overline{\alpha}\overline{\beta}$ of $\overline{\alpha}$ and $\overline{\beta}$ in $\pi_1(N_{g-1,n}, x_\mu)$ is represented by a simple loop on $N_{g-1,n}$,*

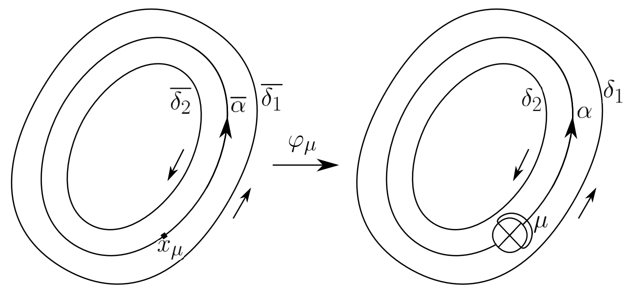


FIGURE 3. If the orientations of α, $\mathcal{N}_{N_{g,n}}(\delta_1)$ and $\mathcal{N}_{N_{g,n}}(\delta_2)$ are as above, then we have $Y_{\mu,\alpha} = t_{\delta_1} t_{\delta_2}^{-1}$. The x-mark as in the figure means the boundary of D_μ identified the antipodal points of ∂D_μ.

and $\alpha\beta$ is a simple closed curve on $N_{g,n}$ which is the image of the representative of $\overline{\alpha}\overline{\beta}$ with respect to the blowup of $N_{g-1,n}$ on D_μ. Then we have

$$Y_{\mu,\alpha\beta} = Y_{\mu,\alpha} Y_{\mu,\beta}.$$

Finally, we recall the following relation between a Dehn twist and a Y-homeomorphism.

Lemma 2.10. *Let α be a two-sided simple closed curve on $N_{g,n}$ which transversely intersect with a one-sided simple closed curve μ on $N_{g,n}$ at one point and let δ be the boundary of $\mathcal{N}_{N_{g,n}}(\alpha \cup \mu)$. Then we have*

$$Y_{\mu,\alpha}^2 = t_\delta^\varepsilon,$$

where ε is 1 or -1, and ε depends on the orientations of α and $\mathcal{N}_{N_{g,n}}(\delta)$ (See Figure 4).

Lemma 2.10 follows from relations in Lemma 2.1, Lemma 2.8 and Lemma 2.9.

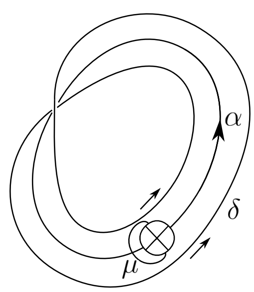


FIGURE 4. If the orientations of α and $\mathcal{N}_{N_{g,n}}(\delta)$ are as above, then we have $Y_{\mu,\alpha}^2 = t_{\delta_1}$.

2.3. **Stukow's finite presentation for $\mathcal{M}(N_{g,n})$.** Let $e_i : D_i' \hookrightarrow \Sigma_0$ for $i = 1, 2, \ldots, g+1$ be smooth embeddings of the unit disk $D' \subset \mathbb{C}$ to a 2-sphere Σ_0 such that $D_i := e_i(D')$ and D_j are disjoint for distinct $1 \leq i, j \leq g+1$. Then we take a model of N_g (resp. $N_{g,1}$) as the surface obtained from Σ_0 (resp. $\Sigma_0 - \mathrm{int} D_{g+1}$) by the blowups on $D_1, \ldots, D_g$ and we describe the identification of ∂D_i by the x-mark as in Figure 5, 6. When $n \in \{0, 1\}$, for $1 \leq i_1 < i_2 < \cdots < i_k \leq g$, let $\gamma_{i_1, i_2, \ldots, i_k}$ be the simple closed curve on $N_{g,n}$ as in Figure 5. Then we define the simple closed curves $\alpha_i := \gamma_{i,i+1}$ for $i = 1, \ldots, g-1$, $\beta := \gamma_{1,2,3,4}$ and $\mu_1 := \gamma_1$ (See Figur 6), and the mapping classes $a_i := t_{\alpha_i}$ for $i = 1, \ldots, g-1$, $b := t_\beta$ and $y := Y_{\mu_1,\alpha_1}$. Then the following finite presentation for $\mathcal{M}(N_{g,n})$ is obtained by Lickorish [8] for $(g,n) = (2,0)$, Stukow [12] for $(g,n) = (2,1)$, Birman-Chillingworth [1] for $(g,n) = (3,0)$ and Stukow [13] for the other (g,n) such that $g \geq 3$ and $n \in \{0,1\}$.

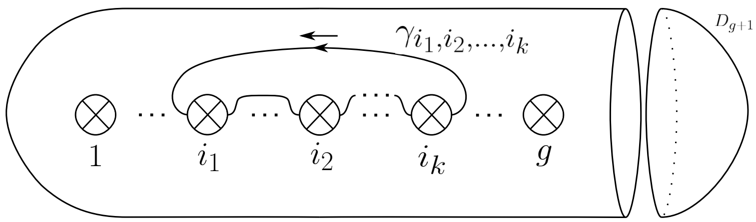


FIGURE 5. Simple closed curve $\gamma_{i_1,i_2,\ldots,i_k}$ on $N_{g,n}$.

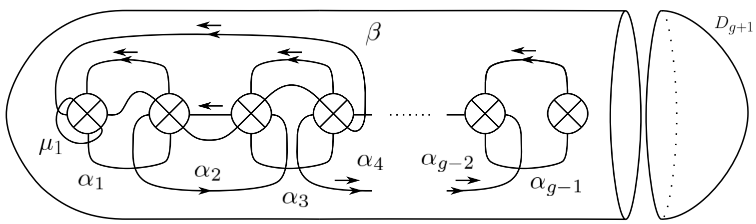


FIGURE 6. Simple closed curves $\alpha_1, \ldots, \alpha_{g-1}$, β and μ_1 on $N_{g,n}$.

