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On differentiability of implicitly defined function in semi-parametric profile likelihood estimation

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In this paper, we study the differentiability of implicitly defined functions which we encounter in the profile likelihood estimation of parameters in semi-parametric models. Scott and Wild (*Biometrika* **84** (1997) 57–71; *J. Statist. Plann. Inference* **96** (2001) 3–27) and Murphy and van der Vaart (*J. Amer. Statist. Assoc.* **95** (2000) 449–485) developed methodologies that can avoid dealing with such implicitly defined functions by parametrizing parameters in the profile likelihood and using an approximate least favorable submodel in semi-parametric models. Our result shows applicability of an alternative approach presented in Hirose (*Ann. Inst. Statist. Math.* **63** (2011) 1247–1275) which uses the direct expansion of the profile likelihood.

Keywords: efficiency; efficient information bound; efficient score; implicitly defined function; profile likelihood; semi-parametric model

1. Introduction

Consider a general semi-parametric model

$$\mathcal{P} = \{p_{\theta,\eta}(x)\colon\ \theta \in \Theta, \eta \in H\},$$

where $p_{\theta,\eta}(x)$ is a density function on the sample space $\mathcal{X}$ which depends on a finite-dimensional parameter θ and an infinite-dimensional parameter η. We assume that the set Θ of the parameter θ is an open subset of R^d and the set H is a convex subset of a Banach space $\mathcal{B}$.

Once observations $X_1, \ldots, X_n$ are generated from the model, the log-likelihood is given by

$$\ell_n(\theta,\eta) = n^{-1}\sum_{i=1}^{n} \log p_{\theta,\eta}(X_i) = \int \log p_{\theta,\eta}(x)\,\mathrm{d}F_n(x), \qquad (1.1)$$

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where F_n is the empirical c.d.f. based on the observations. In the profile likelihood approach, we find a function $\eta_{\theta,F}$ of the parameter θ and a c.d.f. F as the maximizer of the log-likelihood given θ such that

$$\eta_{\theta,F_n} = \arg\max_{\eta} \int \log p_{\theta,\eta}(x)\,\mathrm{d}F_n(x). \tag{1.2}$$

Then the profile (log)-likelihood is given by

$$\int \log p_{\theta,\eta_{\theta,F_n}}(x)\,\mathrm{d}F_n(x). \tag{1.3}$$

In this paper, we consider the situation when the function $\eta_{\theta,F}$ is given as the solution to the operator equation of the form

$$\eta = \Psi_{\theta,F}(\eta). \tag{1.4}$$

Murphy, Rossini and van der Vaart [10] encountered this type of implicitly defined function in their maximum likelihood estimation problem in the proportional odds model. According to them, "because $\hat{H}_\beta$ is not an explicit function of β, we are unable to differentiate the profile log-likelihood explicitly in β to form an estimator of Σ" (here $\hat{H}_\beta$ is the maximizer of the log-likelihood $\ell_n(\beta, H)$ given β, H is the baseline odds of failure and Σ is the efficient information). The authors (Murphy, Rossini and van der Vaart [10]) used a numerical approximation to the problem. In the first example (Example 1) given below, we present a modified version of the proportional odds model and give an example of implicitly defined function there.

Scott and Wild [13, 14] also encountered implicitly defined functions in their estimation problem with data from various outcome-dependent sampling design. They proposed a method of re-parametrization of profile-likelihood so that the log-likelihood is an explicitly defined function in terms of the parameters in the re-parametrized model. Their estimators turned out to be efficient and Hirose and Lee [7] showed conditions under which re-parametrization gives efficient estimation in a context of multiple-sample semiparametric model.

Another way to avoid dealing with implicitly defined functions is developed by Murphy and van der Vaart [11]. The paper proved the efficiency of profile likelihood estimation by introducing an approximate least favorable sub-model to express the upper and lower bounds for the profile log-likelihood. Since these two bounds have the same expression for the asymptotic expansion, so does the one for the profile log-likelihood. The advantage of the approach is that it does not need to deal with implicitly defined functions which we discussed in the current paper. Disadvantage of Murphy and van der Vaart [11] are (1) it needs to find an approximate least favorable submodel in each example which may be difficult to find in some cases; (2) no-bias condition (equation (3.4) in Murphy and van der Vaart [11]) is assumed in the main theorem and it needs to be verified in examples to which the main theorem is applied. In their "Discussion", they commented "It appears difficult to derive good approximations to a least favorable path for such models, and given such approximation it is unclear how one would verify the no-bias condition".

Hirose [6] used direct asymptotic expansion of the profile likelihood to show the efficiency of the profile likelihood estimator. The result in the paper (Theorem 1 in Hirose [6]) does not assume the no-bias condition and, under the assumptions given there, the no-bias condition (equation (4) in Hirose [6]) is proved (therefore, verification of the no-bias condition is not required in examples). In the approach, we cannot avoid dealing with implicitly defined functions of the form given in (1.4) in some applications. The purpose of this paper is to study the properties of these function such as differentiability so that the method in Hirose [6] is applicable to those applications. The results in Hirose [6] are summarized in Section 6.

In Section 2, we give examples of implicitly defined functions. The main results are presented in Section 3. In Sections 4 and 5, the main results are applied to the examples. In Section 6.1, we demonstrate how the result of the paper (the differentiability of implicitly defined functions in semi-parametric models) can be applied in a context of asymptotic linear expansion of the maximum profile likelihood estimator in a semi-parametric model.

2. Examples

2.1. Example 1 (semi-parametric proportional odds model)

The original asymptotic theory for maximum likelihood estimator in the semi-parametric proportional odds model is developed in Murphy, Rossini and van der Vaart [10]. We present a modified version of the model in Kosorok [9].

In this model, we observe $X = (U, \delta, Z)$, where $U = T \wedge C$, $\delta = 1_{\{U=T\}}$, $Z \in R^d$ is a covariate vector, T is a failure time and C is a right censoring time. We assume C and T are independent given Z.

The proportional odds regression model is specified by the survival function of T given Z of the form

$$S(t|Z) = \frac{1}{1 + \mathrm{e}^{\beta' Z} A(t)},$$

where $A(t)$ is nondecreasing function on $[0, \tau]$ with $A(0) = 0$. τ is the limit of censoring distribution such that $P(C > \tau) = 0$ and $P(C = \tau) > 0$. The distribution of Z and C are uninformative of S and $\operatorname{var} Z$ is positive definite.

Define the counting process $N(t) = \delta 1_{\{U \leq t\}}$ and at risk process $Y(t) = 1_{\{U \geq t\}}$. We assume $P\{\delta Y(t) = 1\} > 0$ for each $t \in [0, \tau]$.

Let F_n be the empirical process for i.i.d. observation (U_i, δ_i, Z_i), $i = 1, \ldots, n$. Then the log-likelihood on page 292 in Kosorok [9] can be written as

$$\ell_n(\beta, A) = \int \{\delta(\beta' Z + \log a(U)) - (1 + \delta) \log(1 + \mathrm{e}^{\beta' Z} A(U))\} \, \mathrm{d}F_n,$$

where $a(t) = \mathrm{d}A(t)/\mathrm{d}t$.

Consider one-dimensional sub-models for A defined by the map

$$t \to A_t(u) = \int_0^u (1 + th(s))\,\mathrm{d}A(s),$$

where $h(s)$ is an arbitrary total variation bounded cadlag function on $[0, \tau]$. By differentiating the log-likelihood function $\ell_n(\beta, A_t)$ with respect to t at $t = 0$, we obtain the score operator

$$B_n(\beta, A)(h) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} \ell_n(\beta, A_t) = \int \left\{ \delta h(U) - (1+\delta) \frac{\mathrm{e}^{\beta' Z} \int_0^U h(u)\,\mathrm{d}A(u)}{1 + \mathrm{e}^{\beta' Z} A(U)} \right\} \mathrm{d}F_n.$$

Choose $h(u) = 1_{\{u \le t\}}$, then

$$B_n(\beta, A)(h) = \int N(t)\,\mathrm{d}F_n - \int \left\{ \int_0^U W(u; \beta, A)\,\mathrm{d}A(u) \right\} \mathrm{d}F_n,$$

where $N(t)$ and $Y(t)$ are defined above and

$$W(u; \beta, A) = \frac{(1+\delta)\mathrm{e}^{\beta' Z} Y(u)}{1 + \mathrm{e}^{\beta' Z} A(U)}. \tag{2.1}$$

The solution $\hat{A}_{\beta, F_n}$ to the equation $B_n(\beta, A)(h) = 0$ is of the form

$$\hat{A}_{\beta, F_n}(u) = \int_0^u \frac{E_{F_n}\,\mathrm{d}N(s)}{E_{F_n} W(s; \beta, \hat{A}_{\beta, F_n})}, \tag{2.2}$$

where $E_{F_n}\,\mathrm{d}N(s) = \int \mathrm{d}N(s)\,\mathrm{d}F_n$ and $E_{F_n} W(s; \beta, \hat{A}_{\beta, F_n}) = \int W(s; \beta, \hat{A}_{\beta, F_n})\,\mathrm{d}F_n$.

