A GENERALIZATION OF WATSON TRANSFORMATION AND REPRESENTATIONS OF TERNARY QUADRATIC FORMS

JANGWON JU, INHWAN LEE AND BYEONG-KWEON OH

Abstract. Let L be a positive definite (non-classic) ternary $\mathbb{Z}$-lattice and let p be a prime such that a $\frac{1}{2}\mathbb{Z}_p$-modular component of L_p is nonzero isotropic and $4 \cdot dL$ is not divisible by p. For a nonnegative integer m, let $\mathcal{G}_{L,p}(m)$ be the genus with discriminant $p^m \cdot dL$ on the quadratic space $L^{p^m} \otimes \mathbb{Q}$ such that for each lattice $T \in \mathcal{G}_{L,p}(m)$, a $\frac{1}{2}\mathbb{Z}_p$-modular component of T_p is nonzero isotropic, and T_q is isometric to $(L^{p^m})_q$ for any prime q different from p. Let $r(n, M)$ be the number of representations of an integer n by a $\mathbb{Z}$-lattice M. In this article, we show that if $m \leqslant 2$ and n is divisible by p only when $m = 2$, then for any $T \in \mathcal{G}_{L,p}(m)$, $r(n, T)$ can be written as a linear summation of $r(pn, S_i)$ and $r(p^3n, S_i)$ for $S_i \in \mathcal{G}_{L,p}(m+1)$ with an extra term in some special case. We provide a simple criterion on when the extra term is necessary, and we compute the extra term explicitly. We also give a recursive relation to compute $r(n, T)$, for any $T \in \mathcal{G}_{L,p}(m)$, by using the number of representations of some integers by lattices in $\mathcal{G}_{L,p}(m + 1)$ for an arbitrary integer m.

1. Introduction

For a positive definite (non-classic) integral ternary quadratic form

$$f(x_1, x_2, x_3) = \sum_{1 \leqslant i \leqslant j \leqslant 3} a_{ij} x_i x_j \qquad (a_{ij} \in \mathbb{Z})$$

and an integer n, we define a set $R(n, f) = \{(x_1, x_2, x_3) \in \mathbb{Z}^3 : f(x_1, x_2, x_3) = n\}$, and $r(n, f) = |R(n, f)|$. It is well known that $R(n, f)$ is always finite if f is positive definite. The theta series $\theta_f(z)$ of f is defined by

$$\theta_f(z) = \sum_{n=0}^{\infty} r(n, f) e^{2\pi i n z},$$

which is a modular form of weight $\frac{3}{2}$ and some character with respect to a certain congruence subgroup. Finding a closed formula for $r(n, f)$ or finding all integers n such that $r(n, f) \neq 0$ for an arbitrary ternary form f are quite old problems which are still widely open. As a simplest case, Gauss showed that if f is a sum of three squares, then $r(n, f)$ is a multiple of the Hurwitz-Kronecker class number.

Though it seems to be quite difficult to find a closed formula for $r(n, f)$, some various relations between $r(n, f)$'s are known. One of the important relations is

2000 *Mathematics Subject Classification.* Primary 11E12, 11E20.

Key words and phrases. Representaions of ternary quadratic forms, Watson transformation.

This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2014R1A1A2056296).

the Minkowski-Siegel formula. Let $O(f)$ be the group of isometries of f and $o(f) = |O(f)|$. The weight $w(f)$ of f is defined by $w(f) = \sum_{[f']\in \text{gen}(f)} \frac{1}{o(f')}$, where $[f']$ is the equivalence class containing f'. The Minkowski-Siegel formula says that the weighted sum of the representations by quadratic forms in the genus is, in principle, the product of local densities, that is,

$$\frac{1}{w(f)} \sum_{[f']\in \text{gen}(f)} \frac{r(n,f')}{o(f')} = c^* \prod_p \alpha_p(n, f_p),$$

where the constant c^* can easily be computable and α_p is the local density depending only on the local structure of f over $\mathbb{Z}_p$. Hence if the class number of f is one, then we have a closed formula on $r(n,f)$. As a natural modification of the Minkowski-Siegel formula, it was proved in [6] and [12] that the weighted sum of the representations of quadratic forms in the spinor genus is also equal to the product of local densities except spinor exceptional integers (see also [11] for spinor exceptional integers).

For any prime $p \nmid 2df$, the action of Hecke operators $T(p^2)$ on the theta series of the quadratic form f gives

$$r(p^2n, f) + \left(\frac{-ndf}{p}\right) r(n,f) + p \cdot r\left(\frac{n}{p^2}, f\right) = \sum_{[f']\in \text{gen}(f)} \frac{r^*(p^2f', f)}{o(f')} r(n, f').$$

Here, if n is not divisible by p^2, then $r\left(\frac{n}{p^2}, f\right) = 0$, and $r^*(p^2f', f)$ is the number of primitive representations of p^2f' by f. For details, see [1] and [5].

Another important relation comes from the Watson transformation. If a unimodular component of the ternary form f in a Jordan decomposition over $\mathbb{Z}_p$ is anisotropic, then one may easily show that

$$r(pn, f) = r(pn, \Lambda_p(f)),$$

where $\Lambda_p(f)$ is defined in Section 2. Hence the theta series of f completely determines the theta series of $\lambda_p(f)$. Unfortunately if a unimodular component of the ternary form f over $\mathbb{Z}_p$ is isotropic, one cannot expect such a nice relation. In this article, we consider the case when a unimodular component of the ternary form f over $\mathbb{Z}_p$ is isotropic.

The subsequence discussion will be conducted in the more adapted geometric language of quadratic spaces and lattices. The term "lattice" will always refer to a positive definite non-classic integral $\mathbb{Z}$-lattice on an n-dimensional positive definite quadratic space over $\mathbb{Q}$. Here, a $\mathbb{Z}$-lattice is said to be *non-classic* if the norm ideal $\mathfrak{n}(L)$ of L is contained in $\mathbb{Z}$. Let $L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \cdots + \mathbb{Z}x_n$ be a $\mathbb{Z}$-lattice of rank n. We write

$$L \simeq (B(x_i, x_j)).$$

The right hand side matrix is called a *matrix presentation* of L. Any unexplained notations and terminologies can be found in [7] or [8].

Let V be a (positive definite) ternary quadratic space and let L be a (non-classic) ternary $\mathbb{Z}$-lattice on V. Let p be a prime such that $L_p \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \epsilon \rangle$, where

$\epsilon \in \mathbb{Z}_p^\times$. For any nonnegative integer m, let $\mathcal{G}_{L,p}(m)$ be a genus on a quadratic space W such that each $\mathbb{Z}$-lattice $T \in \mathcal{G}_{L,p}(m)$ satisfies

$$T_p \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \epsilon p^m \rangle \quad \text{and} \quad T_q \simeq (L^{p^m})_q \ \text{ for any } q \neq p.$$

Here $W = V$ if m is even, $W = V^p$ otherwise. The aim of this article is to show that if $T \in \mathcal{G}_{L,p}(m)$ for $m = 0$ or 1, then there are rational numbers a_i, b_i such that

$$r(n,T) = \sum_{[S_i] \in \mathcal{G}_{L,p}(m+1)} \big(a_i r(pn, S_i) + b_i r(p^3 n, S_i)\big) + (\text{some extra term}).$$

In Section 4, we prove this statement in each case and compute the rational numbers a_i's, b_i's and the extra term explicitly. For the case when $m = 2$, we give an example such that the above statement does not hold, and prove that the above statement still holds for $m = 2$ if we additionally assume that n is divisible by p. In the case when $m \geqslant 3$, we show that under some restriction, the above statement holds if we replace $r(n,T)$ by $r(p^2n,T) - pr(n,T)$, and for any integer n not divisible by p, both $r(n,T)$ and $r(pn,T)$ can be written as a linear summation of $r(pn,S)$'s and $r(n,S)$'s, respectively, for $S \in \mathcal{G}_{L,p}(m+1)$.

In some cases, the extra term in the above equation can be removed. To determine when it happens, we need to know some structure of the graph $\mathfrak{G}_{L,p}(m)$ defined by the equivalence classes in $\mathcal{G}_{L,p}(m)$ and $\mathcal{G}_{L,p}(m+1)$. The definition and basic facts on the graph $\mathfrak{G}_{L,p}(m)$ will be treated in Section 3.

For any integer a, we say that $\frac{a}{2}$ is divisible by a prme p if p is odd and $a \equiv 0 \pmod{p}$, or $p = 2$ and $a \equiv 0 \pmod{4}$.

2. A generalization of Watson transformation

Let L be a ternary $\mathbb{Z}$-lattice. Recall that we are assuming that a (quadratic) $\mathbb{Z}$-lattice is non-classic and positive definite. For any prime p, the λ_p-transformation (or Watson transformation) is defined as follows:

$$\Lambda_p(L) = \{x \in L : Q(x+z) \equiv Q(z) \pmod{p} \text{ for all } z \in L\}.$$

Let $\lambda_p(L)$ be the primitive lattice obtained from $\Lambda_p(L)$ by scaling $V = L \otimes \mathbb{Q}$ by a suitable rational number. Assume that p is odd. If the unimodular component in a Jordan decomposition of L_p is anisotropic, it is well known that

$$R(pn, L) = R(pn, \Lambda_p(L)). \tag{2.1}$$

Hence $r(n, \lambda_p(L)) = r(pn, L)$ if $p\mathbb{Z}_p$-modular component of L_p is nonzero, and $r(n, \lambda_p(L)) = r(p^2n, L)$ otherwise. One may easily show that (2.1) still holds for $p = 2$ unless

$$L_2 \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \alpha \rangle, \quad (\alpha \in \mathbb{Z}_2).$$

The readers are referred to [3] for more properties of the operators Λ_p.

Let L be a ternary $\mathbb{Z}$-lattice and let p be a fixed prime. In the remaining of this section, we always assume that in a Jordan splitting of L_p,

$$\textit{the } \frac{1}{2}\mathbb{Z}_p\textit{-modular component is non-zero isotropic.} \tag{2.2}$$

The purpose of this article is to find similar results to (2.1) under this assumption. To do this, we generalize Watson's transformation in various directions. Since

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \perp \langle \delta \rangle \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle 5\delta \rangle \ \text{ over } \mathbb{Z}_2$$

for any $\delta \in \mathbb{Z}_2^\times$, any $\mathbb{Z}$-lattice L such that L_2 is isometric to the above will also be considered when $p = 2$.

Definition 2.1. Assume that p is odd. For $\epsilon = 0$ or ± 1, we define

$$S_p(\epsilon, L) = \left\{ x \in L \;\middle|\; \left(\frac{Q(x)}{p} \right) = \epsilon \right\}.$$

We also define $S_2(0, L) = \{x \in L : Q(x) \equiv 0 \pmod 2\}$ and $S_2(*, L) = L - S_2(0, L)$.

Let $\mathfrak{B} = \{x_1, x_2, x_3\}$ be a (ordered) basis of a ternary $\mathbb{Z}$-lattice L and p be a prime. We define a natural projection map

$$\phi_{\mathfrak{B}} : L - pL \to (L/pL)^* \to \mathbb{P}^2,$$

where $\mathbb{P}^2$ is the 2-dimensional projective space over the finite field $\mathbb{F}_p$. The set $\phi_{\mathfrak{B}}(S_p(\epsilon, L) - pL)$ is denoted by $s_p^{\mathfrak{B}}(\epsilon, L)$ for any $\epsilon \in \{0, 1, -1\}$ if p is odd and $\epsilon \in \{0, *\}$ otherwise. If the basis $\mathfrak{B}$ is obvious, we will omit it. For each element $\mathbf{s} \in \mathbb{P}^2$, we define a $\mathbb{Z}$-sublattice $L_{\mathbf{s}} := \phi_{\mathfrak{B}}^{-1}(\mathbf{s}) \cup pL$ of L, and

$$\Omega_p(\epsilon, L) = \{L_{\mathbf{s}} \mid \mathbf{s} \in s_p^{\mathfrak{B}}(\epsilon, L)\}.$$

Note that if $T : \mathfrak{B} \to \mathfrak{C}$ is the transition matrix between ordered bases, then one may easily show that $T(s_p^{\mathfrak{B}}(\epsilon, L)) = s_p^{\mathfrak{C}}(\epsilon, L)$. Hence the set $\Omega_p(\epsilon, L)$ is independent of choices of the basis for L.

Lemma 2.2. *Assume that a ternary $\mathbb{Z}$-lattice L and a prime p satisfies the condition (2.2). If $4dL_p \in \mathbb{Z}_p^\times$, then*

$$|s_p(0, L)| = p + 1, \quad |s_p(\pm 1, L)| = \frac{p\left(p \pm \left(\frac{-dL}{p}\right)\right)}{2} \quad \textit{and} \quad s_2(*, L) = 4$$

and

$$|s_p(0, L)| = 2p + 1, \quad |s_p(1, L)| = |s_p(-1, L)| = \frac{p(p-1)}{2} \quad \textit{and} \quad s_2(*, L) = 2,$$

otherwise.