Theorem 2.11 ([8], [1], [12], [13])**.** *For $(g,n) = (2,0)$, $(2,1)$ and $(3,0)$, we have the following presentation for $\mathcal{M}(N_{g,n})$:*

$$\begin{aligned}
\mathcal{M}(N_2) &= \langle a_1, y \mid a_1^2 = y^2 = (a_1y)^2 = 1 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \\
\mathcal{M}(N_{2,1}) &= \langle a_1, y \mid y a_1 y^{-1} = a_1^{-1} \rangle, \\
\mathcal{M}(N_3) &= \langle a_1, a_2, y \mid a_1a_2a_1 = a_2a_1a_2, y^2 = (a_1y)^2 = (a_2y)^2 = (a_1a_2)^6 = 1 \rangle.
\end{aligned}$$

If $g \geq 4$ and $n \in \{0,1\}$ or $(g,n) = (3,1)$, then $\mathcal{M}(N_{g,n})$ admits a presentation with generators $a_1, \ldots, a_{g-1}, y$, and b for $g \geq 4$. The defining relations are

(A1) $[a_i, a_j] = 1$ *for $g \geq 4$, $|i-j| > 1$,*
(A2) $a_ia_{i+1}a_i = a_{i+1}a_ia_{i+1}$ *for $i = 1, \ldots, g-2$,*
(A3) $[a_i, b] = 1$ *for $g \geq 4$, $i \neq 4$,*
(A4) $a_4ba_4 = ba_4b$ *for $g \geq 5$,*
(A5) $(a_2a_3a_4b)^{10} = (a_1a_2a_3a_4b)^6$ *for $g \geq 5$,*
(A6) $(a_2a_3a_4a_5a_6b)^{12} = (a_1a_2a_3a_4a_5a_6b)^9$ *for $g \geq 7$,*
(A9a) $[b_2, b] = 1$ *for $g = 6$,*
(A9b) $[a_{g-5}, b_{\frac{g-2}{2}}] = 1$ *for $g \geq 8$ even,*
where $b_0 = a_1$, $b_1 = b$ and
$b_{i+1} = (b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3}b_i)^5(b_{i-1}a_{2i}a_{2i+1}a_{2i+2}a_{2i+3})^{-6}$
for $1 \leq i \leq \frac{g-4}{2}$,

(B1) $y(a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}) = (a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1})y$ *for* $g \geq 4$,
(B2) $y(a_2a_1y^{-1}a_2^{-1}ya_1a_2)y = a_1(a_2a_1y^{-1}a_2^{-1}ya_1a_2)a_1$,
(B3) $[a_i, y] = 1$ *for* $g \geq 4$, $i = 3, \dots, g-1$,
(B4) $a_2(ya_2y^{-1}) = (ya_2y^{-1})a_2$,
(B5) $ya_1 = a_1^{-1}y$,
(B6) $byby^{-1} = \{a_1a_2a_3(y^{-1}a_2y)a_3^{-1}a_2^{-1}a_1^{-1}\}\{a_2^{-1}a_3^{-1}(ya_2y^{-1})a_3a_2\}$ *for* $g \geq 4$,
(B7) $[(a_4a_5a_3a_4a_2a_3a_1a_2ya_2^{-1}a_1^{-1}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_5^{-1}a_4^{-1}), b] = 1$ *for* $g \geq 6$,
(B8) $\{(ya_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})b(a_4a_3a_2a_1y^{-1})\}\{(a_1^{-1}a_2^{-1}a_3^{-1}a_4^{-1})b^{-1}(a_4a_3a_2a_1)\}$
$= \{(a_4^{-1}a_3^{-1}a_2^{-1})y(a_2a_3a_4)\}\{a_3^{-1}a_2^{-1}y^{-1}a_2a_3\}\{a_2^{-1}ya_2\}y^{-1}$ *for* $g \geq 5$,
(C1b) $(a_1a_2\cdots a_{g-1})^g = 1$ *for* $g \geq 4$ *even and* $n = 0$,
(C2) $[a_1, \rho] = 1$ *for* $g \geq 4$ *and* $n = 0$,
where $\rho = (a_1a_2\cdots a_{g-1})^g$ *for* g *odd and*
$\rho = (y^{-1}a_2a_3\cdots a_{g-1}ya_2a_3\cdots a_{g-1})^{\frac{g-2}{2}}y^{-1}a_2a_3\cdots a_{g-1}$ *for* g *even,*
(C3) $\rho^2 = 1$ *for* $g \geq 4$ *and* $n = 0$,
(C4a) $(y^{-1}a_2a_3\cdots a_{g-1}ya_2a_3\cdots a_{g-1})^{\frac{g-1}{2}} = 1$ *for* $g \geq 4$ *odd and* $n = 0$,

where $[x_1, x_2] = x_1x_2x_1^{-1}x_2^{-1}$.

3. Presentation for $\mathcal{M}(N_{g,n})$

The main theorem in this paper is as follows:

Theorem 3.1. *For $g \geq 3$ and $n \in \{0, 1\}$ or $(g, n) = (2, 1)$, $\mathcal{M}(N_{g,n})$ has the following presentation:*
generators: $\{t_c \mid c :$ *two-sided s.c.c. on* $N_{g,n}\}$
$\cup\{Y_{\mu,\alpha} \mid \mu :$ *one-sided s.c.c. on* $N_{g,n}$, $\alpha :$ *s.c.c. on* $N_{g,n}$, $|\mu \cap \alpha| = 1\}$.
Set the generating set by X.
relations:

(0) $t_c = 1$ *when* c *bounds a disk or a Möbius band in* $N_{g,n}$,
(I) *All the braid relations*

$$\begin{cases} (i) & ft_cf^{-1} = t_{f(c)}^{\varepsilon_{f(c)}} \quad \text{for } f \in X, \\ (ii) & fY_{\mu,\alpha}f^{-1} = Y_{f(\mu),f(\alpha)}^{\varepsilon_{f(\alpha)}} \quad \text{for } f \in X, \end{cases}$$

(II) *All the 2-chain relations,*
(III) *All the lantern relations,*
(IV) *All the relations in Lemma 2.9, i.e.* $Y_{\mu,\alpha\beta} = Y_{\mu,\alpha}Y_{\mu,\beta}$,
(V) *All the relations in Lemma 2.8, i.e.* $Y_{\mu,\alpha} = t_{\delta_1}^{\varepsilon_{\delta_1}}t_{\delta_2}^{\varepsilon_{\delta_2}}$.

Remark that Relations (V) are superfluous by rewriting Relations (I)(ii) and (I V) as words of Dehn twists and Y-homeomorphisms.