Let F be a generic notation for the c.d.f., and if we let

$$\Psi_{\beta, F}(A) = \int_0^u \frac{E_F\,\mathrm{d}N(s)}{E_F W(s; \beta, A)}, \tag{2.3}$$

then (2.2) is a solution to the operator equation $A = \Psi_{\beta, F_n}(A)$, here $E_F\,\mathrm{d}N(s) = \int \mathrm{d}N(s)\,\mathrm{d}F$ and $E_F W(s; \beta, \hat{A}_{\beta, F}) = \int W(s; \beta, \hat{A}_{\beta, F})\,\mathrm{d}F$. More detailed treatment of this example can be found in [9], Section 15.3, pages 291–303. We continue this example in Section 4.

2.2. Example 2 (continuous outcome with missing data)

This example is studied in Weaver and Zhou [19] and Song, Zhou and Kosorok [17]. Suppose the underlying data generating process on the sample space $\mathcal{Y} \times \mathcal{X}$ is a model

$$\mathcal{Q} = \{p(y, x; \theta) = f(y|x; \theta) g(x) \colon\ \theta \in \Theta, g \in \mathcal{G}\}. \tag{2.4}$$

Here, $f(y|x;\theta)$ is a conditional density of Y given X which depends on a finite-dimensional parameter θ, $g(x)$ is an unspecified density of X which is an infinite-dimensional nuisance parameter. We assume the set $\Theta \subset R^d$ is an open set containing a neighborhood of the true value θ_0 and $\mathcal{G}$ is the set of density function of x containing the true value $g_0(x)$. We assume the variable Y is a continuous variable.

We consider a situation when there are samples for which we observe complete observation (Y, X) and for which we observe only Y. Let R_i be the indicator variable for the ith observation defined by

$$R_i = \begin{cases} 1, & \text{if } X_i \text{ is observed,} \\ 2, & \text{if } X_i \text{ is not observed.} \end{cases}$$

Then the index set for the complete observations is $V = \{i\colon\ R_i = 1\}$ and the index set for the incomplete observations is $\overline{V} = \{i\colon\ R_i = 2\}$. (In the paper Song, Zhou and Kosorok [17] $R_i = 0$ was used for subjects X_i is not observed.) Let $n_V = |V|$, $n_{\overline{V}} = |\overline{V}|$ be the total number of complete observations and incomplete observations, respectively.

Weaver and Zhou [19] and Song, Zhou and Kosorok [17] consider the likelihood of the form

$$L_n(\theta, g) = \prod_{i \in V} \{f(Y_i|X_i;\theta) g(X_i)\} \prod_{i \in \overline{V}} f_Y(Y_i;\theta, g), \tag{2.5}$$

where

$$f_Y(y;\theta, g) = \int_{\mathcal{X}} f(y|x;\theta) g(x)\,\mathrm{d}x. \tag{2.6}$$

The log-likelihood, the $1/n$ times log of (2.5) is

$$\ell_n(\theta, g) = \frac{n_V}{n} \frac{1}{n_V} \sum_{i \in V} \{\log f(y_i|x_i;\theta) + \log g(x_i)\} + \frac{n_{\overline{V}}}{n} \frac{1}{n_{\overline{V}}} \sum_{i \in \overline{V}} \log f_Y(y_i;\theta, g).$$

For the proof in the later part of the paper, we introduce notation: let F_{1n} and F_{2n} be the empirical c.d.f.s based on the samples in V and $\overline{V}$, respectively; denote $w_{1n} = n_V/n$, $w_{2n} = n_{\overline{V}}/n$ and let $F_n = \sum_{s=1}^{2} w_{sn} F_{sn}$ be the empirical c.d.f. for the combined samples in $V \cup \overline{V}$.

Then the log-likelihood can be expressed as

$$\ell_n(\theta, g) = w_{1n} \int \{\log f(y|x;\theta) + \log g(x)\}\,\mathrm{d}F_{1n} + w_{2n} \int \log f_Y(y;\theta, g)\,\mathrm{d}F_{2n}.$$

To find the maximizer of $\ell_n(\theta, g)$, we treat $g(x)$ as probability mass function on the observed values $\{x_i\colon\ i \in V\}$. Denote $g_i = g(x_i)$, $i \in V$. The derivative of the log-likelihood with respect to g_i is

$$\frac{\partial}{\partial g_i} \ell_n(\theta, g) = w_{1n} \frac{\int 1_{\{x = x_i\}}\,\mathrm{d}F_{1n}}{g_i} + w_{2n} \int \frac{f(y|x_i;\theta)}{f_Y(y;\theta, g)}\,\mathrm{d}F_{2n},$$

here, for the discrete g, $f_Y(y;\theta,g)=\sum_{i\in V} f(y|x_i;\theta)g_i$.

Let λ be a Lagrange multiplier to account for $\sum_{i\in V} g_i=1$. Set $\frac{\partial}{\partial g_i}\ell_n(\theta,g)+\lambda=0$. Multiply by g_i and sum over $i\in V$ to get $w_{1n}+w_{2n}+\lambda=0$. Therefore, $\lambda=-(w_{1n}+w_{2n})=-1$ and $\frac{\partial}{\partial g_i}\ell_n(\theta,g)-1=0$. By rearranging this equation, we obtain

$$\hat{g}_i=\frac{w_{1n}\int 1_{\{x=x_i\}}\,\mathrm{d}F_{1n}}{1-w_{2n}\int f(y|x_i;\theta)/f_Y(y;\theta,\hat{g})\,\mathrm{d}F_{2n}}.$$

This is exactly equation (3) in Song, Zhou and Kosorok [17]. Since the $\hat{g}_i$ is a function of θ and $F_n=\sum_{s=1}^2 w_{sn}F_{sn}$, it can be written as

$$\hat{g}_{\theta,F_n}(x_i)=\frac{w_{1n}(\partial_x\int \mathrm{d}F_{1n})(x_i)}{1-w_{2n}\int f(y|x_i;\theta)/f_Y(y;\theta,\hat{g}_{\theta,F_n})\,\mathrm{d}F_{2n}},\qquad i\in V, \tag{2.7}$$

where $\partial_x=\frac{\partial}{\partial x}$ (see Note below for the notation $\partial_x\int \mathrm{d}F_1$). This is a solution to the equation $g=\Psi_{\theta,F_n}(g)$ with

$$\Psi_{\theta,F}(g)=\frac{w_1\partial_x\int \mathrm{d}F_1}{1-w_2\int f(y|x;\theta)/f_Y(y;\theta,g)\,\mathrm{d}F_2},$$

here $F=\sum_{s=1}^2 w_sF_s$. We continue this example in Sections 5 and 6.1.

Note (Comment on the notation $\partial_x\int \mathrm{d}F_1$). Let us denote $\partial_x=\frac{\partial}{\partial x}$. The Heaviside step function $H(x)=1_{\{x\geq 0\}}$ and the Dirac delta function $\delta(x)$ are related by $\partial_x H(x)=\delta(x)$. Using this, for the joint empirical c.d.f. $F_n(x,y)=\frac{1}{n}\sum_{i=1}^n H(x-x_i)H(y-y_i)$, we have

$$\left(\partial_x\int \mathrm{d}F_n\right)(x)=\frac{1}{n}\sum_{i=1}^n \delta(x-x_i)\int \mathrm{d}H(y-y_i)=\frac{1}{n}\sum_{i=1}^n\delta(x-x_i),$$

where we used $\int \mathrm{d}H(y-y_i)=1$ (since the integral is over all y). For the continuous case, joint c.d.f. $F(x,y)$ and marginal p.d.f. $f(x)$ are related by $(\partial_x\int \mathrm{d}F)(x)=f(x)$. This justifies the notation $\partial_x\int \mathrm{d}F_1$ for both continuous and empirical c.d.f.s.

3. Main results

In this section, we show the differentiability of implicitly defined function which is given as a solution to the operator equation (1.4).

As we stated in the Introduction, we consider a general semi-parametric model

$$\mathcal{P}=\{p_{\theta,\eta}(x):\ \theta\in\Theta,\eta\in H\},$$

where $p_{\theta,\eta}(x)$ is a density function on the sample space $\mathcal{X}$ which depends on a finite-dimensional parameter θ and an infinite-dimensional parameter η. We assume that the set Θ of the parameter θ is an open subset of R^d and the set H is a convex set in a Banach space $\mathcal{B}$, which we may assume the closed linear span of H.

