An adaptive algorithm based on the shifted inverse iteration for the Steklov eigenvalue problem

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Abstract : This paper proposes and analyzes an a posteriori error estimator for the finite element multi-scale discretization approximation of the Steklov eigenvalue problem. Based on the a posteriori error estimates, an adaptive algorithm of shifted inverse iteration type is designed. Finally, numerical experiments comparing the performances of three kinds of different adaptive algorithms are provided, which illustrate the efficiency of the adaptive algorithm proposed here.

Keywords : Steklov eigenvalue problem, finite element, multi-scale discretization, adaptive algorithm, a posteriori error estimate.

AMS subject classifications.65N25, 65N30, 65N15

1 Introduction

In recent years, numerical methods for Steklov eigenvalue problems have attracted more and more scholars' attention (see, e.g.,[3, 4, 5, 6, 9, 13, 21, 26, 27, 28, 31, 36, 40]). It is well known that in the numerical approximation of partial differential equations, the adaptive procedures based on a posteriori error estimates, due to the less computational cost and time, are the mainstream direction and have gained an enormous importance. The aim of this paper is to propose and analyze an a posteriori error estimator for the finite element multi-scale discretization approximation of the Steklov eigenvalue problem, based on which an adaptive algorithm is designed.

As for eigenvalue problems, till now, there are basically three ways to design adaptive algorithms as follows in which the a posterior error estimators are more or less the same but the equations solved in each iteration are different: I. Solve the original eigenvalue problem at each iteration. The convergence and optimality of this adaptive procedure has been studied in [18, 19, 21]. II. Inverse iteration type. [14, 17, 32, 34, 35, 38] have studied and obtained the convergence of this method. III. Shifted inverse iteration type (see [23, 29, 41]). This paper studies the third type of adaptive method for the Steklov eigenvalue problem, and the special features are:

(1) As for the Steklov eigenvalue problem, so far, it has been discussed the first type of adaptive method of combining the a posteriori error estimates and adaptivity (see [21] or Algorithm 4.1 in this paper). As far as our information goes, there has not been any report on the other two kinds of adaptive methods. This paper designs the third type of adaptive algorithm based on the a posterior error estimates (see Algorithm 4.3 in this paper). Here we propose an a posteriori error estimator of residual type and give not only the global upper bound but also the local lower bound of the error which is important for the adaptive procedure.

(2) [29] established and analyzed the a posteriori error estimates of the multi-scale discretization scheme for second order self-adjoint elliptic eigenvalue problems with homogenous Dirichlet boundary value condition by means of the a posteriori error estimates of the associated boundary value problem and left the local lower bound of the error unproved, while this paper studies the a posteriori error estimates of the multi-scale discretization scheme by using the a posteriori error estimates of finite element eigenfunctions directly, and obtains the local lower bound of the error.

(3) Numerical experiments comparing the performances of three kinds of different adaptive methods mentioned above are provided. It can be seen from the numerical results that the adaptive algorithm of shifted inverse iteration type has advantages over the other two kinds. More precisely, comparing with the first type, to achieve the same accurate approximation, our method uses less computational time; and our adaptive algorithm can be used to seek efficiently approximations of any eigenpair of the Steklov eigenvalue problem, however, the algorithm of the second type (see Algorithm 4.2 in this paper) is only suitable to the smallest eigenvalue.

The rest of this paper is organized as follows. In section 2, some preliminaries needed in this paper are introduced. In section 3, a multi-scale discretization scheme is presented, and its a priori and a posteriori error estimates are given and analyzed, respectively. In section 4, three kinds of adaptive algorithms for the Steklov eigenvalue problem are presented, and finally numerical experiments are provided which illustrate the advantages of our algorithm.

2 Preliminaries

Let $H^t(\Omega)$ and $H^t(\partial\Omega)$ denote Sobolev spaces on Ω and $\partial\Omega$ with real order t, respectively. The norm in $H^t(\Omega)$ and $H^t(\partial\Omega)$ are denoted by $\|\cdot\|_t$ and $\|\cdot\|_{t,\partial\Omega}$, respectively. $H^0(\partial\Omega) = L_2(\partial\Omega)$.

In this paper, we will write $a \lesssim b$ to indicate that $a \leqslant Cb$ with $C > 0$ being a constant depending on the data of the problem but independent of meshes generated by the adaptive algorithm.

We consider the following Steklov eigenvalue problem

$$-\Delta u + u = 0 \ \ in\ \Omega, \qquad \frac{\partial u}{\partial n} = \lambda u \ \ on\ \partial\Omega, \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain with θ being the largest inner angle of Ω and $\frac{\partial u}{\partial n}$ is the outward normal derivative.

The weak form of (2.1) is given by: find $\lambda \in \mathbb{R}$, $u \in H^1(\Omega)$ with $\|u\|_1 = 1$, such that

$$a(u,v) = \lambda b(u,v), \quad \forall v \in H^1(\Omega), \tag{2.2}$$

where

$$\begin{aligned} a(u,v) &= \int\limits_{\Omega} \nabla u \cdot \nabla v + uv dx,\ b(u,v) = \int\limits_{\partial\Omega} uv ds, \\ \|u\|_b &= b(u,u)^{\frac{1}{2}} = \|u\|_{0,\partial\Omega}. \end{aligned}$$

It is easy to know that $a(\cdot,\cdot)$ is a symmetric, continuous and $H^1(\Omega)$-elliptic bilinear form on $H^1(\Omega) \times H^1(\Omega)$. So, we use $a(\cdot,\cdot)$ and $\|\cdot\|_a = \sqrt{a(\cdot,\cdot)} = \|\cdot\|_1$ as the inner product and norm on $H^1(\Omega)$, respectively.

Let $\{\pi_h\}$ be a family of regular triangulations of Ω with the mesh diameter h, and $V_h \subset C(\overline{\Omega})$ be a space of piecewise linear polynomials defined on π_h.

The conforming finite element approximation of (2.2) is: find $\lambda_h \in \mathbb{R}$, $u_h \in V_h$ with $\|u_h\|_a = 1$, such that

$$a(u_h,v) = \lambda_h b(u_h,v), \quad \forall v \in V_h. \tag{2.3}$$

Consider the following source problem (2.4) associated with (2.2) and the approximate source problem (2.5) associated with (2.3), respectively.
Find $w \in H^1(\Omega)$, such that

$$a(w,v) = b(f,v), \quad \forall v \in H^1(\Omega). \tag{2.4}$$

Find $w_h \in V_h$, such that

$$a(w_h,v) = b(f,v), \quad \forall v \in V_h. \tag{2.5}$$

From [20] we know that the following regularity estimates hold for (2.4).

Lemma 2.1. If $f \in L_2(\partial\Omega)$, then there exists a unique solution $w \in H^{1+\frac{r}{2}}(\Omega)$ to (2.4), and

$$\|w\|_{1+\frac{r}{2}} \lesssim \|f\|_{0,\partial\Omega}; \tag{2.6}$$

if $f \in H^{\frac{1}{2}}(\partial\Omega)$, then there exists a unique solution $w \in H^{1+r}(\Omega)$ to (2.4), and

$$\|w\|_{1+r} \lesssim \|f\|_{\frac{1}{2},\partial\Omega}, \tag{2.7}$$

where $r = 1$ if Ω is convex, and $r < \frac{\pi}{\theta}$ which can be arbitrarily close to $\frac{\pi}{\theta}$ when Ω is concave.