We set $\overline{X} := \{\overline{f} \mid f \in X\}$, where $\overline{f}$ is an abstract symbol for $f \in X$. Let G be the group whose presentation has the generating set $\overline{X}$ and relations which are obtained from the relations of the presentation in Theorem 3.1 by replacing $f^{\pm 1}$ for $f \in X$ in the relations with $\overline{f}^{\pm 1}$. Denote by $(\overline{0})$, $(\overline{I})$, $(\overline{II})$, $(\overline{III})$, $(\overline{IV})$ and $(\overline{V})$ the relations which are obtained from Relation (0), (I), (II), (III), (IV) and (V) by replacing $f^{\pm 1}$ for $f \in X$ in the relations with $\overline{f}^{\pm 1}$, respectively.

Let $\iota : \Sigma_{h,m} \hookrightarrow N_{g,n}$ be a smooth embedding and let G' be the group whose presentation has all Dehn twists along simple closed curves on $\Sigma_{h,m}$ as generators and Relations (0′), (I′), (II) and (III) in Theorem 2.5. By Theorem 2.5, $\mathcal{M}(\Sigma_{h,m})$ is isomorphic to G', and we have the homomorphism $G' \to G$ defined by the correspondence of t_c to $t_{\iota(c)}^{\varepsilon_{\iota(c)}}$, where $\varepsilon_{\iota(c)} = 1$ if the restriction $\iota|_{\mathcal{N}_{\Sigma_{h,m}}(c)} : \mathcal{N}_{\Sigma_{h,m}}(c) \to \mathcal{N}_{N_{g,n}}(\iota(c))$ is orientation preserving, and $\varepsilon_{\iota(c)} = -1$ if the restriction $\iota|_{\mathcal{N}_{\Sigma_{h,m}}(c)} : \mathcal{N}_{\Sigma_{h,m}}(c) \to \mathcal{N}_{N_{g,n}}(\iota(c))$ is orientation reversing. Then we remark as follows.

Remark 3.2. The composition $\iota_* : \mathcal{M}(\Sigma_{h,m}) \to G$ of the homomorphisms is a homomorphism.

Remark 3.2 means that if a product $t_{c_1}^{\varepsilon_1} t_{c_2}^{\varepsilon_2} \cdots t_{c_k}^{\varepsilon_k}$ of Dehn twists along simple closed curves $c_1, c_2, \ldots, c_k$ on a connected compact orientable subsurface of $N_{g,n}$ is a product of relators on the mapping class group of the orientable subsurface of $N_{g,n}$, then $\overline{t_{c_1}}^{\varepsilon_1} \overline{t_{c_2}}^{\varepsilon_2} \cdots \overline{t_{c_k}}^{\varepsilon_k}$ is a product of relators obtained by Relations $(\overline{0})$, $(\overline{I})$, $(\overline{II})$, $(\overline{III})$.

Set $X^{\pm} := X \cup \{x^{-1} \mid x \in X\}$. By Relation $(\overline{I})$, we have the following lemma.

Lemma 3.3. *For $f \in G$, suppose that $f = \overline{f_1}\ \overline{f_2} \cdots \overline{f_k}$, where $f_1, f_2, \ldots, f_k \in X^{\pm}$. Then we have*

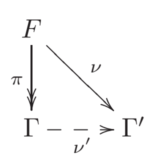
$$\begin{cases} (i) & f\overline{t_c}f^{-1} = \overline{t_{f_1 f_2 \cdots f_k(c)}}^{\varepsilon_{f_1 f_2 \cdots f_k(c)}}, \\ (ii) & f\overline{Y_{\mu,\alpha}}f^{-1} = \overline{Y_{f_1 f_2 \cdots f_k(\mu), f_1 f_2 \cdots f_k(\alpha)}}^{\varepsilon_{f_1 f_2 \cdots f_k(\alpha)}}, \end{cases}$$

where for $f_i \in \{x^{-1} \mid x \in \overline{X}\}$, $\overline{f_i} := \overline{f_i^{-1}}^{-1}$.

The next lemma follows from a argument of the combinatorial group theory (for instance, see [6, Lemma 4.2.1, p42]).

Lemma 3.4. *For groups Γ, Γ' and F, a surjective homomorphism $\pi : F \to \Gamma$ and a homomorphism $\nu : F \to \Gamma'$, we define a map $\nu' : \Gamma \to \Gamma'$ by $\nu'(x) := \nu(\widetilde{x})$ for $x \in \Gamma$, where $\widetilde{x} \in F$ is a lift of x with respect to π (See the diagram below).*

Then if $\ker\pi \subset \ker\nu$, ν' is well-defined and a homomorphism.



Proof of Theorem 3.1. Assume $g \geq 3$ and $n \in \{0, 1\}$ or $(g, n) = (2, 1)$. Then we obtain Theorem 3.1 if $\mathcal{M}(N_{g,n})$ is isomorphic to G. Let $\varphi : G \to \mathcal{M}(N_{g,n})$ be the surjective homomorphism defined by $\varphi(\overline{t_c}) := t_c$ and $\varphi(\overline{Y_{\mu,\alpha}}) := Y_{\mu,\alpha}$.

Set $X_0 := \{a_1, \ldots, a_{g-1}, b, y\} \subset \mathcal{M}(N_{g,n})$ for $g \geq 4$ and $X_0 := \{a_1, \ldots, a_{g-1}, y\} \subset \mathcal{M}(N_{g,n})$ for $g = 2, 3$. Let $F(X_0)$ be the free group which is freely generated by X_0 and let $\pi : F(X_0) \to \mathcal{M}(N_{g,n})$ be the natural projection (by Theorem 2.11). We define the homomorphism $\nu : F(X_0) \to G$ by $\nu(a_i) := \overline{a_i}$ for $i = 1, \ldots, g-1$, $\nu(b) := \overline{b}$ and $\nu(y) := \overline{y}$, and a map $\psi = \nu' : \mathcal{M}(N_{g,n}) \to G$ by $\psi(a_i^{\pm 1}) := \overline{a_i}^{\pm 1}$ for $i = 1, \ldots, g-1$, $\psi(b^{\pm 1}) := \overline{b}^{\pm 1}$, $\psi(y^{\pm 1}) := \overline{y}^{\pm 1}$ and $\psi(f) := \nu(\widetilde{f})$ for the other $f \in \mathcal{M}(N_{g,n})$, where $\widetilde{f} \in F(X_0)$ is a lift of f with respect to π (See the diagram below).

$$\begin{array}{ccc} F(X_0) & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\nu} & \\ \mathcal{M}(N_{g,n}) & \underset{\psi}{\dashrightarrow} & G \end{array}$$

If ψ is a homomorphism, $\varphi \circ \psi = \mathrm{id}_{\mathcal{M}(N_{g,n})}$ by the definition of φ and ψ. Thus it is sufficient for proving that ψ is isomorphism to show that ψ is a homomorphism and surjective.