Definition (Hadamard differentiability). Suppose X and Y are two normed linear spaces and let $T \subset X$. We say that a map $\psi : T \to Y$ is Hadamard differentiable at $x \in T$ if there is a continuous linear map $\mathrm{d}\psi(x) : X \to Y$ such that

$$t^{-1}\{\psi(x_t) - \psi(x)\} \to \mathrm{d}\psi(x)h \qquad \text{as } t \downarrow 0 \tag{3.1}$$

for any map $t \to x_t$ with $x_{t=0} = x$ and $t^{-1}(x_t - x) \to h \in X$ as $t \downarrow 0$. The map $\mathrm{d}\psi(x)$ is called the Hadamard derivative of ψ at x, and is continuous in x (for reference, see Gill [5] and Shapiro [16]).

We denote the second derivative of ψ in the sense of Hadamard by $\mathrm{d}^2\psi(x)$. The usual first and second derivative of a parametric function $\psi(x)$, $x \in R^d$, are denoted by $\dot{\psi}$ and $\ddot{\psi}$.

Note on Hadamard differentiability. The above form of definition of the Hadamard differentiability is due to Fréchet in 1937. M. Sova showed the equivalence of the Hadamard differentiability and the compact differentiability in metrizable linear spaces (Averbukh and Smolyanov [2]). Because of the equivalence, some authors use compact differentiability as definition of Hadamard differentiability (Gill [5], van der Vaart and Wellner [18], Bickel, Klaassen, Ritov and Wellner [3]). In this paper, we use the definition of Hadamard differentiability given by Fréchet.

In addition to the Hadamard differentiability of functions, in Theorem 3.1 below, we assume the following condition.

Additional condition. We say a Hadamard differentiable map $\psi(x)$ satisfies the additional condition at x, if, for each path x_t in some neighborhood of x, there is a bounded and linear map $h \to \mathrm{d}\psi_t^* h$ such that the equality

$$\psi(x_t) - \psi(x) = \mathrm{d}\psi_t^*(x_t - x) \tag{3.2}$$

holds.

For a smooth map x_t with $x_t \to x$ as $t \downarrow 0$, the Hadamard differentiability of the function ψ and the additional condition (3.2) imply that

$$\mathrm{d}\psi_t^* h \to \mathrm{d}\psi(x)h \qquad \text{as } t \downarrow 0, \tag{3.3}$$

where the limit $\mathrm{d}\psi(x)$ is the Hadamard derivative of ψ at x.

Note on additional condition. In many statistics applications, we have the additional condition. For example, for functions $F(x)$ and $g(x)$, the map $\psi : F \to \int g(x)\,\mathrm{d}F(x)$ satisfies the additional condition:

$$\psi(F_t) - \psi(F) = \int g(x)\,\mathrm{d}(F_t - F)(x)$$

here the map $\mathrm{d}\psi_t^*$ in (3.2) is $\mathrm{d}\psi^* h = \int g(x)\,\mathrm{d}h(x)$ which coincides with the Hadamard derivative of ψ. For another example, consider a map $\psi : g \to (\int g(x)\,\mathrm{d}F(x))^{-1}$. Then

$$\psi(g_t) - \psi(g) = \frac{1}{\int g_t(x)\,\mathrm{d}F(x)} - \frac{1}{\int g(x)\,\mathrm{d}F(x)} = \frac{-\int [g_t(x) - g(x)]\,\mathrm{d}F(x)}{\int g_t(x)\,\mathrm{d}F(x) \int g(x)\,\mathrm{d}F(x)},$$

and it shows the map ψ satisfies the additional condition with

$$\mathrm{d}\psi_t^* h = \frac{-\int h(x)\,\mathrm{d}F(x)}{\int g_t(x)\,\mathrm{d}F(x)\int g(x)\,\mathrm{d}F(x)}.$$

If $g_t \to g$ as $t \downarrow 0$, then $\mathrm{d}\psi_t^* h$ converges to the Hadamard derivative of ψ:

$$\mathrm{d}\psi h = \frac{-\int h(x)\,\mathrm{d}F(x)}{(\int g(x)\,\mathrm{d}F(x))^2}.$$

Note on norm used in Theorem 3.1 (below). We treat the set of c.d.f. functions $\mathcal{F}$ on $\mathcal{X}$ as a subset of $\ell^\infty(\mathcal{X})$, the collection of all bounded functions on $\mathcal{X}$. This means the norm on $\mathcal{F}$ is the sup-norm: for $F \in \mathcal{F}$, $\|F\| = \sup_{x\in\mathcal{X}} |F(x)|$. The convex subset H of a Banach space $\mathcal{B}$ has the natural norm from the Banach space and it is also denoted by $\|h\|$ for $h \in H$. For all derivatives in the theorem, we use the operator norm. The open subset Θ of R^d has the Euclidean norm.

Theorem 3.1. *Suppose the map $(\theta, F, \eta) \to \Psi_{\theta,F}(\eta) \in H$, $(\theta, F, \eta) \in \Theta \times \mathcal{F} \times H$, is:*

- (A1) *Two times continuously differentiable with respect to θ and two times Hadamard differentiable with respect to η and Hadamard differentiable with respect to F so that the derivatives $\dot\Psi_{\theta,F}(\eta)$, $\ddot\Psi_{\theta,F}(\eta)$, $\mathrm{d}_\eta\Psi_{\theta,F}(\eta)$, $\mathrm{d}^2_\eta\Psi_{\theta,F}(\eta)$, $\mathrm{d}_\eta\dot\Psi_{\theta,F}(\eta)$ and $\mathrm{d}_F\Psi_{\theta,F}(\eta)$ exist in some neighborhood of the true value (θ_0, η_0, F_0) (where, e.g., $\dot\Psi_{\theta,F}(\eta)$ is the first derivative with respect to θ, and $\mathrm{d}_\eta\Psi_{\theta,F}(\eta)$ is the first derivative with respect to η in the sense of Hadamard. Similarly, the rest is defined). For each derivative, we assume the corresponding additional condition (3.2).*
- (A2) *The true value (θ_0, η_0, F_0) satisfy $\eta_0 = \Psi_{\theta_0,F_0}(\eta_0)$.*
- (A3) *The linear operator $\mathrm{d}_\eta\Psi_{\theta_0,F_0}(\eta_0) : \mathcal{B} \to \mathcal{B}$ has the operator norm $\|\mathrm{d}_\eta\Psi_{\theta_0,F_0}(\eta_0)\| < 1$.*

Then the solution $\eta_{\theta,F}$ to the equation

$$\eta = \Psi_{\theta,F}(\eta) \tag{3.4}$$

exists in an neighborhood of (θ_0, F_0) and it is two times continuously differentiable with respect to θ and Hadamard differentiable with respect to F in the neighborhood. Moreover, the derivatives are given by

$$\dot\eta_{\theta,F} = [I - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})]^{-1}\dot\Psi_{\theta,F}(\eta_{\theta,F}), \tag{3.5}$$

$$\begin{aligned}\ddot\eta_{\theta,F} = [I - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})]^{-1}[\ddot\Psi_{\theta,F}(\eta_{\theta,F}) + \mathrm{d}_\eta\dot\Psi_{\theta,F}(\eta_{\theta,F})\dot\eta^T_{\theta,F}\\ + \mathrm{d}_\eta\dot\Psi^T_{\theta,F}(\eta_{\theta,F})\dot\eta_{\theta,F} + \mathrm{d}^2_\eta\Psi_{\theta,F}(\eta_{\theta,F})\dot\eta_{\theta,F}\dot\eta^T_{\theta,F}]\end{aligned} \tag{3.6}$$

and

$$\mathrm{d}_F\eta_{\theta,F} = [I - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})]^{-1}\,\mathrm{d}_F\Psi_{\theta,F}(\eta_{\theta,F}). \tag{3.7}$$

3.1. Proof of Theorem 3.1

We assumed the derivative $\mathrm{d}_\eta\Psi_{\theta_0,F_0}(\eta_0)$ exists and its operator norm satisfies $\|\mathrm{d}_\eta\Psi_{\theta_0,F_0}(\eta_0)\| < 1$. By continuity of the map $(\theta,\eta,F)\to \mathrm{d}_\eta\Psi_{\theta,F}(\eta)$, there are $\varepsilon>0$ and a neighborhood of (θ_0,η_0,F_0) such that

$$\|\mathrm{d}_\eta\Psi_{\theta,F}(\eta)\| < 1-\varepsilon \tag{3.8}$$

for all (θ,η,F) in the neighborhood. In the following, we assume the parameters (θ,η,F) stay in the neighborhood so that the inequality (3.8) holds.