Then, thanks to Lemma 2.1, from (2.4) we can define the operator $A : L_2(\partial\Omega) \to H^1(\Omega)$ by

$$a(Af,v) = b(f,v), \quad \forall v \in H^1(\Omega). \tag{2.8}$$

Similarly, from (2.5) we define the operator $A_h : L_2(\partial\Omega) \to V_h$ by

$$a(A_h f,v) = b(f,v), \quad \forall v \in V_h. \tag{2.9}$$

It is obvious that $A : H^1(\Omega) \to H^1(\Omega)$ is a self-adjoint operator. In fact, for any $u, v \in H^1(\Omega)$,

$$a(Au, v) = b(u, v) = b(v, u) = a(Av, u) = a(u, Av).$$

Analogously, A_h is also a self-adjoint operator. Observe that Af and A_hf are the exact solution and the finite element solution of (2.4), respectively, and

$$a(Af - A_hf, v) = 0, \forall v \in V_h \subset H^1(\Omega).$$

Define the Ritz-Galerkin projection operator $P_h : H^1(\Omega) \to V_h$ by

$$a(u - P_hu, v) = 0, \quad \forall u \in H^1(\Omega), \forall v \in V_h.$$

Thus, for any $f \in H^1(\Omega)$,

$$a(A_hf - P_h(Af), v) = a(A_hf - Af + Af - P_h(Af), v) = 0, \quad \forall v \in V_h,$$

therefore, $A_hf = P_hAf, \forall f \in H^1(\Omega)$, then $A_h = P_hA$.
From Lemma 2.1 and the interpolation error estimate, we deduce

$$\begin{aligned}
\|A_h - A\|_a &= \sup_{g\in H^1(\Omega)} \frac{\|(A_h - A)g\|_a}{\|g\|_a} \\
&= \sup_{g\in H^1(\Omega)} \frac{\|P_hAg - Ag\|_a}{\|g\|_a} \\
&\lesssim \sup_{g\in H^1(\Omega)} \frac{h^r\|Ag\|_{1+r}}{\|g\|_a} \\
&\lesssim \sup_{g\in H^1(\Omega)} \frac{h^r\|g\|_a}{\|g\|_a} = h^r \to 0 \ (h \to 0). \qquad (2.10)
\end{aligned}$$

It is clear that A_h is a finite rank operator, then A is a completely continuous operator.
From [7] and [12] we know that (2.2) and (2.3) have the following equivalent operator forms, respectively:

$$Au = \mu u, \qquad (2.11)$$

$$A_hu_h = \mu_hu_h, \qquad (2.12)$$

where $\mu = \frac{1}{\lambda}, \mu_h = \frac{1}{\lambda_h}$. In this paper, μ and μ_h, λ and λ_h are all called eigenvalues.

Suppose that μ_k is the kth eigenvalue of A and the algebraic multiplicity of μ_k is equal to q, $\mu_k = \mu_{k+1} = \cdots = \mu_{k+q-1}$. Let $M(\mu_k)$ be the space spanned by all eigenfunctions corresponding to μ_k of A and $M_h(\mu_k)$ be the direct sum of eigenspaces corresponding to all eigenvalues of A_h that converge to μ_k. Let $\widehat{M}(\mu_k) = \{v : v \in M(\mu_k), \|v\|_a = 1\}$, $\widehat{M}_h(\mu_k) = \{v : v \in M_h(\mu_k), \|v\|_a = 1\}$. We also write $M(\lambda_k) = M(\mu_k)$, $M_h(\lambda_k) = M_h(\mu_k)$, $\widehat{M}(\lambda_k) = \widehat{M}(\mu_k)$ and $\widehat{M}_h(\lambda_k) = \widehat{M}_h(\mu_k)$.

Denote

$$\begin{aligned}
\sigma(h) &= \sup_{f\in H^1(\Omega), \|f\|_a=1} \inf_{v\in V_h} \|Af - v\|_a, \\
\rho(h) &= \sup_{f\in L_2(\partial\Omega), \|f\|_{0,\partial\Omega}=1} \inf_{v\in V_h} \|Af - v\|_a, \\
\delta_h(\lambda_k) &= \sup_{u\in \widehat{M}(\lambda_k)} \inf_{v\in V_h} \|u - v\|_a.
\end{aligned}$$

It is obvious that

$$\delta_h(\lambda_k) \lesssim \sigma(h) \lesssim \rho(h). \tag{2.13}$$

The following a priori error estimates have been obtained in [5, 30]:

Lemma 2.2. Let $\lambda_{k,h}$ and λ_k be the kth eigenvalue of (2.3) and (2.2), respectively. Then

$$\lambda_k \leq \lambda_{k,h} \leq \lambda_k + C\delta_h^2(\lambda_k); \tag{2.14}$$

for any eigenfunction $u_{k,h}$ corresponding to $\lambda_{k,h}$, satisfying $\|u_{k,h}\|_a = 1$, there exists $u_k \in M(\lambda_k)$ such that

$$\|u_k - u_{k,h}\|_a \lesssim \delta_h(\lambda_k); \tag{2.15}$$

$$\|u_k - u_{k,h}\|_{0,\partial\Omega} \lesssim \rho(h)\delta_h(\lambda_k); \tag{2.16}$$

$$\|u_k - u_{k,h}\|_{-\frac{1}{2},\partial\Omega} \lesssim \sigma(h)\delta_h(\lambda_k); \tag{2.17}$$

for any $u_k \in \widehat{M}(\lambda_k)$, there exists $u_h \in M_h(\lambda_k)$ such that

$$\|u_h - u_k\|_a \lesssim \delta_h(\lambda_k). \tag{2.18}$$

The following lemma states a crucial property (but straightforward) of eigenvalue and eigenfunction approximation.

Lemma 2.3. Let (λ, u) be an eigenpair of (2.2), then for any $v \in H^1(\Omega)$, $\|v\|_b \neq 0$, the Rayleigh quotient $\frac{a(v,v)}{\|v\|_b^2}$ satisfies

$$\frac{a(v,v)}{\|v\|_b^2} - \lambda = \frac{\|v-u\|_a^2}{\|v\|_b^2} - \lambda\frac{\|v-u\|_b^2}{\|v\|_b^2}. \tag{2.19}$$

Proof. See, for instance, Lemma 9.1 in [7] for details. □

Our analysis is based on the following crucial property of the shifted-inverse iteration in finite element method (see Lemma 4.1 in [41]).

Lemma 2.4. Let (μ_0, u_0) be an approximation for (μ_k, u_k) where μ_0 is not an eigenvalue of A_h, and $u_0 \in V_h$ with $\|u_0\|_a = 1$. Suppose that

(C1) $dist(u_0, M_h(\mu_k)) \leq \frac{1}{2}$;

(C2) $|\mu_0 - \mu_k| \leq \frac{\rho}{4}$, $|\mu_{j,h} - \mu_j| \leq \frac{\rho}{4}$ for $j = k-1, k, k+q (j \neq 0)$, where $\rho = \min\limits_{\mu_j \neq \mu_k} |\mu_j - \mu_k|$ is the separation constant of the eigenvalue μ_k;

(C3) $u' \in V_h, u_k^h \in V_h$ satisfy

$$(\mu_0 - A_h)u' = u_0, \quad u_k^h = \frac{u'}{\|u'\|_a}. \tag{2.20}$$

Then

$$dist(u_k^h, M_h(\mu_k)) \leq \frac{4}{\rho} \max_{k\leq j\leq k+q-1} |\mu_0 - \mu_{j,h}| dist(u_0, M_h(\mu_k)). \tag{2.21}$$

3 A posteriori error estimate for multi-scale discretizations

In [10, 11, 29, 41], a multi-scale discretization scheme based on shifted inverse iteration has been established and its a priori error estimate has been proved. In this section, we will discuss the a posteriori error estimates for this multi-scale discretization scheme, which together with the adaptivity leads to the adaptive algorithm in the next section.

3.1 Multi-scale discretization scheme and a priori error estimates

Traditional multigrid methods based on the shifted inverse iteration has already been used for solving a given discretization scheme (see, e.g., [22, 37]) while in this paper we apply the multi-scale discretization scheme based on the shifted inverse iteration to solve a differential equation directly.

Let $\{\pi_{h_i}\}_1^l$ be a family of regular triangulations, and $\{V_{h_i}\}_1^l$ be the conforming finite element spaces defined on $\{\pi_{h_i}\}_1^l$, and let $\pi_H = \pi_{h_1}$, $V_H = V_{h_1}$, $\pi_h = \pi_{h_l}$, $V_h = V_{h_l}$. Assume that the following condition for meshes holds (see Condition 4.3 in [41]).

Condition 3.1. Suppose that $\varepsilon \in (0, 1)$ be a given number, there exists $t_i \in (1, 3-\varepsilon]$, $i = 2, 3, \cdots$ such that $\delta_{h_i}(\lambda) = \delta_{h_{i-1}}(\lambda)^{t_i}$ and $\delta_{h_i}(\lambda) \to 0$ $(i \to \infty)$.

Note that Condition 3.1 is closely related the saturation assumption for the approximation of piecewise polynomials. Recently, it was proved in [24, 25] that the saturation assumption holds for the quasi-uniform mesh, which is an essential advance.