3.1. **Proof that ψ is a homomorphism.** When (g, n) is either $(2, 1)$ or $(3, 0)$, relations of the presentation in Theorem 2.11 are obtained from Relations $(\overline{0})$, $(\overline{I})$, $(\overline{II})$, $(\overline{IV})$ and $(\overline{V})$, clearly. Thus by Lemma 3.4, ψ is a homomorphism.

Assume $g \geq 4$ or $(g, n) = (3, 1)$. By Lemma 3.4, if the relations of the presentation in Theorem 2.11 are obtained from Relations $(\overline{0})$, $(\overline{I})$, $(\overline{II})$, $(\overline{III})$, $(\overline{IV})$ and $(\overline{V})$, then ψ is a homomorphism.

The group generated by $a_1, \dots, a_{g-1}$ and b with Relations (A1)-(A9b) as defining relations is isomorphic to $\mathcal{M}(\Sigma_{h,1})$ (resp. $\mathcal{M}(\Sigma_{h,2})$) for $g = 2h+1$ (resp. $g = 2h+2$) by Theorem 3.1 in [11], and Relations (A1)-(A9b) are relations on the mapping class group of the orientable subsurface $\mathcal{N}_{N_{g,n}}(\alpha_1 \cup \cdots \cup \alpha_{g-1})$ of $N_{g,n}$. Hence Relations (A1)-(A9b) are obtained from Relations $(\overline{0})$, $(\overline{I})$, $(\overline{II})$, $(\overline{III})$ by Remark 3.2.

Stukow [13] gave geometric interpretations for Relations (B1)-(B8) in Section 4 in [13]. By the interpretation, Relations (B1), (B2), (B3), (B4), (B5), (B7) are obtained from Relations $(\overline{I})$ (Use Lemma 3.3), Relation (B6) is obtained from Relations $(\overline{0})$, $(\overline{I})$, $(\overline{III})$, $(\overline{IV})$ and $(\overline{V})$ (Use Lemma 2.10 and Lemma 3.3), and Relation (B8) is obtained from Relations $(\overline{I})$, $(\overline{IV})$ and $(\overline{V})$ (Use Lemma 3.3). Thus ψ is a homomorphism when $n = 1$.

We assume $n = 0$. By Remark 3.2, k-chain relations are obtained from Relations $(\overline{0})$, $(\overline{I})$, $(\overline{II})$ and $(\overline{III})$ for each k. Relation (C1b) is interpreted in G as follows.

$$(a_1 a_2 \cdots a_{g-1})^g \overset{(\overline{0}),(\overline{I}),(\overline{II}),(\overline{III})}{=} t_{\gamma_{1,2,\dots,g}} t^{-1}_{\gamma_{1,2,\dots,g}} = 1.$$

Thus Relation (C1b) is obtained from Relations $(\overline{0})$, $(\overline{I})$, $(\overline{II})$ and $(\overline{III})$.

Relation (C2) is obtained from Relations $(\overline{I})$ by Lemma 3.3, clearly.

When g is odd, by using the $(g-1)$-chain relation, Relation (C3) is interpreted in G as follows.

$$\rho^2 = (a_1 a_2 \cdots a_{g-1})^{2g} \overset{(\overline{0}),(\overline{I}),(\overline{II}),(\overline{III})}{=} t^{\varepsilon}_{\partial \mathcal{N}_{N_g}(\gamma_{1,2,\dots,g})} \overset{(\overline{0})}{=} 1,$$

where ε is 1 or -1. Note that $\mathcal{N}_{N_g}(\gamma_{1,2,\dots,g})$ is a Möbius band in N_g. Thus Relation (C3) is obtained from Relations $(\overline{0})$, $(\overline{I})$, $(\overline{II})$ and $(\overline{III})$ when g is odd.

When g is even, we rewrite the left-hand side ρ^2 of Relation (C3) by braid relations. Set $A := a_2 a_3 \cdots a_{g-1}$. Note that

$$Y_{\mu_1,\gamma_{1,2,3}}(a_2 \cdots a_{2i} a_2 \cdots a_{2i-1} Y_{\mu_1,\gamma_{1,2,\dots,2i-1}} a^{-1}_{2i-1} \cdots a^{-1}_2 a^{-1}_{2i} \cdots a^{-1}_2) = Y_{\mu_1,\gamma_{1,2,\dots,2i+1}}$$

for $i = 2, \dots, \frac{g-2}{2}$ by Relation $(\overline{I})$, $(\overline{IV})$, and then we have

$$\begin{aligned} & \rho \\ = \quad & y^{-1} A (yA\underleftarrow{y^{-1}}A)^{\frac{g-2}{2}} \end{aligned}$$