Proof. Since everything is trivial for $p = 2$, we assume that p is odd. For the unimodular case, see Theorem 1.3.2 of [7]. Assume that L_p is not unimodular. Fix an ordered basis $\mathfrak{B} = \{x_1, x_2, x_3\}$ of L such that

$$(B(x_i, x_j)) \equiv \text{diag}(1, -1, p^{\text{ord}_p(dL)}\delta) \pmod{p^{\text{ord}_p(dL)+1}},$$

for some $\delta \in \mathbb{Z} - p\mathbb{Z}$. Note that such a basis always exists by the Weak Approximation Theorem. Assume that $x = a_1x_1 + a_2x_2 + a_3x_3 \in S_p(0, L)$. Then $a_1^2 \equiv a_2^2 \pmod p$. Therefore

$$s_p^{\mathfrak{B}}(0, L) = \{(0, 0, 1), (1, \pm 1, d)\}, \qquad \text{where } d \in \mathbb{F}_p.$$

The lemma follows from this. The case when $\epsilon = \pm 1$ can be done in a similar manner. ☐

Lemma 2.3. *Under the same assumptions given above, assume that p is an odd prime. If $\epsilon \neq 0$ or $\epsilon = 0$ and L_p is unimodular, then every $\mathbb{Z}$-lattice $M \in \Omega_p(\epsilon, L)$ is contained in one genus. Furthermore for the former case,*

$$M_q \simeq \begin{cases} \langle \delta, -p^2\delta, -p^2dL \rangle & \text{if } q = p, \\ L_q & \text{otherwise,} \end{cases}$$

where $\delta \in \mathbb{Z}_p^\times$ such that $\left(\frac{\delta}{p}\right) = \epsilon$ and,

$$M_q \simeq \begin{cases} \langle p, -p, -p^2dL \rangle & \text{if } q = p, \\ L_q & \text{otherwise,} \end{cases}$$

for the latter case. If L_p is not unimodular and $\epsilon = 0$ then every $\mathbb{Z}$-lattice $M \in \Omega_p(0, L)$ is exactly contained in two genera. More precisely

$$M_q \simeq \begin{cases} \langle p^2, -p^2, -dL \rangle \ \text{ or } \ \langle p, -p, -p^2dL \rangle & \text{if } q = p, \\ L_q & \text{otherwise.} \end{cases}$$

Proof. Let $L = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3$ and $M \in \Omega_p(\epsilon, L)$. Since $pL \subset M$, we may assume without loss of generality that $M = \mathbb{Z}(x_1 + b_2x_2 + b_3x_3) + \mathbb{Z}(px_2) + \mathbb{Z}(px_3)$. First assume that $\epsilon \neq 0$. Then we may further assume that $\left(\frac{Q(x_1+b_2x_2+b_3x_3)}{p}\right) = \epsilon$. Since $Q(x_1 + b_2x_2 + b_3x_3) \in \mathbb{Z}_p^\times$,

$$M_p \simeq \langle Q(x_1 + b_2x_2 + b_3x_3) \rangle \perp m_p$$

for some binary sublattice m_p of M_p whose scale is $p^2\mathbb{Z}_p$. The assertion follows from this. Assume that $\epsilon = 0$ and L_p is unimodular. In this case we may assume that $Q(x_1 + b_2x_2 + b_3x_3) \in p\mathbb{Z}_p$. Then $B(x_1 + b_2x_2 + b_3x_3, x_2)$ or $B(x_1 + b_2x_2 + b_3x_3, x_3)$ is a unit in $\mathbb{Z}_p$, for L_p is unimodular. The assertion follows from this.

Finally assume that L_p is not unimodular and $\epsilon = 0$. In this case we may assume that the ordered basis $\mathfrak{B} = \{x_1, x_2, x_3\}$ satisfies every condition in Lemma 2.2. Then by a direct computation we know $L_{(0,0,1)} \in \Omega_p(0, L)$ satisfies the first local property and the others satisfy the second local property. ☐

Lemma 2.4. *Under the same assumptions given above, assume that $p = 2$. Let M be a $\mathbb{Z}$-lattice in $\Omega_2(\epsilon, L)$. If $-4dL_2 = \delta \in \mathbb{Z}_2^\times$, then*

$$M_2 \simeq \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle 4\delta \rangle & \text{if } \epsilon = 0, \\ \langle 1, -1, 4\delta \rangle \quad \text{or} \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \perp \langle \delta \rangle & \text{otherwise,} \end{cases}$$

and $M_q \simeq L_q$ for any prime $q \neq 2$. If $-4dL_2 = \delta \in 2\mathbb{Z}_2$, then

$$M_2 \simeq \begin{cases} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \perp \langle 4\delta \rangle \quad \text{or} \quad \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \perp \langle \delta \rangle & \text{if } \epsilon = 0, \\ \langle 1, -1, 4\delta \rangle & \text{otherwise}, \end{cases}$$

and $M_q \simeq L_q$ for any prime $q \neq 2$.

Proof. The proof is quite similar to the above. □

Lemma 2.5. *Assume that a ternary $\mathbb{Z}$-lattice L and a prime p satisfies the condition (2.2). For any positive integer n such that $\left(\frac{n}{p}\right) = \epsilon$,*

$$r(n, L) = \sum_{M \in \Omega_p(\epsilon, L)} r(n, M) - (|s_p(\epsilon, L)| - 1) r(n, pL).$$

This equality also holds for $p = 2$ if either $\epsilon = 0$ and n is even or $\epsilon = *$ and n is odd.

Proof. The lemma follows from the facts that

$$\{x \in S_p(\epsilon, L) - pL \mid Q(x) = n, \quad \phi(x) = s\} = \{x \in L_s \mid Q(x) = n\} - R(n, pL),$$

and

$$L_s \cap L_t = pL \qquad \text{if and only if} \qquad s \neq t,$$

for any $s, t \in \mathbb{P}^2$. □

Under the same assumptions given above, one may easily show that $dM = p^4 dL$ for any $M \in \Omega_p(\epsilon, L)$. Furthermore $L/M \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$.

Remark 2.6. If a $\frac{1}{2}\mathbb{Z}_p$-modular component of L_p is zero or anisotropic, the above lemma implies the equation (2.1). So we may consider the above lemma as a natural generalization of Watson's transformation.

Let L and ℓ be ternary $\mathbb{Z}$-lattices such that $d\ell = p^4 dL$. We define

$$\tilde{R}(\ell, L) = \{\sigma : \ell \to L \mid L/\sigma(\ell) \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}\} \text{ and } \tilde{r}(\ell, L) = |\tilde{R}(\ell, L)|.$$

One may easily show that $|\{M \in \Omega_p(\epsilon, L) \mid M \simeq \ell\}| = \tilde{r}(\ell, L)/o(\ell)$ for any $\epsilon \in \{0, \pm 1\}$ or $\epsilon \in \{0, *\}$.

Lemma 2.7. *For any ternary $\mathbb{Z}$-lattices ℓ and L such that $d\ell = p^4 dL$, we have*

$$\tilde{r}(\ell, L) = r(p\ell^{\#}, L^{\#}) = r(pL, \ell).$$

Proof. Assume that $T \in \tilde{R}(\ell, L)$. Then $T^t M_L T = M_\ell$ and pT^{-1} is an integral matrix. Since

$$(pT^{-1}) M_L^{-1} (pT^{-1})^t = p^2 M_\ell^{-1},$$

$(pT^{-1})^t \in R(p\ell^{\#}, L^{\#})$. Conversely if $S^t M_L^{-1} S = p^2 M_\ell^{-1}$, then $d(S) = \pm p$. Hence pS^{-1} is an integral matrix and $(pS^{-1})^t \in \tilde{R}(\ell, L)$. This completes the proof. □

Assume that a ternary $\mathbb{Z}$-lattice L and a prime p satisfies the condition (2.2). In the remaining of this section, we additionally assume that $\text{ord}_p(4\cdot dL)\geqslant 2$. Let $K=\lambda_p(L)$ and let

$$\text{gen}_p^K(L)=\{L'\in\text{gen}(L):\lambda_p(L')\simeq K\}.$$

For any integer n, we also define

$$r(n,\text{gen}_p^K(L))=\sum_{\substack{[L']\in\text{gen}(L)\\ \lambda_p(L')\simeq K}}\frac{r(n,L')}{o(L')}.$$

In fact, every $\mathbb{Z}$-lattice in $\text{gen}_p^K(L)$ is isometric to one of $\mathbb{Z}$-lattices in

$$\Gamma_p^L(\Lambda_p(L))=\{M\subset K\mid M\in\text{gen}(L)\}.$$

Furthermore, the isometry group $O(K)$ acts on $\Gamma_p^L(\Lambda_p(L))$. Each orbit under this action consists of all isometric lattices in $\Gamma_p^L(\Lambda_p(L))$, and hence there are exactly $\frac{o(K)}{o(L)}$ lattices that are isometric to L in $\Gamma_p^L(\Lambda_p(L))$. There are exactly p^2+p+1 sublattices of K with index p. They are, in fact,

$$K_0=\mathbb{Z}(px_1)+\mathbb{Z}x_2+\mathbb{Z}x_3,\quad K_{1,u}=\mathbb{Z}(x_1+ux_2)+\mathbb{Z}(px_2)+\mathbb{Z}x_3\ (0\leqslant u\leqslant p-1)$$

and

$$K_{2,\alpha,\beta}=\mathbb{Z}(x_1+\alpha x_3)+\mathbb{Z}(x_2+\beta x_3)+\mathbb{Z}(px_3)\ (0\leqslant\alpha,\beta\leqslant p-1).$$

Among these sublattices of K, there are exactly $\frac{p(p+1)}{2}$ lattices (p^2 lattices) that are contained in the genus of L if $\text{ord}_p(4\cdot dL)=2$ ($\text{ord}_p(4\cdot dL)\geqslant 3$, respectively) (for details, see [4]).

Proposition 2.8. *Assume that $\mathbb{Z}$-lattices L and K and a prime p satisfies the above condition. Then for any integer n not divisible by p, we have*

$$r(n,gen_p^K(L))=\begin{cases}\dfrac{p-\left(\frac{-ndK}{p}\right)}{2}\dfrac{r(n,K)}{o(K)} & \text{if } p\neq 2 \text{ and } ord_p(4\cdot dL)=2,\\ \dfrac{r(n,K)-r(n,\Lambda_1(K))}{o(K)} & \text{if } p=2 \text{ and } ord_p(4\cdot dL)=2,\\ p\dfrac{r(n,K)}{o(K)} & \text{if } ord_p(4\cdot dL)\geqslant 3,\end{cases}$$

where $\Lambda_1(K)=\{x\in K:B(x,K)\subset\mathbb{Z}\}$ is a sublattice of K.

Proof. Since proofs are quite similar to each other, we only provide the proof of the first case. Assume that $Q(x_1)=n$ for some $x_1\in K$. We will count the number of lattices containing the vector x_1 in $\Gamma_p^L(\Lambda_p(L))$. Note that for any vector $y\in K$ and any integer d not divisible by p, $dy\in M$ if and only if $y\in M$ for any $M\in\Gamma_p^L(\Lambda_p(L))$. Hence we may assume that x_1 is a primitive vector in K. Then there is a basis $\{x_1,x_2,x_3\}$ of K such that for some integer t not divisible by p,

$$(B(x_i,x_j))\equiv\text{diag}(n,n,t)\pmod p.$$

Among all sublattices of K with index p that are contained in the genus of L, those $\mathbb{Z}$-lattices containing x_1 are $K_{2,0,\beta}$, for any β satisfying $\left(\frac{-n^2-n\beta^2dK}{p}\right)=1$, and $K_{1,0}$

only when $\left(\frac{-ndK}{p}\right)=1$. Therefore one may easily show that the total number of such lattices is $\frac{p-\left(\frac{-ndK}{p}\right)}{2}$. The proposition follows from

$$\sum_{M\in\Gamma_p^L(\lambda_p(L))} r(n,M)=\sum_{[M]\in\text{gen}_p^K(L)}\frac{o(K)}{o(M)}r(n,M)=\frac{p-\left(\frac{-ndK}{p}\right)}{2}r(n,K).$$

This completes the proof. □

Proposition 2.9. *Under the same assumption given above, if n is divisible by p, then we have*

$$r(n,gen_p^K(L))=\begin{cases} p\dfrac{r(n,K)}{o(K)}+\dfrac{p(p-1)}{2}\dfrac{r\left(\frac{n}{p^2},K\right)}{o(K)} & \text{if } ord_p(4\cdot dL)=2,\\ p\dfrac{r(n,K)}{o(K)}+p^2\dfrac{r\left(\frac{n}{p^2},K\right)}{o(K)}-p\dfrac{r(n,\Lambda_p(K))}{o(K)} & \text{otherwise.}\end{cases}$$

Proof. First we define

$$R^*(n,K)=\{x\in K\mid Q(x)=n,\ x \text{ is primitive as a vector in } K_p\},$$

$r^*(n,K)=|R^*(n,K)|$, and $r^\diamond(n,K)=r(n,K)-r^*(n,K)$. Let $x_1\in K$ be a vector such that $Q(x_1)=n$. We will compute the number of lattices containing x_1 in $\Gamma_p^L(\Lambda_p(L))$. By the similar reasoning to the above, we may assume that there is a primitive vector $\widetilde{x_1}\in K$ and a nonnegative integer k such that $x_1=p^k\widetilde{x_1}$. If $k>0$, then x_1 is contained in all lattices in $\Gamma_p^L(\Lambda_p(L))$.