Scheme 1 (multi-scale discretization scheme).
Step 1. Solve (2.3) on π_H: find $\lambda_{k,H} \in \mathbb{R}, u_{k,H} \in V_H$ such that $\|u_{k,H}\|_a = 1$ and

$$a(u_{k,H}, v) = \lambda_{k,H} b(u_{k,H}, v), \quad \forall v \in V_H.$$

Step 2. $u_k^{h_1} \Leftarrow u_{k,H}$, $\lambda_k^{h_1} \Leftarrow \lambda_{k,H}$, $i \Leftarrow 2$.
Step 3. Solve a linear system on the π_{h_i}: find $\tilde{u} \in V_{h_i}$ such that

$$a(\tilde{u}, v) - \lambda_k^{h_{i-1}} b(\tilde{u}, v) = b(u_k^{h_{i-1}}, v), \quad \forall v \in V_{h_i}.$$

And set $u_k^{h_i} = \frac{\tilde{u}}{\|\tilde{u}\|_a}$.
Step 4. Compute the Rayleigh quotient

$$\lambda_k^{h_i} = \frac{a(u_k^{h_i}, u_k^{h_i})}{b(u_k^{h_i}, u_k^{h_i})}.$$

Step 5. If $i = l$, then output $(\lambda_k^{h_l}, u_k^{h_l})$, i.e., (λ_k^h, u_k^h), stop. Else, $i \Leftarrow i + 1$, and return to Step 3.

(λ_k^h, u_k^h) is used as an approximation for the kth eigenpair, (λ_k, u_k), of (2.2).

The following a priori error estimates can be obtained by using the proof argument in [10, 41].

Lemma 3.1. Let (λ^{h_l}, u^{h_l}) be an approximate eigenpair obtained by Scheme 1. Suppose that the condition 3.1 holds and h_1, i.e. H is properly small. Then there exists $u_k \in M(\lambda_k)$ such that the following error estimates hold:

$$\|u_k^{h_l} - u_k\|_a \lesssim \delta_{h_l}(\lambda_k), \tag{3.1}$$

$$|\lambda_k^{h_l} - \lambda_k| \lesssim \|u_k^{h_l} - u_k\|_a^2, \quad l \geq 2. \tag{3.2}$$

3.2 The a posteriori error estimates

There exists many publications on the a posteriori error estimates (see, e.g., [1, 2, 8, 15, 33, 39]). Especially, [5] studied the a posteriori error estimate of finite element for the Steklov eigenvalue problem. Here we discuss the a posteriori error estimate of multi-scale discretization for the Steklov eigenvalue problem.

For any element $T \in \pi_h$ with diameter h_T, let $\mathcal{E}_T$ denote the set of edges of T, and

$$\mathcal{E} = \bigcup_{T\in\pi_h} \mathcal{E}_T.$$

We decompose $\mathcal{E} = \mathcal{E}_\Omega \cup \mathcal{E}_{\partial\Omega}$ where ε_Ω and $\mathcal{E}_{\partial\Omega}$ refer to interior edges and edges on the boundary $\partial\Omega$, respectively. For each $\ell \in \mathcal{E}_\Omega$, we choose an arbitrary unit normal vector n_ℓ and denote the two triangles sharing this edge by T_{in} and T_{out}, where n_ℓ points outwards T_{in}.
For $v_h \in V_h$ we set

$$[[\frac{\partial v_h}{\partial n_l}]]_\ell = \bigtriangledown(v_h|_{T_{out}}) \cdot n_\ell - \bigtriangledown(v_h|_{T_{in}}) \cdot n_\ell.$$

Let $I_h^c : H^1(\Omega) \to V_h$ be a Clément interpolation operator, then from [5] or Lemma 1.4 in [39] the following error estimates for I_h^c are valid:

$$\|v - I_h^c v\|_{0,T} \le C h_T \|v\|_{1,\widetilde{T}}, \tag{3.3}$$

$$\|v - I_h^c v\|_{0,\ell} \le C|l|^{\frac{1}{2}} \|v\|_{1,\widetilde{\ell}}, \tag{3.4}$$

where $\widetilde{T}$ is the union of all elements sharing a vertex with T and $\widetilde{\ell}$ is the union of all elements sharing a vertex with ℓ.
Let the eigenvectors $\{u_{j,h_l}\}_k^{k+q-1}$ be an orthonormal basis of $M_{h_l}(\lambda_k)$, and denote

$$u^* = \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l}) u_{j,h_l}. \tag{3.5}$$

It follows from Lemma 2.2 that there exist $\{u_j^0\}_k^{k+q-1} \subset M(\lambda_k)$ making $u_{j,h_l} - u_j^0$ satisfy (2.15), (2.16) and (2.17). Let

$$u = \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l}) u_j^0, \tag{3.6}$$

then $u \in M(\lambda_k)$ and

$$u - u^* = \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l})(u_j^0 - u_{j,h_l}). \tag{3.7}$$

Let

$$\widehat{\lambda}_{k,h_l} = \frac{1}{q} \sum_{j=k}^{k+q-1} \lambda_{j,h_l}.$$

For each $\ell \in \mathcal{E}$ we define the jump residual:

$$J_\ell(u^*) \quad = \quad \begin{cases} \frac{1}{2}[[\frac{\partial u^*}{\partial n_l}]]_\ell & \ell \in \mathcal{E}_\Omega, \\ \widehat{\lambda}_{k,h_l} u^* - \frac{\partial u^*}{\partial n_\ell} & \ell \in \mathcal{E}_{\partial\Omega}. \end{cases} \tag{3.8}$$

$$J_\ell(u_k^{h_l}) \quad = \quad \begin{cases} \frac{1}{2}[[\frac{\partial u_k^{h_l}}{\partial n_\ell}]]_\ell & \ell \in \mathcal{E}_\Omega, \\ \lambda_k^{h_l} u_k^{h_l} - \frac{\partial u_k^{h_l}}{\partial n_\ell} & \ell \in \mathcal{E}_{\partial\Omega}. \end{cases} \tag{3.9}$$

Now, the local error indicator is defined as

$$\eta_T(v) = (h_T^2 \|v\|_{0,T}^2 + \sum_{\ell \in \varepsilon_T} |\ell| \|J_\ell(v)\|_{0,\ell}^2)^{1/2}, \tag{3.10}$$

then the global error estimator is given by

$$\eta_\Omega(v) = (\sum_{T \in \pi_h} \eta_T(v)^2)^{1/2}. \tag{3.11}$$

We split the error $u - u_k^{h_l} = (u - u^*) + (u^* - u_k^{h_l})$. Next, we will estimate the first term $e = u - u^*$.

We know that for any $v \in V_{h_l}$

$$\begin{aligned} &\int_\Omega \nabla u^* \cdot \nabla v + \int_\Omega u^* v \\ &= \int_\Omega \nabla(\sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l}) u_{j,h_l}) \cdot \nabla v + \int_\Omega \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l}) u_{j,h_l} v \\ &= \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l})(\int_\Omega \nabla u_{j,h_l} \cdot \nabla v + \int_\Omega u_{j,h_l} v) \\ &= \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l}) \lambda_{j,h_l} \int_{\partial\Omega} u_{j,h_l} v \\ &= \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l})(\lambda_{j,h_l} - \widehat{\lambda}_{k,h_l}) \int_{\partial\Omega} u_{j,h_l} v + \widehat{\lambda}_{k,h_l} \int_{\partial\Omega} u^* v, \end{aligned} \tag{3.12}$$

thus the error $e = u - u^*$ satisfies

$$\begin{aligned} &\int_\Omega \nabla e \cdot \nabla v + \int_\Omega ev = \int_{\partial\Omega} \lambda_k uv - \int_\Omega \nabla u^* \cdot \nabla v - \int_\Omega u^* v \\ &= \int_{\partial\Omega} \lambda_k uv - \widehat{\lambda}_{k,h_l} \int_{\partial\Omega} u^* v \\ &- \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l})(\lambda_{j,h_l} - \widehat{\lambda}_{k,h_l}) \int_{\partial\Omega} u_{j,h_l} v, \quad \forall v \in V_{h_l}. \end{aligned} \tag{3.13}$$

On the other hand, for any $v \in H^1(\Omega)$, we have

$$\int_\Omega \nabla e \cdot \nabla v + \int_\Omega ev = a(u,v) - a(u^*, v) = \int_{\partial\Omega} \lambda_k uv - \sum_T \{\int_{\partial T} \frac{\partial u^*}{\partial n} v + \int_T u^* v\},$$

and so

$$\int_\Omega \nabla e\cdot\nabla v+\int_\Omega ev$$
$$=\sum_T\{-\int_T u^*v+\sum_{\ell\in\mathcal{E}_T\cap\mathcal{E}_{\partial\Omega}}\int_\ell(\widehat{\lambda}_{k,h_l}u^*-\frac{\partial u^*}{\partial n_\ell})v+\frac{1}{2}\sum_{\ell\in\mathcal{E}_T\cap\mathcal{E}_\Omega}\int_\ell[[\frac{\partial u^*}{\partial n_\ell}]]_\ell v\}$$
$$+\int_{\partial\Omega}\lambda_k uv-\int_{\partial\Omega}\widehat{\lambda}_{k,h_l}u^*v,\quad \forall v\in H^1(\Omega). \tag{3.14}$$

The following lemma provides the global upper bound of e.