$$
\begin{aligned}
&\overset{(\overline{I})}{=} && y^{-1}A(ya_2y^{-1}a_3\cdots a_{g-1}A)^{\frac{g-2}{2}}\\
&= && y^{-1}A(\underline{y(a_2y^{-1}a_2^{-1})}A^2)^{\frac{g-2}{2}}\\
&\overset{(\overline{I}),(\overline{IV})}{=} && y^{-1}A(Y_{\mu_1,\gamma_{1,2,3}}A^2)^{\frac{g-2}{2}}.\\
&= && y^{-1}AY_{\mu_1,\gamma_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma_{1,2,3}}A^2\underleftarrow{Y_{\mu_1,\gamma_{1,2,3}}}A^2\\
&\overset{(\overline{I})}{=} && y^{-1}AY_{\mu_1,\gamma_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma_{1,2,3}}Aa_2a_3Y_{\mu_1,\gamma_{1,2,3}}a_4\cdots a_{g-1}A^2\\
&= && y^{-1}AY_{\mu_1,\gamma_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma_{1,2,3}}A(\underleftarrow{a_2a_3Y_{\mu_1,\gamma_{1,2,3}}a_3^{-1}a_2^{-1}})A^3\\
&\overset{(\overline{I})}{=} && y^{-1}AY_{\mu_1,\gamma_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma_{1,2,3}}a_2a_3a_4(a_2a_3Y_{\mu_1,\gamma_{1,2,3}}a_3^{-1}a_2^{-1})a_5\cdots a_{g-1}A^3\\
&= && y^{-1}AY_{\mu_1,\gamma_{1,2,3}}A^2\cdots \underline{Y_{\mu_1,\gamma_{1,2,3}}(a_2a_3a_4a_2a_3Y_{\mu_1,\gamma_{1,2,3}}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_2^{-1})}A^4\\
&\overset{(\overline{I}),(\overline{IV})}{=} && y^{-1}AY_{\mu_1,\gamma_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma_{1,2,3}}A^2\underleftarrow{Y_{\mu,\gamma_{1,2,3,4,5}}}A^4\\
&\overset{(\overline{I})}{=} && y^{-1}AY_{\mu_1,\gamma_{1,2,3}}A^2\cdots \underline{Y_{\mu_1,\gamma_{1,2,3}}}\\
& && \underline{\cdot(a_2a_3a_4a_5a_6a_2a_3a_4a_5Y_{\mu,\gamma_{1,2,3,4,5}}a_5^{-1}a_4^{-1}a_3^{-1}a_2^{-1}a_6^{-1}a_5^{-1}a_4^{-1}a_3^{-1}a_2^{-1})}A^6\\
&\overset{(\overline{I}),(\overline{IV})}{=} && y^{-1}AY_{\mu_1,\gamma_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma_{1,2,3,4,5,6,7}}A^6\\
&\vdots && \\
&= && y^{-1}AY_{\mu_1,\gamma_{1,2,\dots,g-1}}A^{g-2}\\
&= && \underline{y^{-1}\cdot AY_{\mu_1,\gamma_{1,2,\dots,g-1}}A^{-1}}\cdot A^{g-1}\\
&\overset{(\overline{I}),(\overline{IV})}{=} && Y_{\mu_1,\gamma_{1,2,\dots,g}}A^{g-1}.
\end{aligned}
$$

Since $Y_{\mu_1,\gamma_{1,2,\dots,g}}$ commutes with a_i for $i=2,\dots,g-1$, and $\partial\mathcal{N}_{N_g}(\mu_1\cup\gamma_{1,2,\dots,g}) = \partial\mathcal{N}_{N_g}(\alpha_2\cup\cdots\cup\alpha_{g-1})$ (See Figure 7), we have

$$
\begin{aligned}
\rho^2 &= Y_{\mu_1,\gamma_{1,2,\dots,g}}A^{g-1}Y_{\mu_1,\gamma_{1,2,\dots,g}}A^{g-1}\\
&\overset{(\overline{I})}{=} Y^2_{\mu_1,\gamma_{1,2,\dots,g}}A^{2g-2}\\
&\overset{(\overline{0}),(\overline{I}),(\overline{II}),(\overline{III})}{=} Y^2_{\mu_1,\gamma_{1,2,\dots,g}}t_{\partial\mathcal{N}_{N_g}(\alpha_2\cup\cdots\cup\alpha_{g-1})}\\
&\overset{\text{Lem. 2.10}}{=} t^{-1}_{\partial\mathcal{N}_{N_g}(\alpha_2\cup\cdots\cup\alpha_{g-1})}t_{\partial\mathcal{N}_{N_g}(\alpha_2\cup\cdots\cup\alpha_{g-1})}\\
&= 1.
\end{aligned}
$$

Thus Relation (C3) is obtained from Relations $(\overline{0})$, $(\overline{I})$, $(\overline{II})$, $(\overline{IV})$ and $(\overline{V})$ when g is even.

Finally, we also rewrite the left-hand side $(y^{-1}a_2a_3\cdots a_{g-1}ya_2a_3\cdots a_{g-1})^{\frac{g-1}{2}}$ of Relation (C4a) by braid relations. Remark that g is odd. For $1\leq i_1<i_2<\cdots<i_k\leq g$, we denote by $\gamma'_{i_1,i_2,\dots,i_k}$ the simple closed curve on $N_{g,n}$ as in Figure 8. Note that

$$
Y_{\mu_1,\gamma'_{1,2,\dots,2i+1}} = Y_{\mu_1,\gamma'_{1,2,3}}(a_2\cdots a_{2i}a_2\cdots a_{2i-1}Y_{\mu_1,\gamma'_{1,2,\dots,2i-1}}a_{2i-1}^{-1}\cdots a_2^{-1}a_{2i}^{-1}\cdots a_2^{-1})
$$

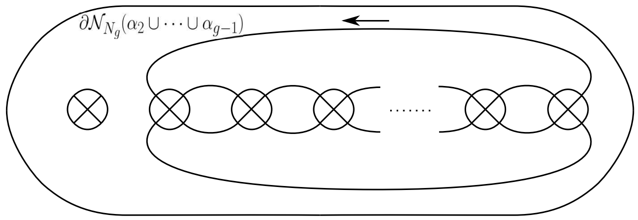


FIGURE 7. Simple closed curve $\partial\mathcal{N}_{N_g}(\alpha_2 \cup \cdots \cup \alpha_{g-1})$ on N_g.

for $i = 2, \ldots, \frac{g-1}{2}$ and $\partial\mathcal{N}_{N_g}(\mu_1 \cup \gamma_{1,2,\ldots,g}) = d_1 \sqcup d_2$, and by a similar argument for Relation (C3) when g is even, we have

$$\begin{aligned}
& (y^{-1}a_2a_3\cdots a_{g-1}ya_2a_3\cdots a_{g-1})^{\frac{g-1}{2}} \\
= \;& (y^{-1}A\,\underleftarrow{y}\,A)^{\frac{g-1}{2}} \\
\overset{(\overline{I})}{=} \;& (\underline{y^{-1}(a_2ya_2^{-1})}A^2)^{\frac{g-1}{2}} \\
\overset{(\overline{I}),(\overline{IV})}{=} \;& (Y_{\mu_1,\gamma'_{1,2,3}}A^2)^{\frac{g-1}{2}} \\
= \;& Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma'_{1,2,3}}A^2\underleftarrow{Y_{\mu_1,\gamma'_{1,2,3}}}A^2 \\
\overset{(\overline{I})}{=} \;& Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma'_{1,2,3}}A^2 \\
& \underline{\cdot Y_{\mu_1,\gamma'_{1,2,3}}(a_2a_3a_4a_2a_3Y_{\mu_1,\gamma'_{1,2,3}}a_3^{-1}a_2^{-1}a_4^{-1}a_3^{-1}a_2^{-1})}A^4 \\
\overset{(\overline{I}),(\overline{IV})}{=} \;& Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots Y_{\mu_1,\gamma'_{1,2,3}}A^2\underleftarrow{Y_{\mu_1,\gamma'_{1,2,3,4,5}}}A^4 \\
\overset{(\overline{I})}{=} \;& Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots \underline{Y_{\mu_1,\gamma'_{1,2,3}}} \\
& \underline{\cdot(a_2a_3a_4a_5a_6a_2a_3a_4a_5Y_{\mu,\gamma'_{1,2,3,4,5}}a_5^{-1}a_4^{-1}a_3^{-1}a_2^{-1}a_6^{-1}a_5^{-1}a_4^{-1}a_3^{-1}a_2^{-1})}A^6 \\
\overset{(\overline{I}),(\overline{IV})}{=} \;& Y_{\mu_1,\gamma'_{1,2,3}}A^2\cdots Y_{\mu,\gamma'_{1,2,3,4,5,6,7}}A^6 \\
\vdots \;& \\
= \;& Y_{\mu,\gamma_{1,2,\ldots,g}}A^{g-1} \\
\overset{(\overline{II})}{=} \;& Y_{\mu,\gamma_{1,2,\ldots,g}}t_{d_1}t_{d_2} \\
\overset{(\overline{V})}{=} \;& t_{d_1}^{-1}t_{d_2}^{-1}t_{d_1}t_{d_2} \\
\overset{(\overline{I})}{=} \;& 1,
\end{aligned}$$