Assume that $k=0$. If $\text{ord}_p(4\cdot dL)=2$, then there is a basis $\{x_1,x_2,x_3\}$ of K such that

$$(B(x_i,x_j))\equiv\begin{pmatrix}0&b&0\\b&0&0\\0&0&e\end{pmatrix}\pmod p,$$

where $2b$ and e are integers not divisible by p. Among all sublattices of K with index p that are contained in the genus of L, those $\mathbb{Z}$-lattices containing x_1 are $K_{2,0,\beta}$ for any β. Therefore if $\text{ord}_p(4\cdot dL)=2$, we have

$$\begin{aligned}\sum_{[M]\in\text{gen}_p^K(L)}\frac{o(K)}{o(M)}r(n,M)&=p\cdot r^*(n,K)+\frac{p(p+1)}{2}r^\diamond(n,K)\\&=p\cdot r(n,K)+\frac{p(p-1)}{2}r\left(\frac{n}{p^2},K\right).\end{aligned}$$

Suppose that $\text{ord}_p(4\cdot dL)\geqslant 3$. If there is a vector $y\in K$ such that $2B(x_1,y)\not\equiv 0 \pmod p$, then there are exactly p lattices in $\Gamma_p^L(\Lambda_p(L))$ containing x_1. However if $2B(x_1,K)\subset p\mathbb{Z}$, then there does not exist a lattice in $\Gamma_p^L(\Lambda_p(L))$ that contains x_1. Note that

$$|\{x\in R^*(n,K)\mid 2B(x,K)\subset p\mathbb{Z}\}|=r(n,\Lambda_p(K))-r^\diamond(n,K).$$

Therefore we have

$$\sum_{[M]\in\text{gen}_p^K(L)}\frac{o(K)}{o(M)}r(n,M)=p(r(n,K)-r(n,\Lambda_p(K)))+p^2\cdot r^\diamond(n,K).$$

This completes the proof. □

3. Finite (multi-) graphs and ternary quadratic forms

Let V be a (positive definite) ternary quadratic space and let L be a (non-classic) ternary $\mathbb{Z}$-lattice on V. Let p be a prime such that $L_p \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \epsilon \rangle$, where $\epsilon \in \mathbb{Z}_p^\times$. For any nonnegative integer m, let $\mathcal{G}_{L,p}(m)$ be a genus on W such that each $\mathbb{Z}$-lattice $T \in \mathcal{G}_{L,p}(m)$ satisfies

$$T_p \simeq \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle \epsilon p^m \rangle \quad \text{and} \quad T_q \simeq (L^{p^m})_q \ \text{ for any } q \neq p.$$

Here $W = V$ if m is even, $W = V^p$ otherwise.

Lemma 3.1. *Let $T \in \mathcal{G}_{L,p}(m)$ and $S \in \mathcal{G}_{L,p}(m+1)$ be ternary $\mathbb{Z}$-lattices. Then we have*

$$\sum_{[N] \in \mathcal{G}_{L,p}(m+1)} \frac{\tilde{r}(N^p, T)}{o(N)} = \begin{cases} p+1 & \text{if } m = 0, \\ 2p & \text{otherwise} \end{cases} \quad \text{and} \quad \sum_{[M] \in \mathcal{G}_{L,p}(m)} \frac{r(M^p, S)}{o(M)} = 2.$$

Proof. Note that $\sum_{[N] \in \mathcal{G}_{L,p}(m+1)} \frac{\tilde{r}(N^p,T)}{o(N)}$ is the number of sublattices X of T such that

$$T/X \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \quad \text{and} \quad X^{\frac{1}{p}} \in \mathcal{G}_{L,p}(m+1).$$

Hence the first equality is a direct consequence of Lemmas 2.2, 2.3 and 2.4.

To prove the second equality, it suffices to show that there are exactly two sublattices of S with index p whose norm is $p\mathbb{Z}$. By Weak Approximation Theorem, there exists a basis $\{x_1, x_2, x_3\}$ for S such that

$$(B(x_i, x_j)) \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle p^{m+1}\delta \rangle \pmod{p^{m+2}},$$

where δ is an integer not divisible by p. Then for the following two sublattices defined by

$$\Gamma_{p,1}(S) = \mathbb{Z}px_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad \Gamma_{p,2}(S) = \mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}x_3,$$

one may easily show that $\Gamma_{p,i}(S)^{\frac{1}{p}} \in \mathcal{G}_{L,p}(m)$ for any $i = 1, 2$. Furthermore, norms of all the other sublattices of S with index p are not contained in $p\mathbb{Z}$. This completes the proof. □

Now we define a multi-graph $\mathfrak{G}_{L,p}(m)$ as follows: the set of vertices in $\mathfrak{G}_{L,p}(m)$ is the set of equivalence classes in $\mathcal{G}_{L,p}(m)$, say, $\{[T_1], [T_2], \dots, [T_h]\}$. The set of edges is exactly the set of equivalence classes in $\mathcal{G}_{L,p}(m+1)$, say, $\{[S_1], [S_2], \dots, [S_k]\}$. For each equivalence class $[S_w] \in \mathcal{G}_{L,p}(m+1)$, two vertices contained in the edge named by $[S_w]$ are defined by $[\Gamma_{p,1}(S_w)^{\frac{1}{p}}]$ and $[\Gamma_{p,2}(S_w)^{\frac{1}{p}}]$, where the lattice $\Gamma_{p,i}(S_w)^{\frac{1}{p}}$ that is defined in Lemma 3.1 is contained in $\mathcal{G}_{L,p}(m)$. Note that the graph $\mathfrak{G}_{L,p}(m)$

is, in general, a multi-graph that might have a loop. We define an $h \times k$ integer matrix $\mathfrak{M}_{L,p}(m) = (m_{ij})$ as follows:

$$m_{ij} = \begin{cases} 2 & \text{if } [S_j] \text{ is a loop of the vertex } [T_i], \\ 1 & \text{if } [S_j] \text{ is not a loop of the vertex } [T_i], \text{ though it contains } [T_i], \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\mathfrak{M}_{L,p}(m)$ is the incidence matrix of $\mathfrak{G}_{L,p}(m)$ if the graph $\mathfrak{G}_{L,p}(m)$ is simple.

For any $\mathbb{Z}$-lattice $T \in \mathcal{G}_{L,p}(m)$, we define

$$\Phi_p(T) = \{S \in \mathcal{G}_{L,p}(m+1) : \Gamma_{p,i}(S)^{\frac{1}{p}} = T \text{ for some } i = 1,2\}$$

and

$$\Psi_p(T) = \{M \in \mathcal{G}_{L,p}(m+2) : \lambda_p(M) = T\}.$$

Then Lemma 3.1 implies that $|\Phi_p(T)| = p+1$ if $m = 0$, $|\Phi_p(T)| = 2p$ otherwise.

Lemma 3.2. *Let $T \in \mathcal{G}_{L,p}(0)$ and $S, S' \in \Phi_p(T)$ $(S \neq S')$ be ternary $\mathbb{Z}$-lattices on V and V^p, respectively. Then there is a unique $\mathbb{Z}$-lattice $M \in \Psi_p(T)$ such that $\{\Gamma_{p,1}(M)^{\frac{1}{p}}, \Gamma_{p,2}(M)^{\frac{1}{p}}\} = \{S, S'\}$.*

Proof. For any $S, S' \in \Phi_p(T)$, we have $pS \subset S'$. Furthermore since $S \neq S'$ and $\text{ord}_p(4dS) = 1$, $S'/pS \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$. Therefore, there is a basis x_1, x_2, x_3 for S' such that

$$S' = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad pS = \mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}p^2x_3$$

and

$$(B(x_i, x_j)) = \begin{pmatrix} p^2a & pb & d \\ pb & pc & e \\ d & e & f \end{pmatrix},$$

where $a, c, f \in \mathbb{Z}$, $b, d, e \in \frac{1}{2}\mathbb{Z}$ and $p \nmid 2d$. Define a $\mathbb{Z}$-lattice

$$M = \left(\mathbb{Z}\left(\frac{x_1}{p}\right) + \mathbb{Z}x_2 + \mathbb{Z}x_3\right)^p \in \mathcal{G}_{L,p}(2).$$

Then one may easily show that $\lambda_p(M) = T$ and $\{\Gamma_{p,1}(M)^{\frac{1}{p}}, \Gamma_{p,2}(M)^{\frac{1}{p}}\} = \{S, S'\}$. As pointed out earlier, the number of $\mathbb{Z}$-lattices $M' \in \mathcal{G}_{L,p}(2)$ such that $\lambda_p(M') = T$ for any $T \in \mathcal{G}_{L,p}(0)$ is $\frac{p(p+1)}{2}$. Furthermore for any such a $\mathbb{Z}$-lattice M', we have $\Gamma_{p,i}(M')^{\frac{1}{p}} \in \Phi_p(T)$ for any $i = 1,2$ and $|\Phi_p(T)| = p+1$. Now the uniqueness of M follows from this observation. $\square$

The above lemma says that if $T \in \mathcal{G}_{L,p}(0)$, then there is always an edge containing $[S]$ and $[S']$ for any $S, S' \in \Phi_p(T)$. However this is not true in general if $T \in \mathcal{G}_{L,p}(m)$ for a positive integer m.

Lemma 3.3. *For a positive integer m, let $T \in \mathcal{G}_{L,p}(m)$ and $S, S' \in \Phi_p(T)$ be ternary $\mathbb{Z}$-lattices on V and V^p, respectively. If*

$$\lambda_p(S) = \Gamma_{p,1}(T)^{\frac{1}{p}} \quad \text{and} \quad \lambda_p(S') = \Gamma_{p,2}(T)^{\frac{1}{p}},$$

then there is a unique $\mathbb{Z}$-lattice $M \in \Psi_p(T)$ such that $\{\Gamma_{p,1}(M)^{\frac{1}{p}}, \Gamma_{p,2}(M)^{\frac{1}{p}}\} = \{S, S'\}$.

Proof. By Weak Approximation Theorem, there is a basis x_1, x_2, x_3 for T such that

$$(B(x_i, x_j)) \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle p^m \delta \rangle \pmod{p^{m+1}},$$

where δ is an integer not divisible by p. We may assume that

$$\Gamma_{p,1}(T)^{\frac{1}{p}} = (\mathbb{Z}px_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3)^{\frac{1}{p}}, \quad \Gamma_{p,2}(T)^{\frac{1}{p}} = (\mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}x_3)^{\frac{1}{p}}.$$

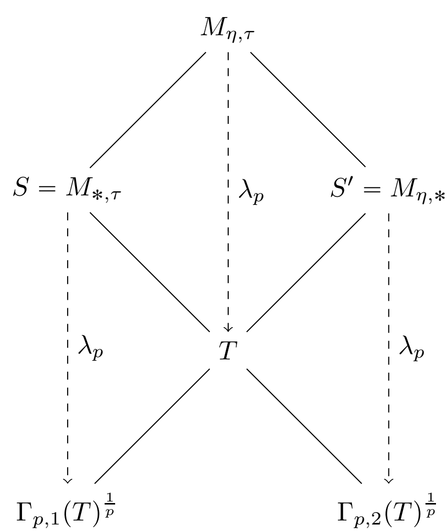
One may easily check that

$$\begin{aligned} \Phi_p(T) = \{M_{*,\beta} =& (\mathbb{Z}px_1 + \mathbb{Z}(x_2 + \beta x_3) + \mathbb{Z}px_3)^{\frac{1}{p}} : 0 \leqslant \beta \leqslant p-1\} \\ &\cup \{M_{\alpha,*} = (\mathbb{Z}(x_1 + \alpha x_3) + \mathbb{Z}px_2 + \mathbb{Z}px_3)^{\frac{1}{p}} : 0 \leqslant \alpha \leqslant p-1\} \end{aligned}$$

and

$$\Psi_p(T) = \{M_{\alpha,\beta} = \mathbb{Z}(x_1 + \alpha x_3) + \mathbb{Z}(x_2 + \beta x_3) + \mathbb{Z}px_3 : 0 \leqslant \alpha, \beta \leqslant p-1\}.$$

Since $\lambda_p(M_{*,\beta}) = \Gamma_{p,1}(T)^{\frac{1}{p}}$ and $\lambda_p(M_{\alpha,*}) = \Gamma_{p,2}(T)^{\frac{1}{p}}$ for any $0 \leqslant \alpha, \beta \leqslant p-1$, there are τ, η such that $S = M_{*,\tau}$ and $S' = M_{\eta,*}$.