Lemma 3.2. The error $e=u-u^*$ satisfies

$$\|e\|_a\lesssim \eta_\Omega(u^*)+\sigma(h_l)\delta_{h_l}(\lambda_k). \tag{3.15}$$

Proof. By (3.13) and (3.14), noting that the definition of $J_\ell(u^*)$, we deduce that

$$\int_\Omega \nabla e\cdot\nabla e+\int_\Omega ee$$
$$=\int_\Omega\nabla e\cdot(\nabla e-\nabla I_h^c e)+\int_\Omega e(e-I_h^c e)+\int_{\partial\Omega}\lambda_k u I_h^c e-\int_{\partial\Omega}\widehat{\lambda}_{k,h_l}u^*I_h^c e$$
$$-\sum_{j=k}^{k+q-1}a(u_k^{h_l},u_{j,h_l})(\lambda_{j,h_l}-\widehat{\lambda}_{k,h_l})\int_{\partial\Omega}u_{j,h_l}I_h^c e$$
$$=\sum_T\{-\int_T u^*(e-I_h^c e)+\sum_{\ell\in\mathcal{E}_T}\int_\ell J_\ell(u^*)(e-I_h^c e)\}+\int_{\partial\Omega}(\lambda_k u-\widehat{\lambda}_{k,h_l}u^*)e$$
$$-\sum_{j=k}^{k+q-1}a(u_k^{h_l},u_{j,h_l})(\lambda_{j,h_l}-\widehat{\lambda}_{k,h_l})\int_{\partial\Omega}u_{j,h_l}I_h^c e. \tag{3.16}$$

Then

$$\|e\|_a^2\le\sum_T\|u^*\|_{0,T}\|e-I_h^c e\|_{0,T}+\sum_T\sum_{\ell\in\mathcal{E}_T}\|J_\ell(u^*)\|_{0,\ell}\|e-I_h^c e\|_{0,\ell}$$
$$+|\int_{\partial\Omega}(\lambda_k u-\widehat{\lambda}_{k,h_l}u^*)e|+|\sum_{j=k}^{k+q-1}a(u_k^{h_l},u_{j,h_l})(\lambda_{j,h_l}-\widehat{\lambda}_{k,h_l})\int_{\partial\Omega}u_{j,h_l}I_h^c e|$$
$$\lesssim\sum_T h_T\|u^*\|_{0,T}\|e\|_{1,\widetilde{T}}+\sum_T\sum_{\ell\in\mathcal{E}_T}|\ell|^{\frac{1}{2}}\|J_\ell(u^*)\|_{0,\ell}\|e\|_{1,\widetilde{\ell}}$$
$$+|\int_{\partial\Omega}(\lambda_k u-\widehat{\lambda}_{k,h_l}u^*)e|+|\sum_{j=k}^{k+q-1}a(u_k^{h_l},u_{j,h_l})(\lambda_{j,h_l}-\widehat{\lambda}_{k,h_l})\int_{\partial\Omega}u_{j,h_l}I_h^c e|$$
$$\lesssim\{\sum_T(h_T^2\|u^*\|_{0,T}^2+\sum_{\ell\in\mathcal{E}_T}|\ell|\|J_\ell(u^*)\|_{0,\ell}^2)\}^{\frac{1}{2}}\|e\|_a$$
$$+\|\lambda_k u-\widehat{\lambda}_{k,h_l}u^*\|_{-\frac{1}{2},\partial\Omega}\|e\|_a+\sum_{j=k}^{k+q-1}|\lambda_{j,h_l}-\widehat{\lambda}_{k,h_l}|\|e\|_a. \tag{3.17}$$

By Lemma 2.2, we have

$$\begin{aligned}\|\lambda_k u - \widehat{\lambda}_{k,h_l} u^*\|_{-\frac{1}{2},\partial\Omega} &\lesssim |\lambda_k - \widehat{\lambda}_{k,h_l}| + \|u - u^*\|_{-\frac{1}{2},\partial\Omega} \\ &\lesssim \sigma(h_l)\delta_{h_l}(\lambda_k) \qquad (3.18) \\ |\lambda_{j,h_l} - \widehat{\lambda}_{k,h_l}| &\lesssim \delta_{h_l}(\lambda_k)^2. \qquad (3.19)\end{aligned}$$

Combining (3.17), (3.18) and (3.19), the proof concludes. □

Next, we shall analyze the local lower bound of e.

Lemma 3.3. The error $e = u - u^*$ satisfies

(a) For $T \in \pi_h$, if $\partial T \cap \partial\Omega = \emptyset$, then

$$\eta_T(u^*) \lesssim \|e\|_{1,T^*}, \tag{3.20}$$

where T^* denote the union of T and the triangles sharing an edge with T.

(b) For $T \in \pi_h$, if $\partial T \cap \partial\Omega \neq \emptyset$, then

$$\eta_T(u^*) \lesssim \|e\|_{1,T} + \sum_{\ell \in \mathcal{E}_T \cap \mathcal{E}_{\partial\Omega}} |\ell|^{\frac{1}{2}} \|\lambda_k u - \widehat{\lambda}_{k,h_l} u^*\|_{0,\ell}. \tag{3.21}$$

Proof. First, using the same argument of Lemma 3.1 in [5] and replacing λ_h, u_h and (3.8) in Lemma 3.1 in [5] by $\widehat{\lambda}_{k,h_l}, u^*$ and (3.14) in this paper, respectively, we can prove that

$$h_T\|u^*\|_{0,T} \lesssim \|\nabla e\|_{0,T} + h_T\|e\|_{0,T}. \tag{3.22}$$

Also using the same proof method with that of Lemma 3.2 in [5] and replacing λ_h, u_h and (3.8) in [5] by $\widehat{\lambda}_{k,h_l}, u^*$ and (3.14) here, respectively, we can obtain

$$|\ell|^{\frac{1}{2}}\|J_\ell(u^*)\|_{0,\ell} \lesssim (1 + h_T)\|e\|_{1,T}^2 + |\ell|^{\frac{1}{2}}\|\lambda_k u - \widehat{\lambda}_{k,h_l} u^*\|_{0,\ell}, \quad \ell \in \mathcal{E}_T \cap \mathcal{E}_{\partial\Omega}, \tag{3.23}$$

and

$$|\ell|^{\frac{1}{2}}\|J_\ell(u^*)\|_{0,\ell} \lesssim \|e\|_{1,T_\ell^1 \cup T_\ell^2}, \quad \ell \in \mathcal{E}_T \cap \mathcal{E}_\Omega, \tag{3.24}$$

where $T_\ell^1, T_\ell^2 \in \pi_h$ are the two triangles sharing ℓ.

Then, we get (3.20) immediately by combining (3.22) with (3.24), and obtain (3.21) by combining (3.22) with (3.23). □

Now, we will analyze the error $u_k^{h_l} - u^*$.

Theorem 3.1. Suppose that the conditions of Lemma 3.1 are satisfied, then

$$\|u_k^{h_l} - u^*\|_a \lesssim \delta_{h_l}(\lambda_k)^{\frac{3}{t_l}}. \tag{3.25}$$

Proof. We use Lemma 2.4 to complete the proof. Using the arguments in [41] it is easy to verify that the conditions of Lemma 2.4 are satisfied.

Using the triangle inequality and (2.18), we get

$$dist(u_0, M_{h_l}(\lambda_k)) \leq dist(u_0, \widehat{M}(\lambda_k)) + C\delta_{h_l}(\lambda_k). \tag{3.26}$$

It follows from (2.14) that $\lambda_{k,h_l} \to \lambda_k (h_l \to 0)$, then by the assumption we have

$$
\begin{aligned}
|\mu_0 - \mu_{j,h_l}| &= |\frac{\lambda_k^{h_{l-1}} - \lambda_{j,h_l}}{\lambda_{j,h_l}\lambda_k^{h_{l-1}}}| = |\frac{\lambda_k^{h_{l-1}} - \lambda_k + \lambda_k - \lambda_{j,h_l}}{\lambda_{j,h_l}\lambda_k^{h_{l-1}}}| \\
&\lesssim \delta_{h_{l-1}}^2(\lambda_k) + \delta_{h_l}^2(\lambda_k), \quad j = k, k+1, \cdots, k+q-1. \qquad (3.27)
\end{aligned}
$$

Substituting (3.26) and (3.27) into (2.21), we obtain

$$
dist(u_k^{h_l}, M_{h_l}(\lambda_k)) \lesssim (\delta_{h_{l-1}}^2(\lambda_k) + \delta_{h_l}^2(\lambda_k))(dist(u_0, \widehat{M}(\lambda_k)) + C\delta_{h_l}(\lambda_k)). \quad (3.28)
$$