where simple closed curves d_1 and d_2 are boundary components of $\mathcal{N}_{N_g}(\alpha_2 \cup \cdots \cup \alpha_{g-1})$ as in Figure 9. Therefore Relation (C4a) is obtained from Relations $(\overline{I})$, $(\overline{II})$, $(\overline{IV})$ and $(\overline{V})$, and $\psi : \mathcal{M}(N_{g,n}) \to G$ is a homomorphism.

3.2. **Surjectivity of ψ.** We show that there exist lifts of $\overline{t_c}$'s and $\overline{Y_{\mu,\alpha}}$'s with respect to ψ for cases below, to prove the surjectivity of ψ.

(1) $\overline{t_c}$; c is non-separating and $N_{g,n} - c$ is non-orientable,

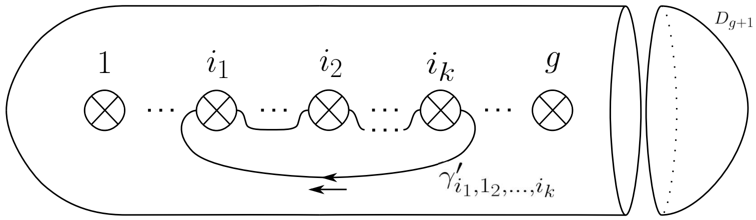


FIGURE 8. Simple closed curve $\gamma'_{i_1,i_2,\dots,i_k}$ on $N_{g,n}$.

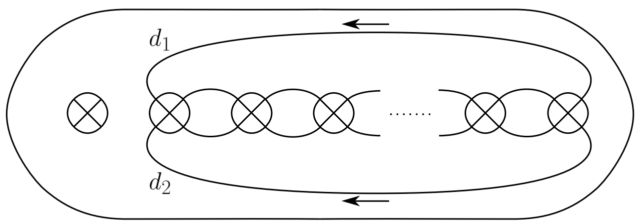


FIGURE 9. Simple closed curve d_1 and d_2 on $N_{g,n}$.

(2) $\overline{t_c}$; c is non-separating and $N_{g,n}-c$ is orientable,
(3) $\overline{t_c}$; c is separating,
(4) $\overline{Y_{\mu,\alpha}}$; α is two-sided and $N_{g,n}-\alpha$ is non-orientable,
(5) $\overline{Y_{\mu,\alpha}}$; α is two-sided and $N_{g,n}-\alpha$ is orientable,
(6) $\overline{Y_{\mu,\alpha}}$; α is one-sided.

Set $X_0^{\pm} := X_0 \cup \{x^{-1} \mid x \in X_0\}$, and for a simple closed curve c on $N_{g,n}$, we denote by $(N_{g,n})_c$ the surface obtained from $N_{g,n}$ by cutting $N_{g,n}$ along c.

Case (1). Since $(N_{g,n})_c$ is diffeomorphic to $N_{g-2,n+2}$ and $g \geq 3$, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \cdots, f_k \in X_0^{\pm}$ such that $f(\alpha_1) = c$. Note that $\psi(f_i) = \overline{f_i} \in \overline{X}^{\pm} = \overline{X} \cup \{x^{-1} \mid x \in \overline{X}\}$ for $i = 1, 2, \dots, k$. Thus we have

$$\begin{aligned}\psi(fa_1f^{-1}) &= \psi(f)\psi(a_1)\psi(f)^{-1}\\ &= \overline{f_1}\,\overline{f_2}\cdots\overline{f_k}\,\overline{a_1}\,\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1}\\ &\overset{\text{Lem. 3.3}}{=} \overline{t_{f(\alpha_1)}}^{\varepsilon}\\ &= \overline{t_c}^{\varepsilon},\end{aligned}$$

where ε is 1 or -1. Thus $fa_1^{\varepsilon}f^{-1}$ is a lift of $\overline{t_c}$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

Case (2). We remark that g is even in this case. When $g = 2$, such a simple closed curve c is unique and $c = \alpha_1$. Thus a_1 is the lift of $\overline{t_c}$ with respect to ψ. When $g = 4$, since $(N_{g,n})_c$ is diffeomorphic to $\Sigma_{1,n+2}$, there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \cdots, f_k \in X_0^{\pm}$ such that $f(\beta) = c$. By a similar argument in Case (1), $fb^{\varepsilon}f^{-1}$ is a lift of $\overline{t_c}$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

Assume $g \geq 6$ even. Then there exists a product $f = f_1 f_2 \cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \cdots, f_k \in X_0^{\pm}$ such that $f(\gamma_{1,2,\dots,g}) = c$. Since $\alpha_1 \cup \alpha_3 \cup \gamma_{5,6,\dots,g} \cup \gamma_{1,2,\dots,g}$ bounds a subsurface of $N_{g,n}$ which is diffeomorphic to $\Sigma_{0,4}$ (See Figure 10), we have $bt_{\gamma_{3,4,\dots,g}}t_{\gamma_{1,2,5,\dots,g}} = t_{\gamma_{1,2,\dots,g}}a_1a_3t_{\gamma_{5,6,\dots,g}}$ by a lantern relation. Note that b, $t_{\gamma_{3,4,\dots,g}}$, $t_{\gamma_{1,2,5,\dots,g}}$, a_1, a_3, $t_{\gamma_{5,6,\dots,g}}$ are Dehn twists of type (1), and $\overline{t_{\gamma_{3,4,\dots,g}}}$,