3.1 Figure

Now, one may easily check that $M_{\eta,\tau}$ is the unique lattice in $\Psi_p(T)$ satisfying

$$\{\Gamma_{p,1}(M_{\eta,\tau})^{\frac{1}{p}}, \Gamma_{p,2}(M_{\eta,\tau})^{\frac{1}{p}}\} = \{M_{*,\tau}, M_{\eta,*}\}.$$

This completes the proof. □

Lemma 3.4. *For an integer $m \geqslant 2$, let $M_1, M_2 \in \mathcal{G}_{L,p}(m)$ be distinct $\mathbb{Z}$-lattices such that $\lambda_p(M_1) = \lambda_p(M_2) = T$. Then there is a path from $[M_1]$ to $[M_2]$ of length 4.*

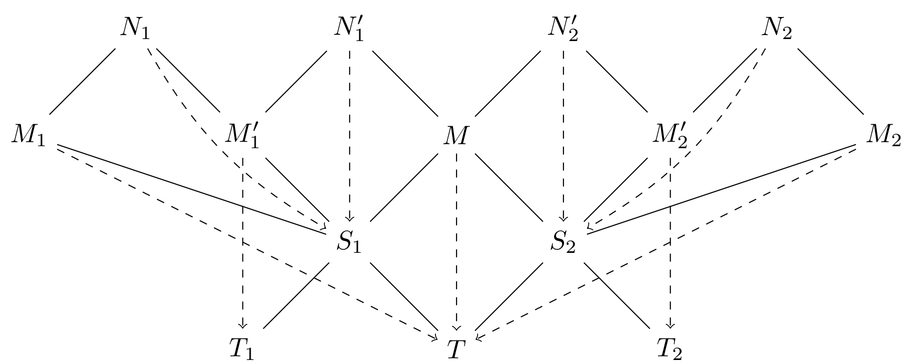
Proof. Note that if $\{\Gamma_{p,1}(M_1), \Gamma_{p,2}(M_1)\} = \{\Gamma_{p,1}(M_2), \Gamma_{p,2}(M_2)\}$, then $M_1 = M_2$. Hence, without loss of generality, we may assume that $S_1 = \Gamma_{p,1}(M_1)^{\frac{1}{p}}$ is different from $S_2 = \Gamma_{p,2}(M_2)^{\frac{1}{p}}$. If $m \geqslant 3$, then

$$\{\lambda_p(\Gamma_{p,1}(M_i)^{\frac{1}{p}}), \lambda_p(\Gamma_{p,2}(M_i)^{\frac{1}{p}})\} = \{\Gamma_{p,1}(T)^{\frac{1}{p}}, \Gamma_{p,2}(T)^{\frac{1}{p}}\}$$

for any $i = 1, 2$. Hence we further assume that $\lambda_p(S_1) \neq \lambda_p(S_2)$. Then by Lemmas 3.2 and 3.3, there is a $\mathbb{Z}$-lattice $M \in \mathcal{G}_{L,p}(m)$ such that $\lambda_p(M) = T$ and $\{\Gamma_{p,1}(M)^{\frac{1}{p}}, \Gamma_{p,2}(M)^{\frac{1}{p}}\} = \{S_1, S_2\}$. We define $\mathbb{Z}$-lattices T_1 and T_2 satisfying

$$\{\Gamma_{p,1}(S_1)^{\frac{1}{p}}, \Gamma_{p,2}(S_1)^{\frac{1}{p}}\} = \{T, T_1\} \quad \text{and} \quad \{\Gamma_{p,1}(S_2)^{\frac{1}{p}}, \Gamma_{p,2}(S_2)^{\frac{1}{p}}\} = \{T, T_2\}.$$

Let $M_i' \in \mathcal{G}_{L,p}(m)$ be a $\mathbb{Z}$-lattice in $\Phi_p(S_i)$ such that $\lambda_p(M_i') = T_i$ for $i = 1, 2$. Then by Lemma 3.3, there are $\mathbb{Z}$-lattices N_1, N_2, N_1', N_2' such that two vertices $[M_i]$ and $[M_i']$ are connected by the edge $[N_i]$, and two vertices $[M]$ and $[M_i']$ are connected by the edge $[N_i']$ for $i = 1, 2$. Therefore two vertices $[M_1]$ and $[M_2]$ are connected by a path of length 4 (see Figure 3.2).



3.2 Figure

The Lemma follows from this. □

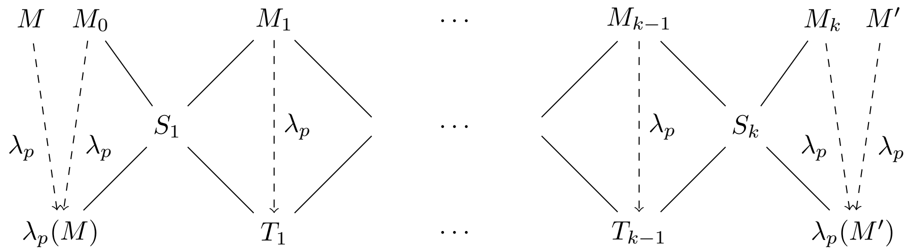
Lemma 3.5. *For an integer $m \geqslant 2$, let $[M], [M']$ be vertices of the graph $\mathfrak{G}_{L,p}(m)$. Then there is a path from $[M]$ to $[M']$ of length $e([M],[M'])$ in $\mathfrak{G}_{L,p}(m)$ if and only if there is a path from $[\lambda_p(M)]$ to $[\lambda_p(M')]$ of length $e([\lambda_p(M)],[\lambda_p(M')])$ in $\mathfrak{G}_{L,p}(m-2)$. Furthermore, in both cases, there is a path satisfying*

$$e([M],[M']) \equiv e([\lambda_p(M)],[\lambda_p(M')]) \pmod 2.$$

Proof. Note that "only if" part is trivial. Assume that $[\lambda_p(M)]$ and $[\lambda_p(M')]$ are connected by a path with edges $[S_1], [S_2], \dots, [S_k]$ as in Figure 3.3, where

$$\{\Gamma_{p,1}(S_i)^{\frac{1}{p}}, \Gamma_{p,2}(S_i)^{\frac{1}{p}}\} = \{T_{i-1}, T_i\}$$

for any $i = 2, 3, \dots, k-1$.



3.3 Figure

Then for any $i = 0, 1, \ldots, k$, there are $\mathbb{Z}$-lattices M_i such that $M_0 \in \Psi_p(\lambda_p(M)) \cap \Phi_p(S_1)$, $M_k \in \Psi_p(\lambda_p(M')) \cap \Phi_p(S_k)$, and $M_j \in \Psi_p(T_j) \cap \Phi_p(S_j) \cap \Phi_p(S_{j+1})$ for any $j = 1, 2, \ldots, k-1$. Now by Lemma 3.3, there are $\mathbb{Z}$-lattices N_i such that

$$\{\Gamma_{p,1}(N_i)^{\frac{1}{p}}, \Gamma_{p,2}(N_i)^{\frac{1}{p}}\} = \{M_{i-1}, M_i\} \quad \text{and} \quad \lambda_p(N_i) = S_i$$

for any $i = 1, 2, \ldots, k$. Since both $[M], [M_0]$ and $[M_k], [M']$ are connected by a path of length 4 by Lemma 3.4, $[M]$ and $[M']$ are connected by a path of length $k+8$. □

We investigate the graph $\mathfrak{G}_{L,p}(0)$ in more detail. Let $T \in \mathcal{G}_{L,p}(0)$ be a $\mathbb{Z}$-lattice. Note that the graph $Z(T,p)$ constructed in [9] is slightly different from our graph (see also [2]). In fact, the graph $Z(T,p)$ is a tree having infinitely many vertices. However our graph is finite and might have a loop. Two vertices $[T_i], [T_j] \in \mathfrak{G}_{L,p}(0)$ are connected by an edge if and only if there are $\mathbb{Z}$-lattices $T_i' \in [T_i]$ and $T_j' \in [T_j]$ such that T_i' and T_j' are connected by an edge in the graph $Z(T,p)$. If two lattices $T_i, T_j \in \mathcal{G}_{L,p}(0)$ are spinor equivalent, then both $[T_i]$ and $[T_j]$ are contained in the same connected component. Moreover, each connected component of $\mathfrak{G}_{L,p}(0)$ contains at most two spinor genera, and it contains only one spinor genus if and only if $\mathbf{j}(p) \in P_D J_{\mathbb{Q}}^T$, where D is the set of positive rational numbers and

$$\mathbf{j}(p) = (j_q) \in J_{\mathbb{Q}} \quad \text{such that } j_p = p \text{ and } j_q = 1 \text{ for any prime } q \neq p.$$

We say that $\mathfrak{G}_{L,p}(0)$ is of O-type if each connected component of $\mathfrak{G}_{L,p}(0)$ contains only one spinor genus, and it is of E-type otherwise. If $\mathfrak{G}_{L,p}(0)$ is of E-type, then adjacent classes are contained in different spinor genera (for details, see [2]), that is, each connect component of the graph $\mathfrak{G}_{L,p}(0)$ is a bipartite graph.

Assume that

$$\mathcal{G}_{L,p}(0) = \{[T_1], [T_2] \ldots, [T_h]\} \quad \text{and} \quad \mathcal{G}_{L,p}(1) = \{[S_1], [S_2], \ldots, [S_k]\} \tag{3.1}$$

are *ordered* sets of equivalence classes in each genus. We define

$$\mathfrak{M} = \left(\frac{r(T_i^p, S_j)}{o(T_i)}\right) \in M_{h,k}(\mathbb{Z}) \text{ and } \mathfrak{N} = \mathfrak{N}_{L,p}(0) = \left(\frac{r(T_i^p, S_j)}{o(S_j)}\right) \in M_{h,k}(\mathbb{Z}).$$

In fact, $\mathfrak{M}$ equals to $\mathfrak{M}_{L,p}(0)$, which is defined earlier. There is a nice relation between $\mathfrak{M}$, $\mathfrak{N}$ and the *Eichler's Anzahlmatrix* $\pi_p(T)$ defined in [5].

Definition 3.6. Under the assumptions given above, the matrix

$$\pi_p(T) = \left(\frac{r(pT_i, T_j)}{o(T_i)} - \delta_{ij}\right) \quad (1 \leqslant i, j \leqslant h)$$

is called the Eichler's Anzahlmatrix of T at p.

Note that $\pi_p(T)$ is independent of the choice of the lattice $T \in \mathcal{G}_{L,p}(0)$.

Lemma 3.7. *For any $\mathbb{Z}$-lattices $T \in \mathcal{G}_{L,p}(0)$ and $S \in \mathcal{G}_{L,p}(1)$, we have $r(S^p, T) = r(T^p, S)$.*

Proof. First we show that $\widetilde{R}(S^p, T) = R(S^p, T)$. Suppose that there is a $\sigma \in R(S^p, T)$ such that $T/\sigma(S^p) \simeq \mathbb{Z}/p^2\mathbb{Z}$. Then there is a basis for T such that

$$T = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \quad \text{and} \quad \sigma(S^p) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}(p^2x_3).$$

Since $\mathfrak{n}(\sigma(S^p)) \subset p\mathbb{Z}$, we have

$$Q(x_1) \equiv Q(x_2) \equiv 2B(x_1, x_2) \equiv 0 \pmod{p}.$$

This is a contradiction to the fact that $4dT$ is not divisible by p. Therefore the lemma follows from Lemma 2.7. □

For $\mathbb{Z}$-lattices X_1, X_2, Y_1 and Y_2, we write $(X_1, X_2) \simeq (Y_1, Y_2)$ if $X_1 \simeq Y_1$ and $X_2 \simeq Y_2$, or $X_1 \simeq Y_2$ and $X_2 \simeq Y_1$.