Observing that

$$
dist(u_k^{h_l}, M_{h_l}(\lambda_k)) = \|u_k^{h_l} - \sum_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l})u_{j,h_l}\|_a,
$$

and recalling that $u^* = \sum\limits_{j=k}^{k+q-1} a(u_k^{h_l}, u_{j,h_l})u_{j,h_l}$, then

$$
\begin{aligned}
\|u_k^{h_l} - u^*\|_a &\lesssim (\delta_{h_{l-1}}^2(\lambda_k) + \delta_{h_l}^2(\lambda_k))(dist(u_0, \widehat{M}(\lambda_k)) + C\delta_{h_l}(\lambda_k)) \\
&\lesssim \delta_{h_{l-1}}(\lambda_k)^3 \lesssim \delta_{h_l}(\lambda_k)^{\frac{3}{t_l}}. \qquad (3.29)
\end{aligned}
$$

The proof is completed. $\square$

Lemma 3.4. Suppose that the conditions of Lemma 3.1 are satisfied, then

$$
\begin{aligned}
|\eta_T(u^*) - \eta_T(u_k^{h_l})| &\lesssim \delta_{h_l}(\lambda_k)^2\|u^*\|_{1,T} + \|u_k^{h_l} - u^*\|_{1,T}, \qquad (3.30) \\
|\eta_\Omega(u^*) - \eta_\Omega(u_k^{h_l})| &\lesssim \delta_{h_l}(\lambda_k)^2\|u^*\|_a + \|u_k^{h_l} - u^*\|_a. \qquad (3.31)
\end{aligned}
$$

Proof. From (3.10) and the triangle inequality, we deduce

$$
|\eta_T(u^*) - \eta_T(u_k^{h_l})| \le (h_T^2\|u^* - u_k^{h_l}\|_{0,T}^2 + \sum_{\ell\in\mathcal{E}_T} |\ell|\|J_\ell(u^*) - J_\ell(u_k^{h_l})\|_{0,\ell}^2)^{1/2}.
$$

From (3.8) and (3.9), if $\ell \in \mathcal{E}_\Omega$, we have

$$
\|J_\ell(u^*) - J_\ell(u_k^{h_l})\|_{0,\ell} = \frac{1}{2}\|[[\frac{\partial(u^* - u_k^{h_l})}{\partial n_\ell}]]_\ell\|_{0,\ell} \lesssim h_T^{-\frac{1}{2}}\|u_k^{h_l} - u^*\|_{1,T},
$$

and if $\ell \in \mathcal{E}_{\partial\Omega}$,

$$
\begin{aligned}
&\|J_\ell(u^*) - J_\ell(u_k^{h_l})\|_{0,\ell} \le \|\widehat{\lambda}_{k,h_l}u^* - \lambda_k^{h_l}u_k^{h_l} - \frac{\partial(u^* - u_k^{h_l})}{\partial n_\ell}\|_{0,\ell} \\
&\lesssim |\widehat{\lambda}_{k,h_l} - \lambda_k^{h_l}|\|u^*\|_{1,T} + \|u_k^{h_l} - u^*\|_{1,T} + h_T^{-\frac{1}{2}}\|u_k^{h_l} - u^*\|_{1,T} \\
&\lesssim \delta_{h_l}(\lambda_k)^2\|u^*\|_{1,T} + \|u_k^{h_l} - u^*\|_{1,T} + h_T^{-\frac{1}{2}}\|u_k^{h_l} - u^*\|_{1,T}.
\end{aligned}
$$

Combining the above three inequalities we obtain (3.30). From the triangle inequality and (3.30) we derive (3.31). $\square$

Combining Lemmas 3.2-3.4 and Theorem 3.1, we give the global bound and the local lower bound of the error.

Theorem 3.2. Suppose that the conditions of Lemma 3.1 are satisfied, then there exists $u_k \in M(\lambda_k)$ such that

$$\|u_k - u_k^{h_l}\|_a \lesssim \eta_\Omega(u_k^{h_l}). \tag{3.32}$$

Proof. Select $u_k \in M(\lambda_k)$ which is given by (3.6), then from Lemma 3.2, Theorem 3.1 and Lemma 3.4 we get

$$\begin{aligned} \|u_k - u_k^{h_l}\|_a &\leq \|u_k - u^*\|_a + \|u^* - u_k^{h_l}\|_a \\ &\lesssim \eta_\Omega(u^*) + \sigma(h_l)\delta_{h_l}(\lambda_k) + \delta_{h_l}(\lambda_k)^{\frac{3}{t_l}} \\ &\lesssim \eta_\Omega(u_k^{h_l}) + \sigma(h_l)\delta_{h_l}(\lambda_k) + \delta_{h_l}(\lambda_k)^{\frac{3}{t_l}}. \end{aligned}$$

Since in the above inequality the first term on the right hand side is the dominant term and the other two terms are of high order, (3.32) is true. □

Theorem 3.3. Suppose that the conditions of Lemma 3.1 are satisfied, then there exists $u_k \in M(\lambda_k)$ such that

$$\eta_\Omega(u_k^{h_l}) \lesssim \|u_k - u_k^{h_l}\|_a. \tag{3.33}$$

Proof. Select $u_k \in M(\lambda_k)$ which is given by (3.6), then from Lemma 3.3, (3.7), (2.14) and (2.16) we have

$$\begin{aligned} \sum_{T\in\pi_{h_l}} \eta_T(u^*)^2 &\lesssim \sum_{T\in\pi_{h_l}} \|e\|_{1,T}^2 + \sum_{\ell\in\mathcal{E}_{\partial\Omega}} |\ell| \|\lambda_k u_k - \widehat{\lambda}_{k,h_l} u^*\|_{0,\ell}^2 \\ &\lesssim \|e\|_a^2 + h_l \|\lambda_k u_k - \widehat{\lambda}_{k,h_l} u^*\|_{0,\partial\Omega}^2 \\ &\lesssim \|e\|_a^2 + h_l(|\lambda_k - \widehat{\lambda}_{k,h_l}|^2 + \|u_k - u^*\|_{0,\partial\Omega}^2) \\ &\lesssim \|e\|_a^2 + h_l \rho(h_l)^2 \delta_{h_l}(\lambda_k)^2, \end{aligned}$$

thus

$$\begin{aligned} \eta_\Omega(u^*) &\lesssim \|e\|_a + h_l^{\frac{1}{2}} \rho(h_l)\delta_{h_l}(\lambda_k) \\ &\leq \|u_k - u_k^{h_l}\|_a + \|u_k^{h_l} - u^*\|_a + h_l^{\frac{1}{2}} \rho(h_l)\delta_{h_l}(\lambda_k), \end{aligned}$$

which together with Theorem 3.1 and Lemma 3.4 leads to

$$\eta_\Omega(u_k^{h_l}) \lesssim \|u_k - u_k^{h_l}\|_a + \delta_{h_l}(\lambda_k)^2 + \delta_{h_l}(\lambda_k)^{\frac{3}{t_l}} + h_l^{\frac{1}{2}} \rho(h_l)\delta_{h_l}(\lambda_k). \tag{3.34}$$

Noting that $\frac{3}{t_l} > 1$, we know the first term on the right hand side of (3.34) is the dominant term and the others are of high order, then we derive (3.33). □

Theorem 3.4. Under the conditions of Lemma 3.1, there exists $u_k \in M(\lambda_k)$ such that

(a) For $T \in \pi_{h_l}$, if $\partial T \cap \partial\Omega = \emptyset$, then

$$\eta_T(u_k^{h_l}) \lesssim \|u_k - u_k^{h_l}\|_{1,T^*} + \|u_k^{h_l} - u^*\|_{1,T^*} + \delta_{h_l}(\lambda_k)^2 \|u^*\|_{1,T}, \tag{3.35}$$

where T^* denote the union of T and the triangles sharing an edge with T.