$\overline{t_{\gamma_{1,2,5,\ldots,g}}}$, $\overline{t_{\gamma_{5,6,\ldots,g}}}$ have lifts h_1, h_2, $h_3 \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\begin{aligned} & \psi(fbh_1h_2a_1^{-1}a_3^{-1}h_3^{-1}f^{-1}) \\ = \; & \overline{f_1}\;\overline{f_2}\cdots\overline{f_k}\;\overline{b}\overline{t_{\gamma_{3,4,\ldots,g}}}\;\overline{t_{\gamma_{1,2,5,\ldots,g}}}\overline{a_1}^{-1}\overline{a_3}^{-1}\overline{t_{\gamma_{5,6,\ldots,g}}}^{-1}\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1} \\ \overset{(\overline{\mathit{III}})}{=} \; & \overline{f_1}\;\overline{f_2}\cdots\overline{f_k}\overline{t_{\gamma_{1,2,\ldots,g}}}\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1} \\ \overset{\text{Lem. 3.3}}{=} \; & \overline{t_c}^{\varepsilon}, \end{aligned}$$

where ε is 1 or -1. Thus $f(bh_1h_2a_1^{-1}a_3^{-1}h_3^{-1})^{\varepsilon}f^{-1}$ is a lift of $\overline{t_c}$ with respect to ψ for some $\{-1,1\}$.

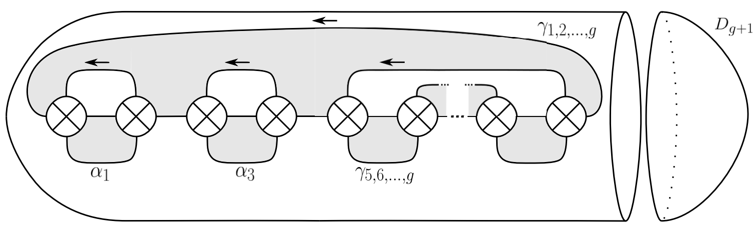


Figure 10. $\alpha_1\cup\alpha_3\cup\gamma_{5,6,\ldots,g}\cup\gamma_{1,2,\ldots,g}$ bound a subsurface of $N_{g,n}$ which is diffeomorphic to $\Sigma_{0,4}$.

Case (3). Let Σ be the component of $(N_{g,n})_c$ which has one boundary component. When Σ is orientable, there exists a k-chain c_1, c_2, $\ldots$, c_k on $N_{g,n}$ such that $\mathcal{N}_{N_{g,n}}(c_1\cup c_2\cup\cdots\cup c_k)=\Sigma$. By the chain relation, $(t_{c_1}^{\varepsilon_1}t_{c_2}^{\varepsilon_2}\cdots t_{c_k}^{\varepsilon_k})^{2k+2}=t_c$ for some ε_1, ε_2, $\ldots$, $\varepsilon_k\in\{-1,1\}$. Note that t_{c_1}, t_{c_2}, $\ldots$, t_{c_k} are Dehn twists of type (1) and $\overline{t_{c_1}}$, $\overline{t_{c_2}}$, $\ldots$, $\overline{t_{c_k}}$ have lifts h_1, h_2, $\ldots$, $h_k\in\mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\begin{aligned} \psi((h_1^{\varepsilon_1}h_2^{\varepsilon_2}\ldots h_k^{\varepsilon_k})^{2k+2}) \; & = \; (\overline{t_{c_1}}^{\varepsilon_1}\overline{t_{c_2}}^{\varepsilon_2}\cdots\overline{t_{c_k}}^{\varepsilon_k})^{2k+2} \\ & \overset{(\overline{0}),(\overline{\mathit{I}}),(\overline{\mathit{II}}),(\overline{\mathit{III}})}{=} \; \overline{t_c}. \end{aligned}$$

Thus $(h_1^{\varepsilon_1}h_2^{\varepsilon_2}\ldots h_k^{\varepsilon_k})^{2k+2}$ is a lift of $\overline{t_c}$ with respect to ψ.

When Σ is non-orientable, we proceed by induction on the genus g' of Σ. For $g'=1$, $\overline{t_c}=1$ by Relation $(\overline{0})$. When $g'=2$, there exists a product $f=f_1f_2\cdots f_k\in\mathcal{M}(N_{g,n})$ of f_1, f_2, $\cdots$, $f_k\in X_0^{\pm}$ such that $f(\partial\mathcal{N}_{N_{g,n}}(\mu_1\cup\alpha_1))=c$. Hence $fy^2f^{-1}=t_c^{\varepsilon}$ for some $\varepsilon\in\{-1,1\}$. Then we have

$$\begin{aligned} \psi(fy^2f^{-1}) \; & = \; \overline{f_1}\;\overline{f_2}\cdots\overline{f_k}\;\overline{y}^2\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1} \\ & \overset{\text{Lem. 2.10}}{=} \; \overline{f_1}\;\overline{f_2}\cdots\overline{f_k}\;\overline{t_{\partial\mathcal{N}_{N_{g,n}}(\mu_1\cup\alpha_1)}}^{\varepsilon}\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1} \\ & \overset{\text{Lem. 3.3}}{=} \; \overline{t_c}^{\varepsilon'}, \end{aligned}$$

where ε' is 1 or -1. Thus $fy^{2\varepsilon'}f^{-1}$ is a lift of $\overline{t_c}$ with respect to ψ for some $\varepsilon'\in\{-1,1\}$.