Proposition 3.8. *Under the notations and assumptions given above, we have*

$$\pi_p(T) + (p+1)I = \mathfrak{M} \cdot \mathfrak{N}^t.$$

Proof. Let $\mathfrak{U}_{ij}$ be the set of sublattices X of T_j such that

$$X \simeq pT_i \quad \text{and} \quad T_j/X \not\simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z},$$

and let $\mathfrak{V}_{ij}$ be the set of sublattices Y of T_j such that

$$Y^{\frac{1}{p}} \in \mathcal{G}_{L,p}(1) \quad \text{and} \quad \left(\Gamma_{p,1}(Y^{\frac{1}{p}}), \Gamma_{p,2}(Y^{\frac{1}{p}})\right) \simeq (T_i^p, T_j^p),$$

where $\Gamma_{p,i}(Y^{\frac{1}{p}})$ is a sublattice of $Y^{\frac{1}{p}}$ with index p defined in Lemma 3.1. Note that $\pi_p(T)_{ij} = |\mathfrak{U}_{ij}|$. Now we define a map $\Phi : \mathfrak{U}_{ij} \mapsto \mathfrak{V}_{ij}$ as follows. Assume that $X \in \mathfrak{U}_{ij}$. Then one may easily show that $T_j/X \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$. Hence there is a basis x_1, x_2, x_3 for T_j such that

$$T_j = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \quad \text{and} \quad X = \mathbb{Z}x_1 + \mathbb{Z}(px_2) + \mathbb{Z}(p^2x_3).$$

Since the integer $4d(T_j)$ is not divisible by p and $Q(x_1) \equiv 0 \pmod{p^2}$, $2B(x_1, x_2) \equiv 0 \pmod{p}$, neither $Q(x_2)$ nor $2B(x_1, x_3)$ is divisible by p. Define $\Phi(X) := Y = \mathbb{Z}x_1 + \mathbb{Z}(px_2) + \mathbb{Z}(px_3)$. Clearly, $Y = \Lambda_p(T_j \cap \frac{1}{p}X)$. Hence it is independent of the choice of basis for T_j. Furthermore one may easily check that $\Phi(X) = Y \in \mathfrak{V}_{ij}$. Conversely, there are exactly two sublattices of $Y^{\frac{1}{p}}$ with index p whose norm is contained in $p\mathbb{Z}$, and one of them is equal to T_j^p. If we define the other one, as a sublattice of Y, by $\Psi(Y)$, then $\Phi \circ \Psi = \Psi \circ \Phi = Id$. Therefore $\pi_p(T)_{ij} = |\mathfrak{V}_{ij}|$. Now from the definition,

$$|\mathfrak{V}_{ij}| = \sum_{w=1}^{k} \frac{r(S_w^p, T_j)}{o(S_w)} \eta_w,$$

where

$$\eta_w = \begin{cases} 1 & \text{if } (\Gamma_{p,1}(S_w), \Gamma_{p,2}(S_w)) \simeq (T_j^p, T_i^p), \\ 0 & \text{otherwise.} \end{cases}$$

Since $r(T_j^p, S_w) = r(S_w^p, T_j)$ by Lemma 3.7,

$$|\mathfrak{V}_{ij}| = \sum_{w=1}^{k} \frac{r(S_w^p, T_j)}{o(S_w)} \left(\frac{r(T_i^p, S_w)}{o(T_i)} - \delta_{ij} \right) = \begin{cases} \sum_{w=1}^{k} \mathfrak{M}_{iw} (\mathfrak{N}^t)_{wj} & \text{if } i \neq j, \\ \sum_{w=1}^{k} \mathfrak{M}_{iw} (\mathfrak{N}^t)_{wj} - (p+1) & \text{if } i = j, \end{cases}$$

by Lemma 3.1. The proposition follows from this. □

The following theorem states that the rank of $\mathfrak{M}_{L,p}(0) = \mathfrak{M}$ is related with some properties of the graph $\mathfrak{G}_{L,p}(0)$.

Theorem 3.9. *The followings are all equivalent:*

(1) $\mathfrak{G}_{L,p}(0)$ *is of O-type;*
(2) *rank*$(\mathfrak{M}) = h$*;*
(3) $\pi_p(T)$ *does not have an eigenvalue* $-(p+1)$*;*
(4) $g^+(\mathcal{G}_{L,p}(0)) = g^+(\mathcal{G}_{L,p}(1))$.

Furthermore, if $\mathfrak{G}_{L,p}(0)$ *is of E-type, then* $g^+(\mathcal{G}_{L,p}(0)) = 2g^+(\mathcal{G}_{L,p}(1))$*, where* $g^+(\mathcal{G}_{L,p}(0))$ *is the number of spinor genera in* $\mathcal{G}_{L,p}(0)$*.*

Proof. **(1) ⇔ (2):** Assume that $\mathfrak{G}_{L,p}(0)$ is of O-type. Without loss of generality, we may assume that $\mathfrak{G}_{L,p}(0)$ is connected, that is, every $\mathbb{Z}$-lattice in $\mathfrak{G}_{L,p}(0)$ is spinor equivalent. It is well known that the rank of an incidence matrix of a connected graph $G(V, E)$ over $\mathbb{F}_2$ is $|V| - 1$. Furthermore if the graph G contains an odd cycle, then the rank of the incidence matrix of G over $\mathbb{Q}$ is equal to the number of vertices. Hence it suffices to show that the graph $\mathfrak{G}_{L,p}(0)$ contains an odd cycle, even though it might contains a loop. Assume that $[T_1]$ and $[T_2]$ be adjacent vertices in $\mathfrak{G}_{L,p}(0)$. Since they are spinor equivalent, there is an isometry $\sigma \in O(V)$ and $\Sigma = (\Sigma_p) \in J'_V$ such that $T_1 = \sigma\Sigma(T_2)$, where $V = \mathbb{Q} \otimes T_1$. Let $\Phi = \{q \in P - \{p\} \mid (\sigma^{-1}(T_1))_q = (T_2)_q\}$ and $\Psi = P - (\Phi \cup \{p\})$, where P is the set of all primes. Now by Strong Approximation Theorem for Rotations, for any $\epsilon > 0$, there is a rotation $\tau \in O'(V)$ such that

$$\|\tau - \Sigma_q\|_q < \epsilon \ \text{ for any } q \in \Psi \quad \text{and} \quad \|\tau\|_q = 1 \ \text{ for any } q \in \Phi.$$

Therefore we have

$$\sigma^{-1}(T_1)_q = \tau(T_2)_q \ \text{ for any } q \neq p \quad \text{and} \quad \Sigma_p \circ \tau^{-1}(\tau(T_2)_p) = \sigma^{-1}(T_1)_p,$$

where $\Sigma_p \circ \tau^{-1} \in O'(V_p)$. Consequently, there is an even integer n and a basis $\{x_1, x_2, x_3\}$ for $\tau(T_2)$ such that

$$\tau(T_2) = \mathbb{Z}x_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3 \quad \text{and} \quad \sigma^{-1}(T_1) = \mathbb{Z}(p^n x_1) + \mathbb{Z}(p^{-n} x_2) + \mathbb{Z}x_3,$$

by Lemma 4.2 of [2]. This implies that there is a path from $[T_1]$ to $[T_2]$ with even edges, and hence the graph $\mathfrak{G}_{L,p}(0)$ contains an odd cycle.

Assume that $\mathfrak{G}_{L,p}(0)$ is of E-type. Since any two adjacent vertices are contained in different spinor genera in this case, it is a bipartite (multi-) graph. Therefore the rank of the matrix $\mathfrak{M}_{L,p}(0)$ is $h-1$.

(2) ⇔ (3) : Note that $\text{rank}(\mathfrak{M}) = \text{rank}(\mathfrak{M}\mathfrak{M}^t)$. Hence the assertion follows directly from Proposition 3.8.

(1) ⇔ (4) : Note that $g^+(\mathcal{L}) = [J_{\mathbb{Q}} : P_D J_{\mathbb{Q}}^{\mathcal{L}}]$ for any genus $\mathcal{L}$ with rank greater than 2. Since

$$P_D J_{\mathbb{Q}}^{\mathcal{G}_{L,p}(1)} = P_D J_{\mathbb{Q}}^{\mathcal{G}_{L,p}(0)} \cup \mathbf{j}(p) \cdot P_D J_{\mathbb{Q}}^{\mathcal{G}_{L,p}(0)},$$

$g^+(\mathcal{G}_{L,p}(1)) = g^+(\mathcal{G}_{L,p}(0))$ if and only if $\mathbf{j}(p) \in P_D J_{\mathbb{Q}}^{\mathcal{G}_{L,p}(0)}$, that is, $\mathfrak{G}_{L,p}(0)$ is of O-type. Furthermore if $\mathfrak{G}_{L,p}(0)$ is of E-type, then $g^+(\mathcal{G}_{L,p}(0)) = 2g^+(\mathcal{G}_{L,p}(1))$. □

Now, we consider the general case. For any positive integer m, we say that a graph $\mathfrak{G}_{L,p}(m)$ is of E-type if m is even and $\mathfrak{G}_{L,p}(0)$ is of E-type, and O-type otherwise.

Assume that $\mathfrak{G}_{L,p}(m)$ is of E-type and $M \in \mathcal{G}_{L,p}(m)$. Since the map $\lambda_p^{\frac{m}{2}} : \text{spn}(K) \to \text{spn}(\lambda_p^{\frac{m}{2}}(K))$ is surjective for any $K \in \mathcal{G}_{L,p}(m)$, there is a $\mathbb{Z}$-lattice $M' \in \mathcal{G}_{L,p}(m)$ such that $M' \notin \text{spn}(M)$ and $[M']$ is connected to $[M]$ by a path by Lemma 3.5. Furthermore, since $g^+(\mathcal{G}_{L,p}(m)) = g^+(\mathcal{G}_{L,p}(0))$ for any even m, every $\mathbb{Z}$-lattice M' satisfying the above condition forms a single spinor genus. From the existence of such a $\mathbb{Z}$-lattice $[M']$, we may define

$$\text{Cspn}(M) = \begin{cases} \text{spn}(M) & \text{if } \mathfrak{G}_{L,p}(m) \text{ is of } O\text{-type}, \\ \text{spn}(M) \cup \text{spn}(M') & \text{otherwise}, \end{cases}$$

Lemma 3.10. *For a $\mathbb{Z}$-lattice $M \in \mathcal{G}_{L,p}(m)$, the set of all vertices in the connected component of $\mathfrak{G}_{L,p}(m)$ containing $[M]$ is the set of equivalence classes in Cspn(M).*

Proof. First, we prove the case when $m = 1$. Assume that $M' \in \text{spn}(M)$. Then there are $\sigma \in P_V$ and $\Sigma \in J_V'$ such that $M' = \sigma\Sigma M$ (see [8]). Since $\Gamma_{p,i}(M)$'s are the only sublattices of M with index p whose norm is $p\mathbb{Z}$, we have

$$\{\sigma\Sigma(\Gamma_{p,1}(M)^{\frac{1}{p}}), \sigma\Sigma(\Gamma_{p,2}(M)^{\frac{1}{p}})\} = \{\Gamma_{p,1}(M')^{\frac{1}{p}}, \Gamma_{p,2}(M')^{\frac{1}{p}}\}.$$

Hence $\Gamma_{p,1}(M)^{\frac{1}{p}} \in \text{spn}(\Gamma_{p,1}(M')^{\frac{1}{p}}) \cup \text{spn}(\Gamma_{p,2}(M')^{\frac{1}{p}})$. Therefore by Lemma 3.2, $[M']$ and $[M]$ are connected by a path in $\mathfrak{G}_{L,p}(1)$. Furthermore, as edges of the graph $\mathfrak{G}_{L,p}(0)$, $[M]$ and $[M']$ are contained in the same connected component. Since the number of connected components in $\mathfrak{G}_{L,p}(0)$ equals to $g^+(\mathcal{G}_{L,p}(1))$ by Theorem 3.9, each spinor genus in $\mathcal{G}_{L,p}(1)$ forms a connected component in $\mathfrak{G}_{L,p}(1)$. Furthermore, since $g^+(\mathcal{G}_{L,p}(2m+1)) = g^+(\mathcal{G}_{L,p}(1))$, $\text{spn}(\lambda_p^{\frac{m}{2}}(M)) = \text{spn}(\lambda_p^{\frac{m}{2}}(M'))$ if and only if $\text{spn}(M) = \text{spn}(M')$ for any $M, M' \in \mathcal{G}_{L,p}(2m+1)$. Therefore by Lemma 3.5, the set of all vertices in the connected component of $\mathfrak{G}_{L,p}(m)$ containing $[M]$ is the set of equivalence classes in Cspn(M) for any odd m. The proof of even case is quite similar to this. □

Theorem 3.11. *For any non-negative integer m, the graph $\mathfrak{G}_{L,p}(m)$ has an odd cycle (including a loop) if and only if $\mathfrak{G}_{L,p}(m)$ is of O-type.*

Proof. We already proved the case when $m = 0$ in Theorem 3.9. Assume that $m = 1$. Let $T \in \mathcal{G}_{L,p}(0)$ be any $\mathbb{Z}$-lattice. Then there are at least three $\mathbb{Z}$-lattices, say S_1, S_2, S_3, in $\Phi_p(T) \cap \mathcal{G}_{L,p}(1)$. Now by Lemma 3.2, $[S_i]$ and $[S_j]$ are connected by an edge for any $1 \leqslant i \neq j \leqslant 3$. Hence the graph $\mathfrak{G}_{L,p}(1)$ contains a cycle of length 3 or a loop. For the general case, we may apply Lemma 3.5 to prove the theorem. □

4. Representations of integers by ternary quadratic forms

Throughout this section, we assume that a $\mathbb{Z}$-lattice L and a prime p satisfies all conditions given in Section 3. For a nonnegative integer m, let $T \in \mathcal{G}_{L,p}(m)$ be a ternary $\mathbb{Z}$-lattice and let $S \in \mathcal{G}_{L,p}(m+1)$ be a ternary $\mathbb{Z}$-lattice such that $r(T^p, S) \neq 0$. This implies that $[T]$ is one of vertices contained in the edge $[S]$ in the graph $\mathfrak{G}_{L,p}(m)$. We assume that

$$\text{(4.1)} \qquad \mathrm{Cspn}(T) = \{[T_1], [T_2] \dots, [T_u]\} \quad \text{and} \quad \mathrm{Cspn}(S) = \{[S_1], [S_2], \dots, [S_v]\}$$

are *ordered* sets of equivalence classes. The aim of this section is to show that if $m \leqslant 2$, then there are rational numbers a_i and b_i such that for any integer n (any integer n divisible by p only when $m = 2$),

$$\text{(4.2)} \qquad r(n, T) = \sum_{i=1}^{v} \left(a_i r(pn, S_i) + b_i r(p^3 n, S_i)\right) + \text{(some extra term)}.$$

For a while, we assume that m is an arbitrary nonnegative integer. The following two propositions will be used repeatedly.