Existence and invertibility. Let $I:\mathcal{B}\to\mathcal{B}$ be the identity operator on the space $\mathcal{B}$. In the neighborhood discussed above, the map $(I-\mathrm{d}_\eta\Psi_{\theta,F}(\eta)):\mathcal{B}\to\mathcal{B}$ has the inverse $(I-\mathrm{d}_\eta\Psi_{\theta,F}(\eta))^{-1}$, which is also a bounded linear map (cf. Kolmogorov and Fomin [8], Theorem 4, page 231). It also follows that there is a neighborhood of (θ_0,η_0,F_0) such that, for each (θ,F), the map $\eta\to\Psi_{\theta,F}(\eta)$ is a contraction mapping in the neighborhood. By Banach's contraction principle (cf. Agarwal, O'Regan and Sahu [1], Theorem 4.1.5, page 178), the solution to the equation (3.4) exists uniquely in the neighborhood.

Differentiability with respect to F. Fix h in an appropriate space and let F_t be a map such that $F_{t=0}=F$, $t^{-1}\{F_t-F\}\to h$ as $t\downarrow 0$. Then, $F_t\to F$ (as $t\downarrow 0$). We aim to find the limit of $t^{-1}\{\eta_{\theta,F_t}-\eta_{\theta,F}\}$ as $t\downarrow 0$.

(Step 1) First step is to show $\eta_{\theta,F_t}\to\eta_{\theta,F}$ as $t\downarrow 0$. Due to equation (3.4), $\eta_{\theta,F}=\Psi_{\theta,F}(\eta_{\theta,F})$ and $\eta_{\theta,F_t}=\Psi_{\theta,F_t}(\eta_{\theta,F_t})$. It follows that

$$\begin{aligned}\{\eta_{\theta,F_t}-\eta_{\theta,F}\} &= \{\Psi_{\theta,F_t}(\eta_{\theta,F_t})-\Psi_{\theta,F}(\eta_{\theta,F})\}\\ &= \{\Psi_{\theta,F_t}(\eta_{\theta,F_t})-\Psi_{\theta,F_t}(\eta_{\theta,F})\}+\{\Psi_{\theta,F_t}(\eta_{\theta,F})-\Psi_{\theta,F}(\eta_{\theta,F})\}.\end{aligned} \tag{3.9}$$

Since the map $F\to\Psi_{\theta,F}(\eta)$ is continuous and $F_t\to F$ (as $t\downarrow 0$), the second term in the right-hand side is

$$\Psi_{\theta,F_t}(\eta_{\theta,F})-\Psi_{\theta,F}(\eta_{\theta,F})=\mathrm{o}(1)\qquad \text{as } t\downarrow 0.$$

By the generalized Taylors theorem for Banach spaces (cf. [20], page 243, Theorem 4C), the first term in the right-hand side is

$$\begin{aligned}\|\Psi_{\theta,F_t}(\eta_{\theta,F_t})-\Psi_{\theta,F_t}(\eta_{\theta,F})\| &\le \sup_{\tau\in[0,1]}\|\mathrm{d}_\eta\Psi_{\theta,F_t}(\eta_{\theta,F}+\tau(\eta_{\theta,F_t}-\eta_{\theta,F}))\|\|\eta_{\theta,F_t}-\eta_{\theta,F}\|\\ &\le (1-\varepsilon)\|\eta_{\theta,F_t}-\eta_{\theta,F}\|,\end{aligned}$$

where the last inequality is due to (3.8).

It follows from (3.9) that

$$\|\eta_{\theta,F_t}-\eta_{\theta,F}\|\le \mathrm{o}(1)+(1-\varepsilon)\|\eta_{\theta,F_t}-\eta_{\theta,F}\|\qquad \text{as } t\downarrow 0.$$

This shows $\eta_{\theta,F_t}\to\eta_{\theta,F}$ as $t\downarrow 0$.

(Step 2) By the Hadamard differentiability of the map $F \to \Psi_{\theta,F}(\eta)$ and the additional condition ((3.2) and (3.3)), there is a linear operator $h \to \mathrm{d}_F\Psi_t^* h$ such that the first term in the right-hand side of (3.9) can be expressed as

$$\{\Psi_{\theta,F_t}(\eta_{\theta,F_t}) - \Psi_{\theta,F}(\eta_{\theta,F_t})\} = \mathrm{d}_F\Psi_t^*(F_t - F),$$

and

$$\mathrm{d}_F\Psi_t^* \to \mathrm{d}_F\Psi_{\theta,F}(\eta_{\theta,F}) \qquad \text{as } t \downarrow 0.$$

Similarly, there is a linear operator $h' \to \mathrm{d}_\eta\Psi_t^* h'$ such that the second term in the right-hand side of (3.9) is

$$\{\Psi_{\theta,F}(\eta_{\theta,F_t}) - \Psi_{\theta,F}(\eta_{\theta,F})\} = \mathrm{d}_\eta\Psi_t^*\{\eta_{\theta,F_t} - \eta_{\theta,F}\}$$

and

$$\mathrm{d}_\eta\Psi_t^* \to \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F}) \qquad \text{as } t \downarrow 0.$$

Altogether, equation (3.9) can be written as

$$\{\eta_{\theta,F_t} - \eta_{\theta,F}\} = \mathrm{d}_F\Psi_t^*(F_t - F) + \mathrm{d}_\eta\Psi_t^*\{\eta_{\theta,F_t} - \eta_{\theta,F}\}.$$

It follows that

$$[I - \mathrm{d}_\eta\Psi_t^*]\{\eta_{\theta,F_t} - \eta_{\theta,F}\} = \mathrm{d}_F\Psi_t^*(F_t - F),$$

where I is the identity operator in the space $\mathcal{B}$.

Since we have the inequality (3.8) and $\mathrm{d}_\eta\Psi_t^* \to \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})$ as $t \downarrow 0$, the inverse $[I - \mathrm{d}_\eta\Psi_t^*]^{-1}$ exists for small $t > 0$. Therefore, when $t^{-1}(F_t - F) \to h$ as $t \downarrow 0$, we have that

$$\begin{aligned} t^{-1}\{\eta_{\theta,F_t} - \eta_{\theta,F}\} &= [I - \mathrm{d}_\eta\Psi_t^*]^{-1}\,\mathrm{d}_F\Psi_t^* t^{-1}(F_t - F) \\ &\to [I - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})]^{-1}\,\mathrm{d}_F\Psi_{\theta,F}(\eta_{\theta,F})h \qquad \text{as } t \downarrow 0. \end{aligned}$$

Since the limit is a bounded and linear map of h, the function $\eta_{\theta,F}(x)$ is Hadamard differentiable with respect to F with the derivative

$$\mathrm{d}_F\eta_{\theta,F} = [I - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})]^{-1}\,\mathrm{d}_F\Psi_{\theta,F}(\eta_{\theta,F}).$$

Differentiability with respect to θ. Similar proof as above can show that, for $t^{-1}(\theta_t - \theta) \to a \in R^d$ as $t \downarrow 0$, we have

$$t^{-1}\{\eta_{\theta_t,F} - \eta_{\theta,F}\} \to [I - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})]^{-1} a^T\dot{\Psi}_{\theta,F}(\eta_{\theta,F}).$$

It follows that the first derivative $\dot{\eta}_{\theta,F}$ of $\eta_{\theta,F}(x)$ with respect to θ is given by

$$a^T\dot{\eta}_{\theta,F} = [I - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})]^{-1} a^T\dot{\Psi}_{\theta,F}(\eta_{\theta,F}). \tag{3.10}$$

Now we show the second derivative of $\eta_{\theta,F}(x)$ with respect to θ. From (3.10), we have

$$a^T\dot\eta_{\theta,F} = a^T\dot\Psi_{\theta,F}(\eta_{\theta,F}) + \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F}).$$

Using this equation, for $t^{-1}(\theta_t-\theta)\to b\in R^d$ as $t\downarrow 0$,

$$\begin{aligned}
&t^{-1}\{a^T\dot\eta_{\theta_t,F} - a^T\dot\eta_{\theta,F}\}\\
&\quad = t^{-1}\{a^T\dot\Psi_{\theta_t,F}(\eta_{\theta_t,F}) - a^T\dot\Psi_{\theta,F}(\eta_{\theta,F})\}\\
&\qquad + t^{-1}\{\mathrm{d}_\eta\Psi_{\theta_t,F}(\eta_{\theta_t,F})(a^T\dot\eta_{\theta_t,F}) - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})\}\\
&\quad = t^{-1}\{a^T\dot\Psi_{\theta_t,F}(\eta_{\theta_t,F}) - a^T\dot\Psi_{\theta,F}(\eta_{\theta_t,F})\} + t^{-1}\{a^T\dot\Psi_{\theta,F}(\eta_{\theta_t,F}) - a^T\dot\Psi_{\theta,F}(\eta_{\theta,F})\}\\
&\qquad + t^{-1}\{\mathrm{d}_\eta\Psi_{\theta_t,F}(\eta_{\theta_t,F})(a^T\dot\eta_{\theta_t,F}) - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta_t,F})(a^T\dot\eta_{\theta_t,F})\}\\
&\qquad + t^{-1}\{\mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta_t,F})(a^T\dot\eta_{\theta_t,F}) - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta_t,F})\}\\
&\qquad + t^{-1}\{\mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta_t,F}) - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})\}.
\end{aligned}$$