(b) For $T \in \pi_{h_l}$, if $\partial T \cap \partial\Omega \neq \emptyset$, then

$$\begin{aligned} \eta_T(u_k^{h_l}) \lesssim{}& \|u_k - u_k^{h_l}\|_{1,T} + \|u_k^{h_l} - u^*\|_{1,T} \\ &+ \delta_{h_l}(\lambda_k)^2 \|u^*\|_{1,T} + \sum_{\ell\in\mathcal{E}_T\cap\mathcal{E}_{\partial\Omega}} |\ell|^{\frac{1}{2}} \|\lambda_k u_k - \widehat{\lambda}_{k,h_l} u^*\|_{0,\ell}. \end{aligned} \tag{3.36}$$

Proof. Select $u_k \in M(\lambda_k)$ which is given by (3.6), then from the triangle inequality we have

$$\|u_k - u^*\|_{1,T^*} \le \|u_k - u_k^{h_l}\|_{1,T^*} + \|u_k^{h_l} - u^*\|_{1,T^*}.$$

From (3.20) we obtain

$$\eta_T(u^*) \lesssim \|u_k - u_k^{h_l}\|_{1,T^*} + \|u_k^{h_l} - u^*\|_{1,T^*},$$

combining with Lemma 3.4 we get (3.35).
Similarly, from (3.21) and Lemma 3.4 we know that (3.36) is valid. □

Remark 3.2. Since

$$\|\lambda_k u_k - \widehat{\lambda}_{k,h_l} u^*\|_{0,\partial\Omega} \lesssim \lambda_k \|u_k - u^*\|_{0,\partial\Omega} + |\lambda_k - \widehat{\lambda}_{k,h_l}| \lesssim \rho(h_l)\delta_{h_l}(\lambda_k) + \delta_{h_l}^2(\lambda_k),$$

according to Remark 3.1 in [5] we know that the term $\sum\limits_{\ell \in \mathcal{E}_T \cap \mathcal{E}_{\partial\Omega}} |\ell|^{\frac{1}{2}} \|\lambda_k u_k - \widehat{\lambda}_{k,h_l} u^*\|_{0,\ell}$ is a higher order term. Similarly, $\|u_k^{h_l} - u^*\|_{1,T^*}$ and $\|u_k^{h_l} - u^*\|_{1,T}$ are higher order terms. In fact, from Theorem 3.1 we have

$$\|u_k^{h_l} - u^*\|_a \lesssim \delta_{h_l}(\lambda_k)^{\frac{3}{t_l}} \quad (\frac{3}{t_l} > 1).$$

And it is obviously that $\delta_{h_l}(\lambda_k)^2 \|u^*\|_{1,T}$ is a higher order term. Therefore, in (3.35) and (3.36) the first term on the right-hand side is the dominant term.

Using the proof method of (4.21) in [41] we can prove that $\|u_k^{h_l} - u_k\|_b$ is an infinitesimal of higher order comparing with $\|u_k^{h_l} - u_k\|_a$, thus, from Lemma 2.3 it is easy to prove that $\lambda_k^{h_l} - \lambda_k = \mathcal{O}(\|u_k^{h_l} - u_k\|_a^2)$. Combining (3.32) with (3.33) we can obtain the following estimates for approximate eigenvalue.

Theorem 3.5. Suppose that the conditions of Lemma 3.1 are satisfied, then

$$\eta_\Omega(u_k^{h_l})^2 \lesssim |\lambda_k^{h_l} - \lambda_k| \lesssim \eta_\Omega(u_k^{h_l})^2. \tag{3.37}$$

Theorem 3.5 shows that $\eta_\Omega(u_k^{h_l})^2$ is a reliable and effective estimator for $\lambda_k^{h_l}$.

4 Adaptive algorithms and numerical experiments

In this section, based on the a posteriori error estimates we will establish an adaptive procedure of shifted inverse iteration type and present another two adaptive methods for the Steklov eigenvalue problems, and report several numerical experiments to compare the efficiency of three different adaptive algorithms and illustrate the efficiency of our adaptive method.

4.1 Adaptive algorithms based on multi-scale discretization

The following adaptive Algorithm 4.1 is usual and standard, which was discussed in [21].

Algorithm 4.1 Choose parameter $0 < \omega < 1$.
Step 1. Pick any initial mesh π_{h_1} with mesh size h_1.
Step 2. Solve (2.3) on π_{h_1} for discrete solution $(\lambda_{k,h_1}, u_{k,h_1})$.
Step 3. Let $l = 1$.
Step 4. Compute the local indicators $\eta_T(u_{k,h_l})$.
Step 5. Construct $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by **Marking Strategy E1** and parameter ω.
Step 6. Refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$ by Procedure **REFINE**.
Step 7. Solve (2.3) on $\pi_{h_{l+1}}$ for discrete solution $(\lambda_{k,h_{l+1}}, u_{k,h_{l+1}})$.
Step 8. Let $l = l + 1$ and go to Step 4.

Marking Strategy E1
Given parameter $0 < \omega < 1$:
Step 1. Construct a minimal subset $\widehat{\pi}_{h_l}$ of π_{h_l} by selecting some elements in π_{h_l} such that

$$\sum_{T \in \widehat{\pi}_{h_l}} \eta_T(u_{k,h_l})^2 \geq \omega \eta_\Omega(u_{k,h_l})^2.$$

Step 2. Mark all the elements in $\widehat{\pi}_{h_l}$.

In Algorithm 4.1 and **Marking Strategy E1**, the a posteriori error estimators $\eta_T(u_{k,h_l})$ and $\eta_\Omega(u_{k,h_l})$ are defined by (3.10), (3.11) and (3.9) with $u_k^{h_l}$ and $\lambda_k^{h_l}$ replaced by u_{k,h_l} and λ_{k,h_l}, respectively.
Based on the work of [17, 32, 35, 38], we give the following adaptive algorithm 4.2 for the Steklov eigenvalue problem.

Algorithm 4.2 Choose parameter $0 < \omega < 1$.
Step 1. Pick any initial mesh π_{h_1}.
Step 2. Solve (2.3) on π_{h_1} for discrete solution $(\lambda_{k,h_1}, u_{k,h_1})$. $u_{k*}^{h_1} \Leftarrow u_{k,h_1}$, $\lambda_{k*}^{h_1} \Leftarrow \lambda_{k,h_1}$.
Step 3. Let $l = 1$.
Step 4. Compute the local indicators $\eta_T(u_{k*}^{h_l})$.
Step 5. Construct $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by **Marking Strategy E2** and parameter ω.
Step 6. Refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$ by Procedure **REFINE**.
Step 7. Find $\tilde{u} \in V_{h_{l+1}}$ such that

$$a(\tilde{u}, \psi) = b(u_{k*}^{h_l}, \psi), \quad \forall \psi \in V_{h_{l+1}};$$

Denote $u_{k*}^{h_{l+1}} = \frac{\tilde{u}}{\|\tilde{u}\|_a}$ and compute the Rayleigh quotient

$$\lambda_{k*}^{h_{l+1}} = \frac{a(u_{k*}^{h_{l+1}}, u_{k*}^{h_{l+1}})}{b(u_{k*}^{h_{l+1}}, u_{k*}^{h_{l+1}})}.$$

Step 8. Let $l = l + 1$ and go to Step 4.

Marking Strategy E2 is the same as **Marking Strategy E1** except that the corresponding a posteriori error estimators are taken as $\eta_T(u_{k*}^{h_l})$ and $\eta_\Omega(u_{k*}^{h_l})$, and $\eta_T(u_{k*}^{h_l})$ and $\eta_\Omega(u_{k*}^{h_l})$ are defined by (3.10), (3.11) and (3.9) with $u_k^{h_l}$ and $\lambda_k^{h_l}$ replaced by $u_{k*}^{h_l}$ and $\lambda_{k*}^{h_l}$, respectively.

Combining the a posteriori error estimator and Scheme 1, we establish the following adaptive algorithm.

Algorithm 4.3 Choose parameter $0 < \omega < 1$.

Step 1. Pick any initial mesh π_{h_1}.

Step 2. Solve (2.3) on π_{h_1} for discrete solution $(\lambda_{k,h_1}, u_{k,h_1})$. $u_k^{h_1} \Leftarrow u_{k,h_1}$, $\lambda_k^{h_1} \Leftarrow \lambda_{k,h_1}$.

Step 3. Let $l = 1$.

Step 4. Compute the local indicators $\eta_T(u_k^{h_l})$.

Step 5. Construct $\widehat{\pi}_{h_l} \subset \pi_{h_l}$ by **Marking Strategy E3** and parameter ω.

Step 6. Refine π_{h_l} to get a new mesh $\pi_{h_{l+1}}$ by Procedure **REFINE**.

Step 7. Find $\tilde{u} \in V_{h_{l+1}}$ such that

$$a(\tilde{u}, \psi) - \lambda_k^{h_l} b(\tilde{u}, \psi) = b(u_k^{h_l}, \psi), \quad \forall \psi \in V_{h_{l+1}};$$

Denote $u_k^{h_{l+1}} = \frac{\tilde{u}}{\|\tilde{u}\|_a}$ and compute the Rayleigh quotient

$$\lambda_k^{h_{l+1}} = \frac{a(u_k^{h_{l+1}}, u_k^{h_{l+1}})}{b(u_k^{h_{l+1}}, u_k^{h_{l+1}})}.$$

Step 8. Let $l = l + 1$ and go to Step 4.