Proposition 4.1. *For any integer n,*

$$\frac{r(pn, S)}{o(S)} = \sum_{i=1}^{u} \frac{r(T_i^p, S)}{o(S)} \frac{r(n, T_i)}{o(T_i)} - \frac{r(pn, \Lambda_p(S))}{o(S)}.$$

Proof. By Weak Approximation Theorem, there exists a basis $\{x_1, x_2, x_3\}$ for S such that

$$(B(x_i, x_j)) \equiv \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \perp \langle p^{m+1}\delta \rangle \pmod{p^{m+2}},$$

where δ is an integer not divisible by p. As in Lemma 3.1, let

$$\Gamma_{p,1}(S) = \mathbb{Z}px_1 + \mathbb{Z}x_2 + \mathbb{Z}x_3, \quad \Gamma_{p,2}(S) = \mathbb{Z}x_1 + \mathbb{Z}px_2 + \mathbb{Z}x_3.$$

Since $Q(x) \equiv a_1 a_2 \pmod p$ for any $x = a_1x_1 + a_2x_2 + a_3x_3 \in S$, we have $Q(x) \equiv 0 \pmod p$ if and only if $a_1 \equiv 0 \pmod p$ or $a_2 \equiv 0 \pmod p$. Hence

$$x \in R(pn, S) \quad \text{if and only if} \quad x \in R(pn, \Gamma_{p,1}(S)) \cup R(pn, \Gamma_{p,2}(S))$$

Furthermore since $\Gamma_{p,1}(S) \cap \Gamma_{p,2}(S) = \Lambda_p(S)$, we have

$$r(pn, S) = r(pn, \Gamma_{p,1}(S)) + r(pn, \Gamma_{p,2}(S)) - r(pn, \Lambda_p(S))$$

for any integer n. Note that $\Gamma_{p,1}(S)$ and $\Gamma_{p,2}(S) \in \mathrm{gen}(T^p)$ are the only sublattices of S that are contained in $\mathrm{gen}(T^p)$. Furthermore, since the edge $[S]$ in $\mathfrak{G}_{L,p}(0)$

contains the vertex $[T]$ by assumption, we have $\Gamma_{p,1}(S)^{\frac{1}{p}}, \Gamma_{p,2}(S)^{\frac{1}{p}} \in \text{Cspn}(T)$. Now for any $\mathbb{Z}$-lattice $T_i \in \text{Cspn}(T)$, the number of sublattices in S that are isometric to T_i^p is $\frac{r(T_i^p,S)}{o(T_i)}$. The proposition follows from this. $\square$

Proposition 4.2. *For any integer n,*

$$\frac{r(pn,T)}{o(T)} = \begin{cases} \displaystyle\sum_{j=1}^{v} \frac{r(S_j^p,T)}{o(T)}\frac{r(n,S_j)}{o(S_j)} - p\cdot\frac{r(n,T^p)}{o(T)} & \text{if } m=0,\\ \displaystyle\sum_{j=1}^{v} \frac{\tilde{r}(S_j^p,T)}{o(T)}\frac{r(n,S_j)}{o(S_j)} + \frac{r(pn,\Lambda_p(T))}{o(T)} - 2p\cdot\frac{r(n,T^p)}{o(T)} & \text{otherwise.}\end{cases}$$

Proof. If we take $\epsilon = 0$ and $L = T$ in Lemma 2.5, then we have

$$r(pn,T) = \sum_{M\in\Omega_p(0,T)} r(pn,M) - (s_p(0,T)-1)r(n,T^p).$$

First, assume that $m = 0$. Let $M \in \Omega_p(0,T)$ be a $\mathbb{Z}$-lattice. Then by Lemmas 2.3 and 2.4,

$$M_p \simeq \begin{pmatrix} 0 & \frac{p}{2}\\ \frac{p}{2} & 0\end{pmatrix} \perp \langle -4p^2dT\rangle \quad \text{and} \quad M_q \simeq T_q \ (q\neq p).$$

Hence $M \in \text{gen}(S^p)$. Furthermore, since $r(T^p, M^{\frac{1}{p}}) = \tilde{r}(M,T) \neq 0$ and $r(T^p,S) = \tilde{r}(S^p,T) \neq 0$ by Lemma 2.7, $M^{\frac{1}{p}} \in \text{Cspn}(S)$ by Lemmas 3.2 and 3.10. Conversely, if $M^{\frac{1}{p}} \in \text{Cspn}(S)$ satisfies $\tilde{r}(M,T) \neq 0$, then M is isometric to a $\mathbb{Z}$-lattice in $\Omega_p(0,T)$. Note that the number of lattices in $\Omega_p(0,T)$ that are isometric to S^p is $\frac{r(S^p,T)}{o(S)}$ and $s_p(0,T) = p+1$. The proof of the case when $m \geqslant 1$ is quite similar to this, except that there is a unique $\mathbb{Z}$-lattice in $\Omega_p(0,T)$ that is not contained in $\text{gen}(S^p)$, which is, in fact, $\Lambda_p(T)$, and $s_p(0,T) = 2p+1$. $\square$

We define

$$\mathcal{M}_{L,p}(m) = \left(\frac{r(T_i^p,S_j)}{o(T_i)}\right) \in M_{u,v}(\mathbb{Z}) \quad \text{and} \quad \mathcal{N}_{L,p}(m) = \left(\frac{r(T_i^p,S_j)}{o(S_j)}\right) \in M_{u,v}(\mathbb{Z}).$$

Note that these two matrices depend on the order of each set $\text{Cspn}(\cdot)$, and $\mathcal{M}_{L,p}(0)$ is one of block diagonal components of $\mathfrak{M}_{L,p}(0)$ if we take a suitable order in (3.1). For any integer n, we define vectors

$$\mathbf{R}(n,\text{Cspn}(T)) = \left(\frac{r(n,T_1)}{o(T_1)}, \frac{r(n,T_2)}{o(T_2)}, \dots, \frac{r(n,T_u)}{o(T_u)}\right)^t,$$

$$\mathbf{R}^\sharp(n,\text{Cspn}(\lambda_p^m(T))) = \left(\frac{r(n,\lambda_p^m(T_1))}{o(T_1)}, \frac{r(n,\lambda_p^m(T_2))}{o(T_2)}, \dots, \frac{r(n,\lambda_p^m(T_u))}{o(T_u)}\right)^t.$$

Similarly, we define $\mathbf{R}(n,\text{Cspn}(S))$ and $\mathbf{R}^\sharp(n,\text{Cspn}(\lambda_p^m(S)))$. If $\text{Cspn}(M) = \text{spn}(M)$, then we use $\mathbf{R}(n,\text{spn}(M))$ rather than $\mathbf{R}(n,\text{Cspn}(M))$.

Theorem 4.3. *Let T and S be ternary $\mathbb{Z}$-lattices satisfying all conditions given above when $m = 0$. If the graph $\mathfrak{G}_{L,p}(0)$ is of O-type, then we have*

$$p\mathbf{R}(n,spn(T^p)) = \mathcal{M}\cdot\mathbf{R}(n,spn(S)) - (\mathcal{M}\cdot\mathcal{N}^t)^{-1}\mathcal{M}\cdot(\mathbf{R}(p^2n,spn(S)) + \mathbf{R}(n,spn(S))).$$

Proof. By Lemma 3.7 and Propositions 4.1, 4.2, we have the following two equalities:

$$\mathbf{R}(pn,\mathrm{spn}(S)) = \mathcal{N}^t\cdot\mathbf{R}(n,\mathrm{spn}(T)) - \mathbf{R}^\sharp(pn,\mathrm{spn}(\Lambda_p(S))), \tag{4.3}$$

$$\mathbf{R}(pn,\mathrm{spn}(T)) = \mathcal{M}\cdot\mathbf{R}(n,\mathrm{spn}(S)) - p\mathbf{R}(n,\mathrm{spn}(T^p)). \tag{4.4}$$

Since $\lambda_p(\lambda_p(S_i)) \simeq S_i$ for any $S_i \in \mathrm{spn}(S)$, we have

$$\mathbf{R}^\sharp(p^2n,\mathrm{spn}(\Lambda_p(S))) = \mathbf{R}(n,\mathrm{spn}(S)).$$

Hence

$$\mathbf{R}(p^2n,\mathrm{spn}(S)) = \mathcal{N}^t\cdot\mathbf{R}(pn,\mathrm{spn}(T)) - \mathbf{R}(n,\mathrm{spn}(S)). \tag{4.5}$$

Note that

$$\mathbf{O}(\mathrm{spn}(T))\cdot\mathcal{N} = \mathcal{M}\cdot\mathbf{O}(\mathrm{spn}(S)),$$

where $\mathbf{O}(\mathrm{spn}(T))$ is the $u\times u$ diagonal matrix with entries $o(T_i)^{-1}$. Furthermore, since we are assuming that $\mathrm{rank}(\mathcal{M}) = u$, the $u\times u$ square matrix $\mathcal{M}\cdot\mathcal{N}^t$ is invertible. Therefore the equation follows directly from (4.4) and (4.5). □

Now assume that $\mathfrak{G}_{L,p}(0)$ is of E-type, then $\mathrm{Cspn}(T)$ consists of two spinor genera and each connected component is a bipartite graph. Hence the rank of the matrix $\mathcal{M}$ is $u-1$ and $\mathcal{M}\cdot\mathcal{N}^t$ is no longer invertible. To get a similar result for an E-type graph, we need to make some adjustments.

Assume that $\mathrm{Cspn}(T) = \mathrm{spn}(T)\cup\mathrm{spn}(\tilde{T})$ and

$$\mathrm{spn}(T) = \{[T_{i_1}],\dots,[T_{i_a}]\},\quad \mathrm{spn}(\tilde{T}) = \{[T_{j_1}],\dots,[T_{j_b}]\},$$

where $\{i_1,i_2,\dots,i_a,j_1,\dots,j_b\} = \{1,2,\dots,u\}$. Note that

$$w(\mathrm{spn}(T')) = \sum_{[K]\in\mathrm{spn}(T')}\frac{1}{o(K)},$$

is independent of T' for any $T'\in\mathrm{gen}(T)$. Define

$$\epsilon_l = \begin{cases} w(\mathrm{spn}(T))^{-1} & \text{if } l\in\{i_1,\dots,i_a\},\\ -w(\mathrm{spn}(T))^{-1} & \text{if } l\in\{j_1,\dots,j_b\},\end{cases}$$

and define a $u\times(v+1)$ matrix $\tilde{\mathcal{N}} = (n_{ij})$ by

$$n_{ij} = \begin{cases} \dfrac{r(T_i^p,S_j)}{o(S_j)} & \text{if } j\leqslant v,\\ \epsilon_i & \text{if } j = v+1.\end{cases}$$

Lemma 4.4. *The rank of the matrix $\tilde{\mathcal{N}}$ defined above is u.*

Proof. Let $\mathbf{n}_i$ be the i-th row vector of the matrix $\tilde{\mathcal{N}}$. Suppose that $\alpha_1\mathbf{n}_1+\cdots+\alpha_u\mathbf{n}_u = 0$ for some integers α_i, that is,

$$\begin{cases} \alpha_1\dfrac{r(T_1^p,S_j)}{o(S_j)}+\cdots+\alpha_u\dfrac{r(T_u^p,S_j)}{o(S_j)} = 0 \text{ for any } j=1,\dots,v,\\ \alpha_1\epsilon_1+\cdots+\alpha_u\epsilon_u = 0.\end{cases} \tag{4.6}$$

For any j such that $1 \leqslant j \leqslant v$, the edge named by $[S_j]$ contains two vertices, one of them, say $[T_{i_e}]$, is contained in $\text{spn}(T)$ and the other, say $[T_{j_f}]$, is contained in $\text{spn}(\tilde{T})$. Hence the first equation in (4.6) implies that

$$\alpha_{i_e}\frac{r(T_{i_e}^p, S_j)}{o(S_j)} + \alpha_{j_f}\frac{r(T_{j_f}^p, S_j)}{o(S_j)} = 0.$$