By the differentiability with respect to θ, the each term in the right-hand side has the limit as follows, as $t\downarrow 0$,

$$\begin{aligned}
t^{-1}\{a^T\dot\Psi_{\theta_t,F}(\eta_{\theta_t,F}) - a^T\dot\Psi_{\theta,F}(\eta_{\theta_t,F})\} &\to a^T\ddot\Psi_{\theta,F}(\eta_{\theta,F})b,\\
t^{-1}\{a^T\dot\Psi_{\theta,F}(\eta_{\theta_t,F}) - a^T\dot\Psi_{\theta,F}(\eta_{\theta,F})\} &\to a^T\,\mathrm{d}_\eta\dot\Psi_{\theta,F}(\eta_{\theta,F})(\dot\eta_{\theta,F}^T b),\\
t^{-1}\{\mathrm{d}_\eta\Psi_{\theta_t,F}(\eta_{\theta_t,F})(a^T\dot\eta_{\theta_t,F}) - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta_t,F})(a^T\dot\eta_{\theta_t,F})\} &\to \{\mathrm{d}_\eta\dot\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})\}^T b,\\
t^{-1}\{\mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta_t,F})(a^T\dot\eta_{\theta_t,F}) - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta_t,F})\} &\to \mathrm{d}_\eta^2\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})(\dot\eta_{\theta,F}^T b),\\
t^{-1}\{\mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta_t,F}) - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})\}&\\
= \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})t^{-1}\{a^T\dot\eta_{\theta_t,F} - a^T\dot\eta_{\theta,F}\},&
\end{aligned}$$

where the last equality is due to the linearity of the operator $\mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F}):\mathcal{B}\to\mathcal{B}$ (the Hadamard derivative of $\Psi_{\theta,F}(\eta_{\theta,F})$ with respect to η).

Using additional condition and the Hadamard differentiability in (A1), by similar argument to the case for the differentiability with respect to F, we can show that

$$\begin{aligned}
&t^{-1}\{a^T\dot\eta_{\theta_t,F} - a^T\dot\eta_{\theta,F}\}\\
&\quad = a^T\ddot\Psi_{\theta,F}(\eta_{\theta,F})b + a^T\,\mathrm{d}_\eta\dot\Psi_{\theta,F}(\eta_{\theta,F})(\dot\eta_{\theta,F}^T b) + \{\mathrm{d}_\eta\dot\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})\}^T b\\
&\qquad + \mathrm{d}_\eta^2\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})(\dot\eta_{\theta,F}^T b) + \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})t^{-1}\{a^T\dot\eta_{\theta_t,F} - a^T\dot\eta_{\theta,F}\} + \mathrm{o}(1).
\end{aligned}$$

By rearranging this, we obtain

$$\begin{aligned}
&[I - \mathrm{d}_\eta\Psi_{\theta,F}(\eta_{\theta,F})]t^{-1}\{a^T\dot\eta_{\theta_t,F} - a^T\dot\eta_{\theta,F}\}\\
&\quad = a^T\ddot\Psi_{\theta,F}(\eta_{\theta,F})b + a^T\,\mathrm{d}_\eta\dot\Psi_{\theta,F}(\eta_{\theta,F})(\dot\eta_{\theta,F}^T b) + \{\mathrm{d}_\eta\dot\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})\}^T b\\
&\qquad + \mathrm{d}_\eta^2\Psi_{\theta,F}(\eta_{\theta,F})(a^T\dot\eta_{\theta,F})(\dot\eta_{\theta,F}^T b) + \mathrm{o}(1),
\end{aligned}$$

and hence, as $t \downarrow 0$,

$$t^{-1}\{a^T \dot{\eta}_{\theta_t,F} - a^T \dot{\eta}_{\theta,F}\} \to a^T \ddot{\eta}_{\theta,F} b,$$

where

$$a^T \ddot{\eta}_{\theta,F} b = [I - \mathrm{d}_\eta \Psi_{\theta,F}(\eta_{\theta,F})]^{-1} [a^T \ddot{\Psi}_{\theta,F}(\eta_{\theta,F}) b + a^T \mathrm{d}_\eta \dot{\Psi}_{\theta,F}(\eta_{\theta,F})(\dot{\eta}_{\theta,F}^T b) + \{\mathrm{d}_\eta \dot{\Psi}_{\theta,F}(\eta_{\theta,F})(a^T \dot{\eta}_{\theta,F})\}^T b + \mathrm{d}_\eta^2 \Psi_{\theta,F}(\eta_{\theta,F})(a^T \dot{\eta}_{\theta,F})(\dot{\eta}_{\theta,F}^T b)].$$

Therefore, $\dot{\eta}_{\theta,F}$ is differentiable with respect to θ with derivative $\ddot{\eta}_{\theta,F}$.

4. Example 1 continued

As an application of the main result (Theorem 3.1), we show existence and differentiability of solution to the operator equation in Example 1.

Theorem 4.1. *Suppose that*

$$E_F\left(\frac{\delta}{1+\delta} W^2(s;\beta,A)\right) > \mathrm{Var}_F\, W(s;\beta,A), \tag{4.1}$$

where $\mathrm{Var}_F\, W(s;\beta,A) = E_F W^2(s;\beta,A) - \{E_F W(s;\beta,A)\}^2$. *Then the solution* $A_{\beta,F}(t)$ *to the operator equation*

$$A = \Psi_{\beta,F}(A)$$

exists in an neighborhood of (β_0, F_0) *and it is two times continuously differentiable with respect to* β *and Hadamard differentiable with respect to* F *in the neighborhood, where the operator* $\Psi_{\beta,F}(A)$ *is given in (2.3).*

For the proof, we verify conditions (A1), (A2) and (A3) in Theorem 3.1 so that the differentiability of the solution is implied by the theorem.

Verification of condition (A1). We show that the map $\Psi_{\beta,F}(A)$ defined by (2.3) is differentiable with respect to β, F and A.

(*The derivative of* $\Psi_{\beta,F}(A)$ *with respect to* F) Suppose a map $t \to F_t$ satisfies $t^{-1}(F_t - F) \to h$ as $t \downarrow 0$.

$$\begin{aligned} t^{-1}\{\Psi_{\beta,F_t}(A) - \Psi_{\beta,F}(A)\} &= t^{-1}\left\{E_{F_t}\int_0^u \frac{\mathrm{d}N(s)}{E_{F_t} W(s;\beta,A)} - E_F \int_0^u \frac{\mathrm{d}N(s)}{E_F W(s;\beta,A)}\right\} \\ &= t^{-1}\left\{E_{F_t}\int_0^u \frac{\mathrm{d}N(s)}{E_{F_t} W(s;\beta,A)} - E_F \int_0^u \frac{\mathrm{d}N(s)}{E_{F_t} W(s;\beta,A)}\right\} \\ &\quad + t^{-1}\left\{E_F\int_0^u \frac{\mathrm{d}N(s)}{E_{F_t} W(s;\beta,A)} - E_F \int_0^u \frac{\mathrm{d}N(s)}{E_F W(s;\beta,A)}\right\}. \end{aligned}$$

After a simple calculation the right-hand side is equal to

$$\begin{aligned}
&\mathrm{d}\Psi_t^*(t^{-1}\{F_t - F\}) \\
&\quad = E_{t^{-1}\{F_t-F\}} \int_0^u \frac{\mathrm{d}N(s)}{E_{F_t} W(s;\beta,A)} - E_F \int_0^u \frac{E_{t^{-1}\{F_t-F\}} W(s;\beta,A)}{E_F W(s;\beta,A) E_{F_t} W(s;\beta,A)} \,\mathrm{d}N(s),
\end{aligned} \tag{4.2}$$

where the notation $E_F f$ means $\int f \,\mathrm{d}F$. The expression (4.2) shows the additional condition (3.2) is satisfied. Moreover, as $t \downarrow 0$, the expression converges to

$$\mathrm{d}_F \Psi_{\beta,F}(A) h = E_h \int_0^u \frac{\mathrm{d}N(s)}{E_F W(s;\beta,A)} - E_F \int_0^u \frac{E_h W(s;\beta,A)}{\{E_F W(s;\beta,A)\}^2} \,\mathrm{d}N(s).$$

This shows the map $F \to \Psi_{\beta,F}(A)$ is Hadamard differentiable at (β, A, F) with derivative $\mathrm{d}_F \Psi_{\beta,F}(A)$ and additional condition satisfied (clearly, the derivative is linear in h, we omit the proof of boundedness of $\mathrm{d}_F \Psi_{\beta,F}(A)$).