Marking Strategy E3 is the same as **Marking Strategy E1** except that the corresponding a posteriori error estimators are taken as $\eta_T(u_k^{h_l})$ and $\eta_\Omega(u_k^{h_l})$, and $\eta_T(u_k^{h_l})$ and $\eta_\Omega(u_k^{h_l})$ are defined by (3.9), (3.10) and (3.11).

4.2 Numerical experiments

We will report two numerical examples to show the performances of Algorithms 4.1-4.3. We use MATLAB 2011b to solve Example 4.1 and Example 4.2. Our program makes use of the package of Chen [16]. We refer to the adaptive program of Chen and take $\omega = 0.25$.

In step 7 of Algorithm 4.1, we use the internal command eigs in MATLAB to perform the eigenvalue computations. The calling format of eigs we used is eigs(K,M,1,sigma), whose function is to solve an eigenvalue which is closest to sigma. In our computation, K and M are stiffness and mass matrices, respectively, sigma is an eigenvalue derived from the last iteration.

For convenience of reading, we specify the following notations used in our tables and figures:

λ_{k,h_l}: The kth eigenvalue derived from the lth iteration obtained by Algorithm 4.1.

$N_{k,l}$: The degrees of freedom of the lth iteration for λ_{k,h_l}.

$CPU_{k,l}(s)$: The CPU time(s) from the program starting to the calculate result of the lth iteration appearing by using Algorithm 4.1.

$\lambda_{k*}^{h_l}$: The kth eigenvalue derived from the lth iteration obtained by Algorithm 4.2.

$N_{k*,l}$: The degrees of freedom of the lth iteration for $\lambda_{k*}^{h_l}$.

$CPU_{k*,l}(s)$: The CPU time(s) from the program starting to the calculate result of the lth iteration appearing by using Algorithm 4.2.

$\lambda_k^{h_l}$: The kth eigenvalue derived from the lth iteration obtained by Algorithm 4.3.

$N_{k+,l}$: The degrees of freedom of the lth iteration for $\lambda_k^{h_l}$.

$CPU_{k+,l}(s)$: The CPU time(s) from the program starting to the calculate result of the lth iteration appearing by using Algorithm 4.3.

$e_i (i = 1, 2, 3)$: The absolute value of the error of approximate eigenvalue obtained by Algorithm 4.i ($i = 1, 2, 3$).

$\eta_i (i = 1, 2, 3)$: The a posteriori error estimator of approximate eigenvalue obtained by Algorithm 4.i ($i = 1, 2, 3$).

Example 4.1. We compute the approximations of the first, the second and the fourth eigenvalue of (2.1) with the triangle linear finite element on $\bar{\Omega} = [0, 1] \times [0, 1]$ by Algorithms 4.1, 4.2 and 4.3, respectively, and the results are shown in Tables 1 and 2. We also depict the error curves and the a posteriori error estimators of Algorithms 4.1, 4.2 and 4.3 in Figs. 1-3.

Table 1: The results of Example 4.1 obtained by Algorithms 4.1 and 4.3

k	l	$N_{k+,l}$	$\lambda_k^{h_l}$	$CPU_{k+,l}(s)$	$N_{k,l}$	λ_{k,h_l}	$CPU_{k,l}(s)$
1	23	386414	0.24007910	237.41	386414	0.24007910	411.07
1	28	776862	0.24007909	510.34	——	————	——-
2	27	500464	1.49230435	340.08	408337	1.49230468	553.05
2	30	756961	1.49230397	531.99	——	————	——-
4	24	404451	2.08265532	250.73	404451	2.08265532	445.37
4	28	776445	2.08265094	476.95	——	————	——-

⋆ The symbol '——' means that the calculation by Algorithm 4.1 cannot proceed since the computer runs out of memory.

Table 2: The results of Example 4.1 obtained by Algorithms 4.2 and 4.3

k	l	$N_{k+,l}$	$\lambda_k^{h_l}$	$CPU_{k+,l}(s)$	$N_{k*,l}$	$\lambda_{k*}^{h_l}$	$CPU_{k*,l}(s)$
1	23	386414	0.24007910	237.41	386267	0.24007911	199.56
1	28	776862	0.24007909	510.34	776224	0.24007909	421.38
2	27	500464	1.49230435	340.08	261640	0.24007911	192.71
2	30	756961	1.49230397	531.99	433359	0.24007910	276.27
4	24	404451	2.08265532	250.73	242547	0.24007911	133.72
4	28	776445	2.08265094	476.95	434294	0.24007910	235.98

Table 3: The results of Example 4.1 obtained by Algorithm 4.2

l	$N_{1*,l}$	$\lambda_{1*}^{h_l}$	l	$N_{2*,l}$	$\lambda_{2*}^{h_l}$	l	$N_{4*,l}$	$\lambda_{4*}^{h_l}$
1	16641	0.24007967	1	16641	1.49234096	1	16641	2.08289558
7	33775	0.24007927	17	141679	1.48357414	7	32621	1.60115184
9	44671	0.24007922	18	146349	1.22503676	8	32798	0.30677585
11	58295	0.24007919	19	146609	0.34911707	9	32945	0.24099836
16	122448	0.24007913	20	146950	0.24316279	10	33637	0.24009151
28	776224	0.24007909	30	433359	0.24007910	28	434294	0.24007910

We can see from Table 1 that compared with Algorithm 4.1, Algorithm 4.3 costs less CPU time to obtain the same accurate approximations. We think the reason is although Algorithm 4.1 and 4.3 all use the shifted inverse iteration method, Step 7 in Algorithm 4.1 is to solve an eigenvalue problem using the command eigs(K,M,1,λ_{k,h_l}), and the iteration times is usually taken as 2 or 3; while Step 7 in Algorithm 4.3 is to solve a linear system and the initial iteration value is taken as $u_k^{h_l}$ which is derived from the last iteration, and the theoretical

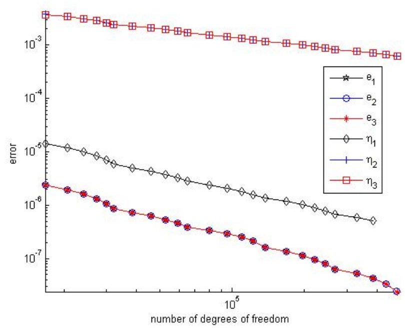

Figure 1: The curves of error and the a posteriori estimators for the first eigenvalue of Example 4.1 in log-log scale

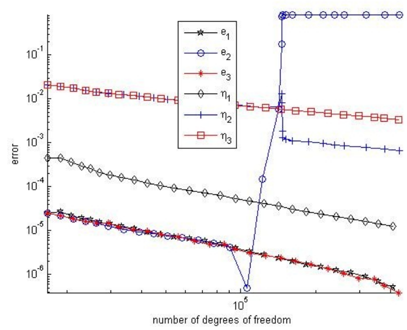

Figure 2: The curves of error and the a posteriori estimators for the second eigenvalue of Example 4.1 in log-log scale

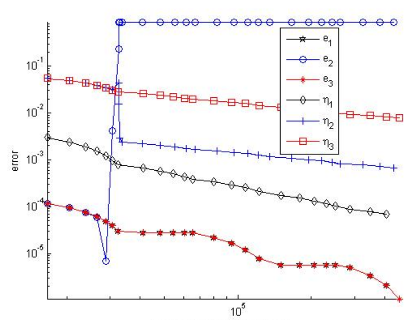

Figure 3: The curves of error and the a posteriori estimators for the fourth eigenvalue of Example 4.1 in log-log scale

analysis assures that there is only one iteration needed.

It can be seen from Table 2 that we can get the same accurate approximation for the first eigenvalue by Algorithms 4.2 and 4.3, and Algorithm 4.2 spends less time. However, it is worthy noting in Table 2 that the approximate eigenvalues $\lambda_{2*}^{h_l}$ and $\lambda_{4*}^{h_l}$ obtained by Algorithm 4.2 are not good approximations for the second and the fourth exact eigenvalue but are close to the first exact one. We also list some results in Table 3 to show this trend (In Figs 2 and 3, the curves of e_2 also reflect this phenomenon). In fact, from the point of numerical algebra, we can view Algorithm 4.2 as the inverse iteration method. And it is well-known that the inverse iteration method is only applicable to solving the smallest eigenvalue. [38] has pointed out that after an orthogonalization procedure, Algorithm 4.2 can be suitable to approximate any eigenpair and not only the first one.