Therefore $\alpha_{i_e} \cdot \alpha_{j_f} \leqslant 0$. Since the subgraph of $\mathfrak{G}_{L,p}(0)$ consisting of vertices in $\text{Cspn}(T)$ is a connected bipartite graph, each α_{i_e} (α_{j_f}) is 0, or it has the same sign to α_{i_1} (α_{j_1}, respectively). Therefore $\alpha_l = 0$ for any $l = 1, \dots, u$ and $\text{rank}(\tilde{\mathcal{N}}) = u$. This completes the proof. □

For a vector $\mathbf{v} = (v_1, \dots, v_n)$, we define $(\mathbf{v}, w_1, \dots, w_s) = (v_1, \dots, v_n, w_1, \dots, w_s)$. Note that the equation (4.5) implies that

$$\tilde{\mathbf{R}} := \tilde{\mathcal{N}}^t \cdot \mathbf{R}(pn, \text{Cspn}(T)) = \begin{pmatrix} \mathbf{R}(p^2n, \text{spn}(S)) + \mathbf{R}(n, \text{spn}(S)) \\ \\ r(pn, \text{spn}(T)) - r(pn, \text{spn}(\tilde{T})) \end{pmatrix}, \tag{4.7}$$

where

$$r(pn, \text{spn}(T)) = \frac{1}{w(\text{spn}(T))} \cdot \sum_{[T_i] \in \text{spn}(T)} \frac{r(pn, T_i)}{o(T_i)}.$$

Theorem 4.5. *If $\mathfrak{G}_{L,p}(0)$ is of E-type, then we have*

$$p\mathbf{R}(n, \mathit{Cspn}(T^p)) = \mathcal{M} \cdot \mathbf{R}(n, \mathit{spn}(S)) - (\tilde{\mathcal{N}} \cdot \tilde{\mathcal{N}}^t)^{-1}\tilde{\mathcal{N}} \cdot \tilde{\mathbf{R}}.$$

Proof. From the above lemma, we know that $\text{rank}(\tilde{\mathcal{N}}) = u$. The theorem follows directly from the equations (4.4) and (4.7). □

Note that $r(pn, \text{spn}(T)) - r(pn, \text{spn}(\tilde{T}))$ can easily be computed by the formula given in [11].

Example 4.6. Let $p = 11$ and $L = \langle 1, 1, 16\rangle$. Then

$$\mathcal{G}_{L,p}(0)/\sim = \left\{ T_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{pmatrix},\ T_2 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 5 \end{pmatrix} \right\},$$

$$\mathcal{G}_{L,p}(1)/\sim = \left\{ S_1 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 6 & -1 \\ 1 & -1 & 11 \end{pmatrix},\ S_2 = \begin{pmatrix} 6 & 2 & 3 \\ 2 & 6 & 1 \\ 3 & 1 & 7 \end{pmatrix} \right\}.$$

One may easily compute that $\mathcal{M} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\mathcal{N} = \begin{pmatrix} 8 & 4 \\ 8 & 4 \end{pmatrix}$. Since $\text{rank}(\mathcal{M}) = 1$, the graph $\mathfrak{G}_{L,p}(0)$ is of E-type by Theorem 3.9. Note that $\tilde{\mathcal{N}} = \begin{pmatrix} 8 & 4 & 16 \\ 8 & 4 & -16 \end{pmatrix}$.

Therefore, by Theorem 4.5, we have

$$\begin{aligned}
11r(n,T_1^{11}) &= \frac{38}{5}r(n,S_1) - \frac{2}{5}r(11^2n,S_1) + \frac{39}{10}r(n,S_2) - \frac{1}{10}r(11^2n,S_2)\\
&\quad - \left(\frac{1}{2}r(11n,T_1) - \frac{1}{2}r(11n,T_2)\right),\\
11r(n,T_2^{11}) &= \frac{38}{5}r(n,S_1) - \frac{2}{5}r(11^2n,S_1) + \frac{39}{10}r(n,S_2) - \frac{1}{10}r(11^2n,S_2)\\
&\quad + \left(\frac{1}{2}r(11n,T_1) - \frac{1}{2}r(11n,T_2)\right).
\end{aligned}$$

Note that by Korollar 2 of [11], one may easily check that

$$r(11n,T_1) - r(11n,T_2) = \begin{cases} 0 & \text{if } n \neq 11m^2,\\ \left(\dfrac{1-(-1)^m}{2}\right)\cdot(-1)^{\frac{m+1}{2}}\cdot 44m & \text{if } n = 11m^2. \end{cases}$$

Theorem 4.7. *Let $T \in \mathcal{G}_{L,p}(1)$ and $S \in \mathcal{G}_{L,p}(2)$ be ternary $\mathbb{Z}$-lattices satisfying $r(T^p,S) \neq 0$. Then we have*

$$(3p^2-p)\cdot r(n,T) = \sum_{[\tilde{S}]\in gen(S)} \frac{\tilde{r}(\tilde{S}^p,T)}{o(\tilde{S})}\left(\frac{3p}{2}\ r(pn,\tilde{S}) - \frac{p}{p-1}\ r(p^3n,\tilde{S})\right)$$

$$+\frac{1}{p-1}\left(o(\Gamma_{p,1}(T)) \sum_{\substack{[\tilde{S}]\in gen(S)\\ \lambda_p(\tilde{S})\simeq\Gamma_{p,1}(T)^{\frac{1}{p}}}} \frac{r(p^3n,\tilde{S})}{o(\tilde{S})} + o(\Gamma_{p,2}(T)) \sum_{\substack{[\tilde{S}]\in gen(S)\\ \lambda_p(\tilde{S})\simeq\Gamma_{p,2}(T)^{\frac{1}{p}}}} \frac{r(p^3n,\tilde{S})}{o(\tilde{S})}\right).$$

Proof. First, we assume that

$$\Phi_p(\lambda_p(S)) = \{T=T_1,T_2,\dots,T_{p+1}\} \quad \text{and} \quad \Psi_p(\lambda_p(S)) = \{S=S_1,S_2,\dots,S_{\frac{p(p+1)}{2}}\}.$$

Without loss of generality, we may assume that $\lambda_p(S) = \Gamma_{p,1}(T)^{\frac{1}{p}}$. Define, for any integer n,

$$\mathbf{R}(n,\Phi_p(\lambda_p(S))) = (r(n,T_1),r(n,T_2),\dots,r(n,T_{p+1}))^t$$

and

$$\mathbf{R}(n,\Psi_p(\lambda_p(S))) = \left(r(n,S_1),r(n,S_2),\dots,r\left(n,S_{\frac{p(p+1)}{2}}\right)\right)^t.$$

We also define a vector $\mathbf{I}(n,\lambda_p(S)) = r(n,\lambda_p(S))\cdot(1,1,\dots,1)^t$ of length $\frac{p(p+1)}{2}$. Now by Proposition 4.1, we have

$$\mathbf{R}(pn,\Psi_p(\lambda_p(S))) = U\cdot\mathbf{R}(n,\Phi_p(\lambda_p(S))) - \mathbf{I}\left(\frac{n}{p},\lambda_p(S)\right),$$

where $U^t \in M_{(p+1)\times\frac{p(p+1)}{2}}(\mathbb{Z})$ is the incidence matrix of the complete graph of order $p+1$ by Lemma 3.2. Therefore $U^tU = (p-1)I + J$ and

$$((U^tU)^{-1}U^t)_{ij} = \begin{cases} \dfrac{1}{p} & \text{if } r(T_i^p,S_j)\neq 0,\\ \dfrac{-1}{p(p-1)} & \text{if } r(T_i^p,S_j) = 0. \end{cases}$$

Here J is a matrix of ones. Therefore we have

$$r(n,T)=\frac{1}{p}\sum_{\mathbf{1}} r(pn,S)-\frac{1}{p(p-1)}\sum_{\mathbf{2}} r(pn,S)+\frac{1}{2}r\left(\frac{n}{p},\lambda_p(S)\right), \tag{4.8}$$

where $\sum_{\mathbf{1}}$ is the summation of all lattices S' in $\Psi_p(\lambda_p(S))$ such that $r(T^p,S')\neq 0$ and $\sum_{\mathbf{2}}$ is the summation of all lattices S' in $\Psi_p(\lambda_p(S))$ such that $r(T^p,S')=0$. We define, for simplicity, $U_1(pn,S)=\sum_{\mathbf{1}} r(pn,S)$ and $U_2(pn,S)=\sum_{\mathbf{2}} r(pn,S)$. Now, by Proposition 2.9, we have

$$\begin{aligned} p\cdot r(pn,\lambda_p(S))+\frac{p(p-1)}{2}r\left(\frac{n}{p},\lambda_p(S)\right) &= o(\lambda_p(S))r(pn,\text{gen}_p^{\lambda_p(S)}(S))\\ &= \sum_{i=1}^{\frac{p(p+1)}{2}} r(pn,S_i)\\ &= U_1(pn,S)+U_2(pn,S). \end{aligned} \tag{4.9}$$

Let $\widetilde{S}$ be a $\mathbb{Z}$-lattice such that $\lambda_p(\widetilde{S})=\Gamma_{p,2}(T)^{\frac{1}{p}}$. We may similarly define $\mathbf{R}(n,\Psi_p(\lambda_p(\widetilde{S})))$, $U_1(pn,\widetilde{S})$ and $U_2(pn,\widetilde{S})$. Then, equations (4.8) and (4.9) hold even if we replace S by $\widetilde{S}$. Furthermore, by Proposition 4.2,

$$\begin{aligned} r(p^2n,T)+(2p-1)r(n,T) &= \sum_{[S']\in\text{gen}(S)} \frac{\tilde{r}((S')^p,T)}{o(S')}r(pn,S')\\ &= U_1(pn,S)+U_1(pn,\widetilde{S}). \end{aligned} \tag{4.10}$$

By combining (4.8)∼(4.10), we have

$$\begin{aligned} \tfrac{3p^2-p}{2}r(n,T) =\ & p(U_1(pn,S)+U_1(pn,\widetilde{S}))-p\left(\tfrac{1}{p}U_1(p^3n,S)-\tfrac{1}{p(p-1)}U_2(p^3n,S)\right)\\ & -\ \tfrac{p(p-1)}{2}\left(\tfrac{1}{p}U_1(pn,S)-\tfrac{1}{p(p-1)}U_2(pn,S)\right)-\tfrac{1}{2}\left(U_1(pn,S)+U_2(pn,S)\right)\\ =\ & \frac{p}{2}U_1(pn,S)+pU_1(pn,\widetilde{S})-\left(U_1(p^3n,S)-\frac{1}{p-1}U_2\left(p^3n,S\right)\right). \end{aligned}$$

Since the above equation holds even if we exchange S for $\widetilde{S}$, we have

$$\begin{aligned} (3p^2-p)r(n,T) =\ & \frac{3p}{2}\left(U_1(pn,S)+U_1(pn,\widetilde{S})\right)-\frac{p}{p-1}\left(U_1(p^3n,S)+U_1(p^3n,\widetilde{S})\right)\\ & +\ \frac{1}{p-1}\left(U_1(p^3n,S)+U_2(p^3n,S)+U_1(p^3n,\widetilde{S})+U_2(p^3n,\widetilde{S})\right). \end{aligned}$$

This completes the proof. □

Remark 4.8. In the above theorem, one may easily check that the sets $\Psi_p(\lambda_p(S))$ and $\Psi_p(\lambda_p(\widetilde{S}))$ are contained in $\text{Cspn}(S)$.

Assume that $m=2$. Recall that $T\in\mathcal{G}_{L,p}(2)$ and $S\in\mathcal{G}_{L,p}(3)$ are ternary $\mathbb{Z}$-lattices satisfying $r(T^p,S)\neq 0$. If we define ϵ_l and $\tilde{\mathcal{N}}$ as before for the E-type, then Lemma 4.4 still holds under this situation.