For the rest the derivatives, the proofs are similar and straightforward, therefore, we omit the proof and just give the derivatives in Appendix B.

Verification of condition (A2). Let F_0 be the true c.d.f. and β_0 be the true value of β. Since the true value A_0 of A is the maximizer of the expected log-likelihood

$$\int \{\delta(\beta_0' Z + \log a(U)) - (1+\delta) \log(1 + \mathrm{e}^{\beta_0' Z} A(U))\} \,\mathrm{d}F_0,$$

the same method to derive the equation (2.2) can be applied to show

$$A_0(u) = \int_0^u \frac{E_{F_0} \,\mathrm{d}N(s)}{E_{F_0} W(s;\beta_0,A_0)} = \Psi_{\beta_0,F_0}(A_0),$$

where $E_{F_0} \,\mathrm{d}N(s) = \int \mathrm{d}N(s) \,\mathrm{d}F_0$, $E_{F_0} W(s;\beta_0,A_0) = \int W(s;\beta_0,A_0) \,\mathrm{d}F_0$ and $\Psi_{\beta,F}(A)$ is defined in (2.3).

Verification of condition (A3). The derivatives $\mathrm{d}_A \Psi_{\beta,F}(A)$ and $\mathrm{d}_A W(s;\beta,A)$ are given in (B.1) and (B.2), respectively, in Appendix B. We consider the *sup*-norm on the space of total variation bounded cadlag functions $h_1(u)$ on $[0,\tau]$. For all $h_1(u)$ such that $\|h_1(u)\| = \sup_{u \in [0,\tau]} |h_1(u)| \le 1$, we have that

$$\begin{aligned}
|\mathrm{d}_A W(s;\beta,A) h_1| &\le \frac{(1+\delta)\mathrm{e}^{2\beta' Z} Y(s) |h_1(U)|}{\{1 + \mathrm{e}^{\beta' Z} A(U)\}^2} \le \frac{(1+\delta)\mathrm{e}^{2\beta' Z} Y(s)}{\{1 + \mathrm{e}^{\beta' Z} A(U)\}^2} \\
&\le \frac{(1+\delta)^2 \mathrm{e}^{2\beta' Z} Y(s)}{\{1 + \mathrm{e}^{\beta' Z} A(U)\}^2} = W^2(s;\beta,A).
\end{aligned}$$

We assumed $P\{\delta Y(s) = 1\} > 0$ for each $s \in [0,\tau]$ so that the last inequality in the above equation is strict inequality with positive probability for each s. This implies

$$W^2(s;\beta,A) - |\mathrm{d}_A W(s;\beta,A) h_1| \ge \frac{\delta}{1+\delta} W^2(s;\beta,A) > 0 \tag{4.3}$$

with positive probability for each s.

Then, by (4.1) and (4.3), we have that, for each s,

$$E_F W^2(s;\beta,A) - E_F|\mathrm{d}_A W(s;\beta,A)h_1| > E_F W^2(s;\beta,A) - \{E_F W(s;\beta,A)\}^2 > 0.$$

It follows that

$$|\mathrm{d}_A \Psi_{\beta,F}(A)h_1| \leq E_F \int_0^u \frac{E_F|\mathrm{d}_A W(s;\beta,A)h_1|}{\{E_F W(s;\beta,A)\}^2}\,\mathrm{d}N(s) < E_F \int_0^u \mathrm{d}N(s) \leq 1.$$

This demonstrates the operator $h_1 \to \mathrm{d}_A W(s;\beta,A)h_1$ has the operator norm smaller than one.

We have completed verification of conditions (A1), (A2) and (A3) in Theorem 3.1. By the theorem it follows that the derivatives of the function (2.2) is given by equations (3.5), (3.6) and (3.7) (needs replacement θ with β and η with A).

5. Example 2 continued

The generic form of c.d.f. for combined samples is $F = \sum_{s=1}^{2} w_s F_s$ where $w_s > 0$, $s = 1, 2$, and $w_1 + w_2 = 1$ and F_1, F_2 are c.d.f.s for the samples in V and $\overline{V}$, respectively.

For $\theta \in R^d$, F and function $g(x)$, define

$$\Psi_{\theta,F}(g) = \frac{\partial_x \int \pi_1(\mathrm{d}F)}{A(x;\theta,g,F)}, \tag{5.1}$$

where $\pi_s : F = \sum_{s'=1}^{2} w_{s'} F_{s'} \to w_s F_s$, $s = 1, 2$, are projections, and

$$A(x;\theta,g,F) = 1 - \int \frac{f(y|x;\theta)}{f_Y(y;\theta,g)} \pi_2(\mathrm{d}F). \tag{5.2}$$

Then the function $g_{\theta,F_n}(x)$ given by (2.7) is the solution to the operator equation

$$g(x) = \Psi_{\theta,F}(g)(x) \tag{5.3}$$

with $F = F_n$.

We show the differentiability of the solution $g_{\theta,F}(x)$ to the equation (5.3) with respect to θ and F.

Theorem 5.1. *Let θ_0, g_0 and $F_0 = \sum_{s=1}^{2} w_{s0} F_{s0}$ be the true values of θ, g and F at which data are generated. We assume that*

$$\frac{w_{20}}{w_{10}} < 1 \tag{5.4}$$

and the function $f(y|x;\theta)$ is twice continuously differentiable with respect to θ. Then the solution $g_{\theta,F}(x)$ to the operator equation (5.3) exists in an neighborhood of (θ_0, F_0) and *it is two times continuously differentiable with respect to θ and Hadamard differentiable with respect to F in the neighborhood.*

To prove the theorem, we verify conditions (A1), (A2) and (A3) in Theorem 3.1 so that the results follows from that theorem.

We denote $f = f(y|x;\theta)$, $f_Y = f_Y(y;\theta,g)$, $A = A(x;\theta,g,F)$, $\dot{f} = \frac{\partial}{\partial\theta} f(y|x;\theta)$, $\ddot{f} = \frac{\partial^2}{\partial\theta\,\partial\theta^T} f(y|x;\theta)$, $\dot{f}_Y = \int \dot{f}(y|x;\theta)g(x)\,\mathrm{d}x$, and $\ddot{f}_Y = \int \ddot{f}(y|x;\theta)g(x)\,\mathrm{d}x$.

Verification of condition (A1). We show that the map $\Psi_{\theta,F}(g)$ is differentiable with respect to θ, F and g.

(*The derivative of $\Psi_{\theta,F}(g)$ with respect to F*) Suppose a map $t \to F_t$ satisfies $t^{-1}(F_t - F) \to h$ as $t \downarrow 0$.

Then

$$\begin{aligned}
&\Psi_{\theta,F_t}(g) - \Psi_{\theta,F}(g) \\
&\quad = \frac{\partial_x \int \pi_1(\mathrm{d}F_t)}{A(x;\theta,g,F_t)} - \frac{\partial_x \int \pi_1(\mathrm{d}F)}{A(x;\theta,g,F)} \\
&\quad = \frac{(\partial_x \int \pi_1[\mathrm{d}(F_t - F)])A(x;\theta,g,F) - (\partial_x \int \pi_1(\mathrm{d}F))\{A(x;\theta,g,F_t) - A(x;\theta,g,F)\}}{A(x;\theta,g,F_t)A(x;\theta,g,F)}.
\end{aligned}$$

By equation (5.2), the right-hand side is equal to

$$\begin{aligned}
&\mathrm{d}_F \Psi_t^*(g)(F_t - F) \\
&\ = \frac{(\partial_x \int \pi_1[\mathrm{d}(F_t - F)])A(x;\theta,g,F) + (\partial_x \int \pi_1(\mathrm{d}F)) \int f(y|x;\theta)/f_Y(y;\theta,g)\pi_2[\mathrm{d}(F_t - F)]}{A(x;\theta,g,F_t)A(x;\theta,g,F)}.
\end{aligned}$$

This shows the additional condition (3.2) is satisfied. Moreover, as $t \downarrow 0$,

$$t^{-1}\{\Psi_{\theta,F_t}(g) - \Psi_{\theta,F}(g)\} = t^{-1}\,\mathrm{d}_F \Psi_t^*(g)(F_t - F) \to \mathrm{d}_F \Psi_{\theta,F}(g)h,$$

where the map $\mathrm{d}_F \Psi_{\theta,F}(g)$ is given by

$$\mathrm{d}_F \Psi_{\theta,F}(g)h = \frac{(\partial_x \int \pi_1(\mathrm{d}h))A(x;\theta,g,F) + (\partial_x \int \pi_1(\mathrm{d}F)) \int f(y|x;\theta)/f_Y(y;\theta,g)\pi_2(\mathrm{d}h)}{\{A(x;\theta,g,F)\}^2}.$$

Hence, the map $F \to \Psi_{\theta,F}(g)$ is Hadamard differentiable at (θ,g,F) with derivative $\mathrm{d}_F \Psi_{\theta,F}(g)$ (clearly, the derivative is linear in h, we omit the proof of boundedness of $\mathrm{d}_F \Psi_{\theta,F}(g)$).