Suppose that $g'\geq 3$ and c' is the separating simple closed curve on $N_{g,n}$ as in Figure 11. Then there exists a product $f=f_1f_2\cdots f_k\in\mathcal{M}(N_{g,n})$ of f_1, f_2, $\cdots$, $f_k\in X_0^{\pm}$ such that $f(c')=c$. Denote by c_i for $i=1$, 2, $\ldots$, 6 the separating simple closed curves on $f(\Sigma)$ as in Figure 11. Note that $c'\cup c_4\cup c_5\cup c_6$ bounds a subsurface of $f(\Sigma)$ which is diffeomorphic to $\Sigma_{0,4}$, and each c_i for $i=1$, 2, $\ldots$, 6 bounds a subsurface of $f(\Sigma)$ which is diffeomorphic to a non-orientable surface of genus $g_i < g'$ with one boundary component. By the inductive assumption, $\overline{t_{c_1}}$, $\overline{t_{c_2}}$, $\overline{t_{c_3}}$, $\overline{t_{c_4}}$ have lifts h_1, h_2, h_3 and $h_4 \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\begin{aligned}\psi(fh_1h_2h_3h_4^{-1}f^{-1}) &= \overline{f_1}\,\overline{f_2}\cdots\overline{f_k}\overline{t_{c_1}}\,\overline{t_{c_2}}\,\overline{t_{c_3}}\,\overline{t_{c_4}}^{-1}\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1}\\ &\stackrel{(\overline{0}),(\overline{I\!I\!I})}{=} \overline{f_1}\,\overline{f_2}\cdots\overline{f_k}\overline{t_{c'}}\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1}\\ &\stackrel{\text{Lem. 3.3}}{=} \overline{t_c}^{\varepsilon},\end{aligned}$$

where ε is 1 or -1. Thus $f(h_1h_2h_3h_4^{-1})^{\varepsilon}f^{-1}$ is a lift of $\overline{t_c}$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

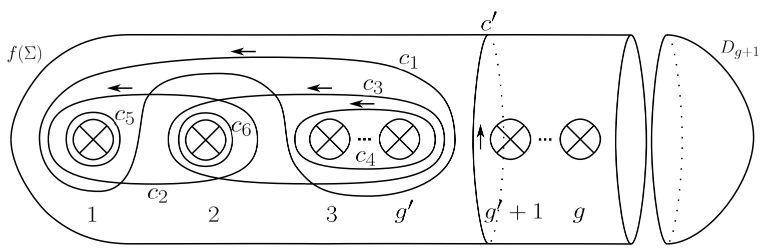


FIGURE 11. Simple closed curves c' and c_i for $i = 1, 2, \ldots, 6$ on $f(\Sigma)$.

Case (4). Since $N_{g,n} - \mathrm{int}\mathcal{N}_{N_{g,n}}(\mu\cup\alpha)$ is diffeomorphic to $N_{g-2,n+1}$ and the two-sided simple closed curve on $N_{2,1}$ is unique, there exists a product $f = f_1f_2\cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \cdots, f_k \in X_0^{\pm}$ such that $f(\alpha_1) = \alpha$ and $f(\mu_1) = \mu$. Thus we have

$$\begin{aligned}\psi(fyf^{-1}) &= \overline{f_1}\,\overline{f_2}\cdots\overline{f_k}\overline{y}\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1}\\ &\stackrel{\text{Lem. 3.3}}{=} \overline{Y_{\mu,\alpha}}^{\varepsilon},\end{aligned}$$

where ε is 1 or -1. Thus $fy^{\varepsilon}f^{-1}$ is a lift of $\overline{Y_{\mu,\alpha}}$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

Case (5). We remark that g is even in this case. Since $N_{g,n} - \mathrm{int}\mathcal{N}_{N_{g,n}}(\mu\cup\alpha)$ is diffeomorphic to $\Sigma_{\frac{g-2}{2},n+1}$ and the two-sided simple closed curve on $N_{2,1}$ is unique, there exists a product $f = f_1f_2\cdots f_k \in \mathcal{M}(N_{g,n})$ of $f_1, f_2, \cdots, f_k \in X_0^{\pm}$ such that $f(\gamma_{1,2,\ldots,g}) = \alpha$ and $f(\mu_1) = \mu$. Note that $Y_{\mu_1,\gamma_{1,2}}, Y_{\mu_1,\gamma_{1,3}}, \ldots, Y_{\mu_1,\gamma_{1,g}}$ are Y-homeomorphisms of type (4), and $\overline{Y_{\mu_1,\gamma_{1,3}}}, \overline{Y_{\mu_1,\gamma_{1,4}}}, \ldots, \overline{Y_{\mu_1,\gamma_{1,g}}}$ have lifts h_3, h_4, $\ldots$, $h_g \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\begin{aligned}&\psi(fh_g\ldots h_4h_3yf^{-1})\\ =\ &\overline{f_1}\,\overline{f_2}\cdots\overline{f_k}\,\overline{Y_{\mu_1,\gamma_{1,g}}}\ldots\overline{Y_{\mu_1,\gamma_{1,4}}}\,\overline{Y_{\mu_1,\gamma_{1,3}}}\overline{y}\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1}\\ \stackrel{(\overline{IV})}{=}\ &\overline{f_1}\,\overline{f_2}\cdots\overline{f_k}\,\overline{Y_{\mu_1,\gamma_{1,2,\ldots,g}}}\,\overline{f_k}^{-1}\cdots\overline{f_2}^{-1}\overline{f_1}^{-1}\\ \stackrel{\text{Lem. 3.3}}{=}\ &\overline{Y_{\mu,\alpha}}^{\varepsilon},\end{aligned}$$

where ε is 1 or -1. Thus $f(h_g\ldots h_4h_3y)^{\varepsilon}f^{-1}$ is a lift of $\overline{Y_{\mu,\alpha}}$ with respect to ψ for some $\varepsilon \in \{-1, 1\}$.

Case (6). Let δ_1, δ_2 be two-sided simple closed curves on $N_{g,n}$ such that $\delta_1 \sqcup \delta_2 = \partial\mathcal{N}_{N_{g,n}}(\mu\cap\alpha)$. By Lemma 2.8, we have $Y_{\mu,\alpha} = t_{\delta_1}^{\varepsilon_1}t_{\delta_2}^{\varepsilon_2}$ for some ε_1 and

$\varepsilon_2 \in \{-1, 1\}$, and by above arguments, $\overline{t_{c_1}}$, $\overline{t_{c_2}}$ have lifts h_1 and $h_2 \in \mathcal{M}(N_{g,n})$ with respect to ψ, respectively. Thus we have

$$\begin{aligned} \psi(h_1^{\varepsilon_1} h_2^{\varepsilon_2}) &= \overline{t_{c_1}}^{\varepsilon_1} \overline{t_{c_2}}^{\varepsilon_2} \\ &\overset{(\overline{V})}{=} \overline{Y_{\mu,\alpha}}. \end{aligned}$$

Thus $h_1^{\varepsilon_1} h_2^{\varepsilon_2}$ is a lift of $\overline{Y_{\mu,\alpha}}$ with respect to ψ and $\psi : \mathcal{M}(N_{g,n}) \to G$ is surjective. We have completed the proof of Theorem 3.1.

□

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