Theorem 4.9. *Let T and S be ternary $\mathbb{Z}$-lattices satisfying all conditions given above. Assume that the graph $\mathfrak{G}_{L,p}(2)$ is of O-type. If n is not divisible by p, then we have*

$$\mathbf{R}(n,spn(T))=(\mathcal{N}\cdot\mathcal{N}^t)^{-1}\mathcal{N}\cdot\mathbf{R}(pn,spn(S)). \tag{4.11}$$

If n is divisible by p, then $\mathbf{R}(n, spn(T))$ is equal to

$$\frac{1}{2p-1}\left(\mathcal{M}\cdot\mathbf{R}(pn, spn(S)) - (\mathcal{N}\cdot\mathcal{N}^t)^{-1}\mathcal{N}\cdot(\mathbf{R}(pn, spn(S)) + \mathbf{R}(p^3n, spn(S)))\right).$$

If $\mathfrak{G}_{L,p}(2)$ is of E-type, then we have

$$\mathbf{R}(n, Cspn(T)) = \begin{cases} (\tilde{\mathcal{N}}\cdot\tilde{\mathcal{N}}^t)^{-1}\tilde{\mathcal{N}}\cdot\tilde{\mathbf{R}}_1 & \text{if } p\nmid n,\\ \dfrac{1}{2p-1}\left(\mathcal{M}\cdot\mathbf{R}(pn, spn(S)) - (\tilde{\mathcal{N}}\cdot\tilde{\mathcal{N}}^t)^{-1}\tilde{\mathcal{N}}\cdot\tilde{\mathbf{R}}_2\right) & \text{otherwise,}\end{cases}$$

where

$$\tilde{\mathbf{R}}_1 = \begin{pmatrix} \mathbf{R}(pn, spn(S)) \\ r(n, spn(T)) - r(n, spn(\tilde{T}))\end{pmatrix},\ \tilde{\mathbf{R}}_2 = \begin{pmatrix} \mathbf{R}(pn, spn(S)) + \mathbf{R}(p^3n, spn(S)) \\ (2p-1)(r(n, spn(\tilde{T})) - r(n, spn(T)))\end{pmatrix}.$$

Proof. The proof is similar to that of Theorem 4.3. First, assume that $\mathfrak{G}_{L,p}(2)$ is of O-type. Since the rank of $\mathcal{N}$ is u, we may define $\mathcal{Z} = (\mathcal{N}\cdot\mathcal{N}^t)^{-1}\mathcal{N}$. From the equation (4.3), we have

$$\mathbf{R}(n, \mathrm{spn}(T)) = \mathcal{Z}\left(\mathbf{R}(pn, spn(S)) + \mathbf{R}^\sharp\left(\frac{n}{p}, \mathrm{spn}(\lambda_p(S))\right)\right), \tag{4.12}$$

and

$$\mathbf{R}(p^2n, \mathrm{spn}(T)) = \mathcal{Z}\left(\mathbf{R}(p^3n, spn(S)) + \mathbf{R}^\sharp\left(pn, \mathrm{spn}(\lambda_p(S))\right)\right). \tag{4.13}$$

If $(\Gamma_{p,1}(S)^{\frac{1}{p}}, \Gamma_{p,2}(S)^{\frac{1}{p}}) \simeq (T_1, T_2)$, then

$$(\Gamma_{p,1}(\lambda_p(S))^{\frac{1}{p}}, \Gamma_{p,2}(\lambda_p(S))^{\frac{1}{p}}) \simeq (\lambda_p(T_1), \lambda_p(T_2)).$$

Hence we have

$$\mathbf{R}^\sharp(pn, \mathrm{spn}(\lambda_p(S))) = \mathcal{N}^t\cdot\mathbf{R}^\sharp(n, \mathrm{spn}(\lambda_p(T))) - \mathbf{R}^\sharp(n, \mathrm{spn}(\lambda_p^2(S))), \tag{4.14}$$

that is,

$$\mathbf{R}^\sharp(n, \mathrm{spn}(\lambda_p(T))) = \mathcal{Z}(\mathbf{R}^\sharp(pn, \mathrm{spn}(\lambda_p(S))) + \mathbf{R}^\sharp(n, \mathrm{spn}(\lambda_p^2(S))). \tag{4.15}$$

By Proposition 4.2, we also have

$$\mathbf{R}(p^2n, \mathrm{spn}(T)) + 2p\,\mathbf{R}(n, \mathrm{spn}(T)) = \mathcal{M}\cdot\mathbf{R}(pn, \mathrm{spn}(S)) + \mathbf{R}^\sharp(n, \mathrm{spn}(\lambda_p(T))). \tag{4.16}$$

If n is not divisible by p, then (4.11) comes directly from (4.12). Assume that n is divisible by p. Since $\lambda_p^3(S) \simeq \lambda_p(S)$, we have

$$\mathbf{R}^\sharp\left(\frac{n}{p}, \mathrm{spn}(\lambda_p(S))\right) = \mathbf{R}^\sharp(n, \mathrm{spn}(\lambda_p^2(S))). \tag{4.17}$$

Therefore, the theorem follows from equations (4.12), (4.13), (4.15) and (4.16).

If we replace $\mathcal{N}$ by $\tilde{\mathcal{N}}$, then the proof of the case when $\mathfrak{G}_{L,p}(2)$ is of E-type is quite similar to this. □

Example 4.10. Let $p = 3$ and let $L = \langle 1,1,2\rangle$. Then $T = \langle 1,2,9\rangle \in \mathcal{G}_{L,p}(2)$ and $S_1 = \langle 1,2,27\rangle \in \mathcal{G}_{L,p}(3)$. In fact, the graph $\mathfrak{G}_{L,p}(2)$ is of O-type and

$$\mathcal{G}_{L,p}(3)/\sim = \left\{ S_1,\ S_2 = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 4 & 2 \\ 1 & 2 & 6 \end{pmatrix},\ S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 1 \\ 0 & 1 & 11 \end{pmatrix},\ S_4 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 1 & 7 \end{pmatrix} \right\}.$$

In this case, one may easily check that there are no rational numbers a_i and b_i satisfying the equation

$$r(n,T) = \sum_{i=1}^{4} a_i \cdot r(3n, S_i) + \sum_{i=1}^{4} b_i \cdot r(27n, S_i) \quad \text{for any integer } n.$$

Finally, assume that $m \geqslant 3$. Let $T \in \mathcal{G}_{L,p}(m)$ and $S \in \mathcal{G}_{L,p}(m+1)$ be $\mathbb{Z}$-lattices such that $r(T^p, S) \neq 0$. We additionally assume that $\mathfrak{G}_{L,p}(m)$ is of O-type. Recall that $\mathcal{M} = \left(\frac{r(T_i^p, S_j)}{o(T_i)}\right)$ and $\mathcal{N} = \left(\frac{r(T_i^p, S_j)}{o(S_j)}\right)$. We define $\mathcal{Z} = (\mathcal{N}\mathcal{N}^t)^{-1}\mathcal{N}$.

Theorem 4.11. *Under the assumptions given above, if n is not divisible by p, then*

$$\mathbf{R}(n, spn(T)) = \mathcal{Z}\left(\mathbf{R}(pn, spn(S))\right) \quad \text{and} \quad \mathbf{R}(pn, spn(T)) = \mathcal{M} \cdot \mathbf{R}(n, spn(S)).$$

For an arbitrary integer n, we have

$$\begin{aligned} & p\mathbf{R}(p^2 n, spn(T)) - p^2\mathbf{R}(n, spn(T)) \\ & = \mathcal{Z}\left(2p\mathbf{R}(p^3 n, spn(S)) + p^2\mathbf{R}(pn, spn(S)) + \mathbf{R}^\flat(pn, spn(S))\right) - p\mathcal{M} \cdot \mathbf{R}(pn, spn(S)), \end{aligned}$$

where

$$\mathbf{R}^\flat(pn, spn(S)) = \left(\frac{o(\lambda_p(S_1))}{o(S_1)} r(pn, gen_p^{\lambda_p(S_1)}(S_1)), \dots, \frac{o(\lambda_p(S_v))}{o(S_v)} r(pn, gen_p^{\lambda_p(S_v)}(S_v))\right)^t.$$

Proof. By Propositions 4.1 and 4.2, we have

$$\mathbf{R}(pn, \operatorname{spn}(S)) = \mathcal{N}^t \cdot \mathbf{R}(n, \operatorname{spn}(T)) - \mathbf{R}^\sharp\left(\frac{n}{p}, \operatorname{spn}(\lambda_p(S))\right), \tag{4.18}$$

and

$$\begin{aligned} \mathbf{R}(pn, \operatorname{spn}(T)) &= \mathcal{M} \cdot \mathbf{R}(n, \operatorname{spn}(S)) \\ & \quad + \mathbf{R}^\sharp\left(\frac{n}{p}, \operatorname{spn}(\lambda_p(T))\right) - 2p \cdot \mathbf{R}\left(\frac{n}{p}, \operatorname{spn}(T)\right). \end{aligned} \tag{4.19}$$

The first two equations follow directly from (4.18) and (4.19).

Now by applying λ_p-transformation to the equation (4.18), we also have

$$\mathbf{R}^\sharp(pn, \operatorname{spn}(\lambda_p(S))) = \mathcal{N}^t \cdot \mathbf{R}^\sharp(n, \operatorname{spn}(\lambda_p(T))) - \mathbf{R}^\sharp\left(\frac{n}{p}, \operatorname{spn}(\lambda_p^2(S))\right). \tag{4.20}$$

Our final ingredient is the following equation which is directly obtained from Proposition 2.9:

$$\begin{aligned} p\mathbf{R}^\sharp(pn, \operatorname{spn}(\lambda_p(S))) + p^2\mathbf{R}^\sharp\left(\frac{n}{p}, \operatorname{spn}(\lambda_p(S))\right) - p\mathbf{R}^\sharp\left(\frac{n}{p}, \operatorname{spn}(\lambda_p^2(S))\right) \\ = \mathbf{R}^\flat(pn, \operatorname{spn}(S)). \end{aligned} \tag{4.21}$$

By multiplying $\mathcal{Z}$ to (4.18), we have

$$\mathbf{R}(n, \mathrm{spn}(T)) = \mathcal{Z}\left(\mathbf{R}(pn, \mathrm{spn}(S)) + \mathbf{R}^{\sharp}\left(\frac{n}{p}, \mathrm{spn}(\lambda_p(S))\right)\right).$$

Hence we have

$$\begin{aligned}2p\mathbf{R}(p^2n, \mathrm{spn}(T)) + p^2\mathbf{R}(n, \mathrm{spn}(T)) &= 2p\mathcal{Z}\left(\mathbf{R}(p^3n, \mathrm{spn}(S)) + \mathbf{R}^{\sharp}(pn, \mathrm{spn}(\lambda_p(S)))\right)\\ &\quad + p^2\mathcal{Z}\left(\mathbf{R}(pn, \mathrm{spn}(S)) + \mathbf{R}^{\sharp}\left(\frac{n}{p}, \mathrm{spn}(\lambda_p(S))\right)\right).\end{aligned}$$

On the other hand, by combining (4.19) and (4.20), we have

$$\begin{aligned}\mathbf{R}(p^2n, \mathrm{spn}(T)) &+ 2p\mathbf{R}(n, \mathrm{spn}(T)) - \mathcal{M}\cdot\mathbf{R}(pn, \mathrm{spn}(S))\\ &= \mathcal{Z}\left(\mathbf{R}^{\sharp}(pn, \mathrm{spn}(\lambda_p(S))) + \mathbf{R}^{\sharp}\left(\frac{n}{p}, \mathrm{spn}(\lambda_p^2(S))\right)\right).\end{aligned}$$

The theorem follows from the above two equations and (4.21). □

References

[1] F. Andrianov, *Clifford algebras and the Shimura lift for theta series*, St. Petersburg Math. J. **12**(2001), 51-81.

[2] J. W. Benham and J. S. Hsia, *On spinor exceptional representations*, Nagoya Math. J. **87**(1982), 247–260.

[3] W. K. Chan and B.-K. Oh, *Finiteness theorems for positive definite n-regular quadratic forms*, Trans. Amer. Math. Soc. **355**(2003), 2385–2396.

[4] W. K. Chan and B.-K. Oh, *Class numbers of ternary quadratic forms*, J. Number Theory **135**(2014), 221–261.

[5] M. Eichler, *Quadratische Formen und orthogonale Gruppen*, Grundlehren Math. Wiss., vol. 63, Springer-Verlag, Berlin-New York, 1974.

[6] M. Kneser, *Darstellungsmaße indefiniter quadratischer Formen*, (German) Math. Z. **77**(1961), 188–194.

[7] Y. Kitaoka, *Arithmetic of quadratic forms*, Cambridge University Press, 1993.

[8] O. T. O'Meara, *Introduction to quadratic forms*, Springer Verlag, New York, 1963.

[9] R. Schulze-Pillot, *Darstellung durch definite ternare quadratische Formen und das Bruhat-Tits-Gebaude der Spingruppe*, Dissertation U, Göttingen 1979.

[10] R. Schulze-Pillot, *Darstellungsmaße von Spinorgeschlechtern ternärer quadratischer Formen*, J. Reine Angew. Math. **352**(1984), 114-132.

[11] R. Schulze-Pillot, *Thetareihen positiv definiter quadratischer Formen*, Invent. Math. **75**(1984), 283-229.

[12] A. Weil, *Sur la théorie des formes quadratiques*, Centre Belge Rech. Math., Colloque Theor. Groupes algebr., Bruxelles 1962, 9–22.

Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea

E-mail address: jjw@snu.ac.kr

Department of Mathematical Sciences, Seoul National University, Seoul 151-747, Korea

E-mail address: lih0905@snu.ac.kr

Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Korea

E-mail address: bkoh@snu.ac.kr