Similarly, other (Hadamard) differentiability of map can be shown. In Appendix C, we list the derivatives without proofs.

Verification of condition (A2). To verify (A2), we show that, at (θ_0, F_0), $g_0(x)$ is a solution to the operator equation (5.3).

Since $\partial_x \int \mathrm{d}F_{10} = \int f(y|x;\theta_0)g_0(x)\,\mathrm{d}y = g_0(x)$, and $\frac{\mathrm{d}F_{20}(y)}{\mathrm{d}y} = f_Y(y;\theta_0,g_0)$, $w_{10}+w_{20} = 1$, we have

$$\begin{aligned}\Psi_{\theta_0,F_0}(g_0)(x) &= \frac{w_{10}\partial_x \int \mathrm{d}F_{10}}{1-w_{20}\int f(y|x;\theta_0)/f_Y(y;\theta_0,g_0)\,\mathrm{d}F_{20}} \\ &= \frac{w_{10}g_0(x)}{1-w_{20}\int f(y|x;\theta_0)/f_Y(y;\theta_0,g_0)f_Y(y;\theta_0,g_0)\,\mathrm{d}y} = g_0(x),\end{aligned} \tag{5.5}$$

where we used $\int f(y|x;\theta)\,\mathrm{d}y = 1$ for each x.

Verification of condition (A3). Let L_1 be the space of all real valued measurable functions $h(x)$ with $\|h\|_1 = \int |h(x)|\,\mathrm{d}x < \infty$. Then L_1 is a Banach space with the norm $\|\cdot\|_1$. The sup-norm is denoted by $\|h\|_\infty = \sup_x |h(x)|$.

The derivatives $\mathrm{d}_g\Psi_{\theta,F}(g)$ and $\mathrm{d}_g A(x;\theta,g,F)$ are, respectively, given in (C.1) and (C.2).

Since $\partial_x \int \pi_1(\mathrm{d}F_0) = w_{10}g_0(x)$, (C.1) implies

$$\mathrm{d}_g\Psi_{\theta_0,F_0}(g_0)h^* = \frac{-w_{10}g_0(x)\,\mathrm{d}_g A(x;\theta_0,g_0,F_0)h^*}{\{A(x;\theta_0,g_0,F_0)\}^2}.$$

By (5.2) together with $\pi_2(\mathrm{d}F_0) = w_{20}f_Y(y;\theta_0,g_0)\,\mathrm{d}y$, and $\int f(y|x;\theta)\,\mathrm{d}y = 1$, for all x, we have

$$A(x;\theta,g_0,F_0) = 1 - \int \frac{f(y|x;\theta_0)}{f_Y(y;\theta_0,g_0)}\pi_2(\mathrm{d}F_0) = 1 - w_{20} = w_{10}.$$

These equations and (C.2) imply

$$\mathrm{d}_g\Psi_{\theta_0,F_0}(g_0)h^* = -\frac{w_{20}}{w_{10}}g_0(x)\int f(y|x;\theta_0)\frac{\int f(y|x;\theta_0)h^*(x)\,\mathrm{d}x}{f_Y(y;\theta_0,g_0)}\,\mathrm{d}y. \tag{5.6}$$

The L_1 norm of (5.6) is

$$\begin{aligned}\|\mathrm{d}_g\Psi_{\theta_0,F_0}(g_0)h^*\|_1 &= \int \left|\frac{w_{20}}{w_{10}}g_0(x)\int f(y|x;\theta_0)\frac{\int f(y|x;\theta_0)h^*(x)\,\mathrm{d}x}{f_Y(y;\theta_0,g_0)}\,\mathrm{d}y\right|\mathrm{d}x \\ &\le \frac{w_{20}}{w_{10}}\int g_0(x)\left(\int f(y|x;\theta_0)\frac{\int f(y|x;\theta_0)|h^*(x)|\,\mathrm{d}x}{f_Y(y;\theta_0,g_0)}\,\mathrm{d}y\right)\mathrm{d}x \\ &= \frac{w_{20}}{w_{10}}\int |h^*(x)|\,\mathrm{d}x \qquad \left(\text{by Fubini's theorem and } \int f(y|x;\theta_0)\,\mathrm{d}y = 1\right) \\ &= \frac{w_{20}}{w_{10}}\|h^*\|_1.\end{aligned}$$

From the calculation above, we see that the operator $h^* \to \mathrm{d}_g\Psi_{\theta_0,F_0}(g_0)h^*$ has the operator norm $\le \frac{w_{20}}{w_{10}}$. Since we assumed $\frac{w_{20}}{w_{10}} < 1$, we have condition (A3).

6. Asymptotic normality of maximum profile likelihood estimator

Hirose [6] showed the efficiency of the maximum profile likelihood estimator in semiparametric models using the direct asymptotic expansion of the profile likelihood. The method gives alternative to the one proposed by Murphy and van der Vaart [11] which uses an asymptotic expansion of approximate profile likelihood. We summarize the results from the paper.

Suppose we have a function $\eta_{\theta,F}$ that depends on (θ, F) such that $\tilde{\ell}_0(x) \equiv \tilde{\ell}_{\theta_0,F_0}(x)$ is the efficient score function, where

$$\tilde{\ell}_{\theta,F}(x) \equiv \frac{\partial}{\partial\theta} \log p_{\theta,\eta_{\theta,F}}(x). \tag{6.1}$$

The theorem below show that if the solution $\hat{\theta}_n$ to the estimating equation

$$\int \tilde{\ell}_{\hat{\theta}_n,F_n}(x)\,\mathrm{d}F_n = 0 \tag{6.2}$$

is consistent then it is asymptotically linear with the efficient influence function $\tilde{I}_0^{-1}\tilde{\ell}_0(x)$ so that

$$n^{-1/2}(\hat{\theta}_n - \theta_0) = \int \tilde{I}_0^{-1}\tilde{\ell}_0(x)\,\mathrm{d}\{n^{-1/2}(F_n - F_0)\} + \mathrm{o}_P(1) \xrightarrow{d} N(0, \tilde{I}_0^{-1}), \tag{6.3}$$

where $N(0, \tilde{I}_0^{-1})$ is a normal distribution with mean zero and variance $\tilde{I}_0^{-1}$. Since $\tilde{I}_0 = E_0(\tilde{\ell}_0\tilde{\ell}_0^T)$ is the efficient information matrix, this demonstrates that the estimator $\hat{\theta}_n$ is efficient.

On the set of c.d.f. functions $\mathcal{F}$, we use the sup-norm, that is, for $F, F_0 \in \mathcal{F}$,

$$\|F - F_0\| = \sup_x |F(x) - F_0(x)|.$$

For $\rho > 0$, let

$$\mathcal{C}_\rho = \{F \in \mathcal{F}\colon\ \|F - F_0\| < \rho\}.$$

Theorem 6.1 (Hirose [6]). *Assumptions:*

(R0) *The function $g_{\theta,F}$ satisfies $g_{\theta_0,F_0} = g_0$ and the function*

$$\tilde{\ell}_0(x) = \tilde{\ell}_{\theta_0,F_0}(x)$$

is the efficient score function where $\tilde{\ell}_{\theta,F}(x)$ is given by (6.1).

(R1) *The empirical process F_n is $n^{1/2}$-consistent, that is, $n^{1/2}\|F_n - F_0\| = \mathrm{O}_P(1)$, and there exists a $\rho > 0$ and a neighborhood Θ of θ_0 such that for each $(\theta, F) \in \Theta \times \mathcal{C}_\rho$, the log-likelihood function $\log p(x; \theta, \hat{g}_{\theta,F})$ is twice continuously differentiable with respect to θ and Hadamard differentiable with respect to F for all x.*

(R2) *The efficient information matrix* $\tilde{I}_0 = E_0(\tilde{\ell}_0 \tilde{\ell}_0^T)$ *is invertible.*

(R3) *There exists a* $\rho > 0$ *and a neighborhood* Θ *of* θ_0 *such that the class of functions* $\{\tilde{\ell}_{\theta,F}(x)\colon\ (\theta, F) \in \Theta \times \mathcal{C}_\rho\}$ *is Donsker with square integrable envelope function, and that the class of functions* $\{\frac{\partial}{\partial\theta}\tilde{\ell}_{\theta,F}(x)\colon\ (\theta, F) \in \Theta \times \mathcal{C}_\rho\}$ *is Glivenko–Cantelli with integrable envelope function.*

Under the assumptions $\{(\mathrm{R0}), (\mathrm{R1}), (\mathrm{R2}), (\mathrm{R3})\}$, *for a consistent solution* $\hat{\theta}_n$ *to the estimating equation (6.2), the equation (6.3) holds.*

6.1. Asymptotic normality and efficiency in Example 2

In this section, we demonstrate how the result of the paper can be used to show the efficiency of profile likelihood estimators in semi-parametric models. We show the efficiency of the estimator in Example 2 (using the result in Section 5). First, we identify the efficient score function in the example. Then we verify conditions (R0)–(R3) in Theorem 6.1. Then the efficiency of the estimator follows from the theorem.