In Fig. 1 we can see that the curves of e_1, e_2 and e_3 are overlapping, and the curves of η_1, η_2 and η_3 are basically parallel to those of e_1, e_2 and e_3, respectively, which shows that our estimator is reliable and efficient. In Fig. 2 and 3 we can see that when the numbers of degrees of freedom are not very large, the curves of e_1, e_2 and e_3 are overlapping and the curves of e_3 and η_3 are basically parallel which illustrate that our estimator is reliable and efficient.

Example 4.2. We compute the three smallest approximate eigenvalues of (2.1) with the triangle linear finite element on $([0,1]\times[0,\frac{1}{2}])\bigcup([0,\frac{1}{2}]\times[\frac{1}{2},1])$ by Algorithms 4.1, 4.2 and 4.3, respectively, and the results are listed in Table 4 and 5. We generate the initial mesh by a uniform triangulation with the diameter $\frac{\sqrt{2}}{128}$, and in Figs. 4-6 we shows the adaptively refined meshes for $\lambda_i (i=1,2,3)$. We also depict the error curves and the a posteriori error estimators of Algorithms 4.1, 4.2 and 4.3 in Figs. 7-9.

Table 4 shows that the efficiency of Algorithm 4.3 is higher than that of Algorithm 4.1. Table 5 illustrates that compared with Algorithm 4.2, our method can be used to solve approximations for any eigenvalue on the L-shaped domain although the CPU time of Algorithm 4.2 is less. In detail, we list some results in Table 6 from which we can see that the approximate eigenvalues $\lambda_{2*}^{h_l}$ and $\lambda_{3*}^{h_l}$ obtained by Algorithm 4.2 are not good approximations for the second and the third exact eigenvalue but are close to the first exact one (in Figs 8 and 9, the curves of e_2 also reflect this phenomenon), and the reason we have analyzed in Example 4.1.

Remark 4.1. In our computation, Algorithm 4.2 and Algorithm 4.3 are performed by solving an eigenvalue problem on the coarsest mesh and then projecting the discrete eigenfunction on the subsequent finer meshes where only one linear system needs to be solved. As for multiple eigenvalues, in recent publication [18], the authors considered the approximation of the eigenspace and summed all the residuals up as for the a posteriori error estimate of eigenspace for the first type adaptive algorithm. Applying this method to Algorithm4.2 and Algorithm 4.3 would be expected.

5 Concluding Remarks

In this paper we propose and analyze an a posteriori error estimator for the finite element multi-scale discretization approximation of the Steklov eigenvalue problem. Based on the a posteriori error estimates, we design an adaptive

Table 4: The results for the three smallest eigenvalues of Example 4.2 obtained by Algorithms 4.1 and 4.3

k	l	$N_{k+,l}$	$\lambda_k^{h_l}$	$CPU_{k+,l}(s)$	$N_{k,l}$	λ_{k,h_l}	$CPU_{k,l}(s)$
1	25	354231	0.18296426	213.98	354231	0.18296426	383.75
1	31	785693	0.18296424	533.99	——	————	——-
2	38	433695	0.89364798	285.66	433695	0.89364798	496.74
2	42	768861	0.89364690	525.16	——	————	——-
3	27	406124	1.68860273	237.45	406124	1.68860273	441.66
3	32	801368	1.68860181	511.28	——	————	——-

Table 5: The results for the three smallest eigenvalues of Example 4.2 obtained by Algorithms 4.2 and 4.3

k	l	$N_{k+,l}$	$\lambda_k^{h_l}$	$CPU_{k+,l}(s)$	$N_{k*,l}$	$\lambda_{k*}^{h_l}$	$CPU_{k*,l}(s)$
1	25	354231	0.18296426	213.98	354231	0.18296426	192.43
1	31	785693	0.18296424	533.99	785693	0.18296424	460.75
2	38	433695	0.89364798	285.66	394017	0.18296425	225.75
2	42	768861	0.89364690	525.16	716738	0.18296425	402.61
3	27	465753	1.68860256	237.45	268505	0.18296426	118.36
3	32	801368	1.68860181	511.28	445339	0.18296425	246.42

Table 6: The results for the three smallest eigenvalues of Example 4.2 obtained by Algorithm 4.2

l	$N_{1*,l}$	$\lambda_{1*}^{h_l}$	l	$N_{2*,l}$	$\lambda_{2*}^{h_l}$	l	$N_{3*,l}$	$\lambda_{3*}^{h_l}$
1	12545	0.18296492	1	12545	0.89423511	1	12545	1.68870013
6	24006	0.18296450	9	12719	0.88859988	6	20431	1.68485494
12	56774	0.18296434	11	13133	0.32003522	8	22958	0.26100111
18	132597	0.18296429	13	13519	0.18326301	10	24108	0.18297591
25	354231	0.18296426	38	394017	0.18296425	27	225747	0.18296426
31	785693	0.18296424	42	716738	0.18296425	32	445339	0.18296425

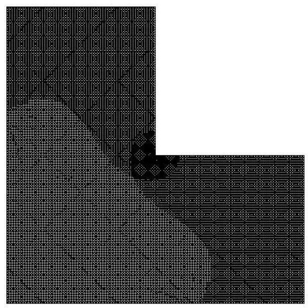

Figure 4: The adaptively refined mesh with 27113 degrees of freedom for λ_1 on $([0,1]\times[0,\frac{1}{2}])\bigcup([0,\frac{1}{2}]\times[\frac{1}{2},1])$ by Algorithm 4.3

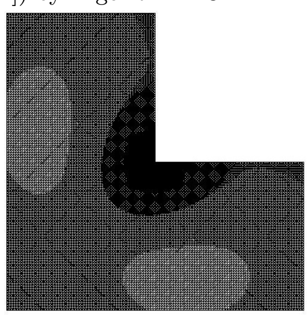

Figure 5: The adaptively refined mesh with 27537 degrees of freedom for λ_2 on $([0,1]\times[0,\frac{1}{2}])\bigcup([0,\frac{1}{2}]\times[\frac{1}{2},1])$ by Algorithm 4.3

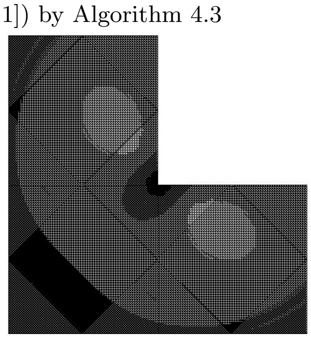

Figure 6: The adaptively refined mesh with 26341 degrees of freedom for λ_3 on $([0,1]\times[0,\frac{1}{2}])\bigcup([0,\frac{1}{2}]\times[\frac{1}{2},1])$ by Algorithm 4.3

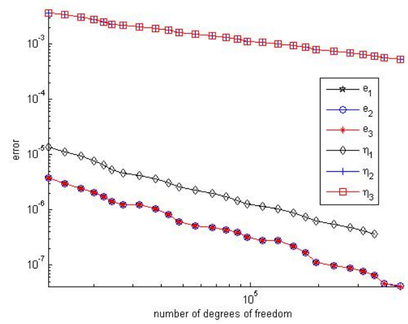

Figure 7: The curves of error and the a posteriori estimators for the first eigenvalue of Example 4.2 in log-log scale

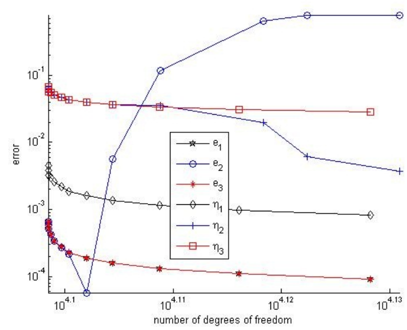

Figure 8: The curves of error and the a posteriori estimators for the second eigenvalue of Example 4.2 in log-log scale

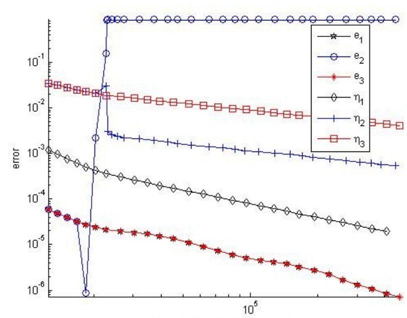

Figure 9: The curves of error and the a posteriori estimators for the third eigenvalue of Example 4.2 in log-log scale

algorithm of shifted-inverse iteration type. With this adaptive algorithm we can seek efficiently approximations of any eigenpair of the Steklov eigenvalue problem.

The techniques used in this paper can also be applied to finite elements approximation of second order self-adjoint elliptic eigenvalue problems. The global upper and lower bound of the error in [29] can also be obtained by using the argument in this paper, moreover, the local lower bound of the error can also be derived.

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