Efficient score function. We show that the function (2.7) (the solution to the equation (5.3)) gives us the efficient score function in Example 2. The log-density function in Example 2 is given by

$$\log p(s, z; \theta, g) = 1_{\{s=1\}}\{\log f(y|x;\theta) + \log g(x)\} + 1_{\{s=2\}} \log f_Y(y; \theta, g), \tag{6.4}$$

where $z = (y, x)$ if $s = 1$ and $z = y$ if $s = 2$, and $f_Y(y; \theta, g)$ is given in (2.6).

Theorem 6.2 (The efficient score function). *Let us denote* $g_{\theta,F_0}(x)$ *as the function (2.7) evaluated at* (θ, F_0)*:*

$$g_{\theta,F_0}(x) = \frac{w_{10}\partial_x \int \mathrm{d}F_{10}}{1 - w_{20}\int f(y|x;\theta)/f_Y(y;\theta, g_{\theta,F_0})\,\mathrm{d}F_{20}}. \tag{6.5}$$

Then the function

$$\tilde{\ell}_{\theta_0,F_0}(s, z) = \frac{\partial}{\partial\theta}\bigg|_{\theta=\theta_0} \log p(s, z; \theta, g_{\theta,F_0}) \tag{6.6}$$

is the efficient score function in the model in Example 2.

Proof. We check conditions (A.1) and (A.2) in Theorem A.1 in the Appendix. Then the claim follows from the theorem.

Condition (A.1) is checked in equation (5.5).

We verify condition (A.2). Let $g_t(x)$ be a path in the space of density functions with $g_{t=0}(x) = g_0(x)$. Define $\alpha_t(x) = g_t(x) - g_0(x)$ and write $\dot{\alpha}_0(x) = \frac{\partial}{\partial t}|_{t=0}\alpha_t(x)$. Then

$$\frac{\partial}{\partial t}\bigg|_{t=0} \int \log p(s, z; \theta, g_{\theta,F_0} + \alpha_t)\,\mathrm{d}F_0$$

$$= \frac{\partial}{\partial t}\bigg|_{t=0} \bigg[w_{10} \int \{\log f(y|x;\theta) + \log(g_{\theta,F_0} + \alpha_t)\} \, \mathrm{d}F_{10}$$

$$+ w_{20} \int \log f_Y(y;\theta, g_{\theta,F_0} + \alpha_t) \, \mathrm{d}F_{20} \bigg]$$

$$= w_{10} \int \frac{\dot{\alpha}_0(x)}{g_{\theta,F_0}(x)} \, \mathrm{d}F_{10} + w_{20} \int \frac{\int f(y|x;\theta)\dot{\alpha}_0(x)\, \mathrm{d}x}{f_Y(y;\theta, g_{\theta,F_0})} \, \mathrm{d}F_{20}$$

$$= \int \dot{\alpha}_0(x) \, \mathrm{d}x = \frac{\partial}{\partial t}\bigg|_{t=0} \int g_t(x)\, \mathrm{d}x = 0 \quad \text{(by (6.5) and since } g_t(x) \text{ is a density).} \qquad \square$$

Efficiency of the profile likelihood estimator. Let $\tilde{\ell}_{\theta,F}(s,x)$ be the score function given by (6.6) with θ_0 and F_0 are replaced by θ and F.

We verify conditions (R0), (R1), (R2) and (R3) of Theorem 6.1 so that we can apply the theorem to show that the solution $\hat{\theta}_n$ to the estimating equation

$$\sum_{s=1}^{2} \sum_{i=1}^{n} \tilde{\ell}_{\hat{\theta}_n, F_n}(s, X_{si}) = 0$$

is asymptotically linear estimator with the efficient influence function, that is, (6.3) holds. This shows the efficiency of the MLE based on the profile likelihood in this example.

Condition (R0). Theorem 6.2 shows that the score function evaluated at (θ_0, F_0) is the efficient score function in Example 2.

Condition (R1). We assume that:

(T1) For all $\theta \in \Theta$, the function $f(y|x;\theta)$ is twice continuously differentiable with respect to θ.

The maps

$$g \to \log g(x)$$

and

$$g \to f_Y(y;\theta,g) = \int_{\mathcal{X}} f(y|x;\theta) g(x) \, \mathrm{d}x$$

are Hadamard differentiable (cf. Gill [5]). It follows that the log-density function $\log p(s,z;\theta,g)$ given by (6.4) is Hadamard differentiable with respect to g and, by assumption (T1), it is also twice continuously differentiable with respect to θ. In the previous section (Section 5), we verified the function $g_{\theta,F}$ is Hadamard differentiable with respect to F and twice continuously differentiable with respect to θ. By the chain rule and product rule of Hadamard differentiable maps, the log-density function $\log p(s,x;\theta,g_{\theta,F})$ is Hadamard differentiable with respect to F and twice continuously differentiable with respect to θ. Therefore, we verified condition (R1).

Derivatives of log-likelihood. The log-density function under consideration is

$$\log p(s,z;\theta,g_{\theta,F}) = 1_{\{s=1\}} \{\log f(y|x;\theta) + \log g_{\theta,F}(x)\} + 1_{\{s=2\}} \log f_Y(y;\theta,g_{\theta,F}). \quad (6.7)$$

The derivative of the log-density with respect to θ is

$$\begin{aligned}\tilde{\ell}_{\theta,F}(s,z) &= \frac{\partial}{\partial\theta}\log p(s,z;\theta,g_{\theta,F}) \\ &= 1_{\{s=1\}}\left\{\frac{\dot{f}}{f}+\frac{\dot{g}_{\theta,F}}{g_{\theta,F}}\right\}+1_{\{s=2\}}\frac{\dot{f}_Y+\mathrm{d}_g f_Y(\dot{g}_{\theta,F})}{f_Y}.\end{aligned}\tag{6.8}$$

The second derivative of the log-density function with respect to θ is

$$\begin{aligned}\frac{\partial}{\partial\theta^T}\tilde{\ell}_{\theta,F}(s,z) &= \frac{\partial^2}{\partial\theta\,\partial\theta^T}\log p(s,z;\theta,g_{\theta,F}) \\ &= 1_{\{s=1\}}\left\{\frac{\ddot{f}}{f}-\frac{\dot{f}\dot{f}^T}{f^2}+\frac{\ddot{g}_{\theta,F}}{g_{\theta,F}}-\frac{\dot{g}_{\theta,F}\dot{g}_{\theta,F}^T}{g_{\theta,F}^2}\right\} \\ &\quad + 1_{\{s=2\}}\left\{\frac{\ddot{f}_Y+\mathrm{d}_g\dot{f}_Y(\dot{g}_{\theta,F})}{f_Y}-\frac{\dot{f}_Y\dot{f}_Y^T+\dot{f}_Y\,\mathrm{d}_g f_Y(\dot{g}_{\theta,F}^T)}{f_Y^2}\right. \\ &\qquad + \frac{\mathrm{d}_g\dot{f}_Y^T(\dot{g}_{\theta,F})+\mathrm{d}_g f_Y(\ddot{g}_{\theta,F})}{f_Y} \\ &\qquad \left. - \frac{\mathrm{d}_g f_Y(\dot{g}_{\theta,F})\dot{f}_Y^T+\mathrm{d}_g f_Y(\dot{g}_{\theta,F})\,\mathrm{d}_g f_Y(\dot{g}_{\theta,F}^T)}{f_Y^2}\right\}.\end{aligned}\tag{6.9}$$

Here, we used the notation $\dot{f}_Y=\dot{f}_Y(y;\theta,g_{\theta,F})$, $\ddot{f}_Y=\ddot{f}_Y(y;\theta,g_{\theta,F})$, $\mathrm{d}_g f_Y(g_{\theta,F})=\int f(y|x;\theta)g_{\theta,F}(x)\,\mathrm{d}x$, and $\mathrm{d}_g\dot{f}_Y(g_{\theta,F})=\int \dot{f}(y|x;\theta)g_{\theta,F}(x)\,\mathrm{d}x$.

Condition (R2). We assume that:

(T2) There is no $a\in R^d$ such that $a^T\frac{\dot{f}}{f}(y|x;\theta)$ is constant in y for almost all x.

The term $\frac{\dot{g}_{\theta,F}}{g_{\theta,F}}(x,\theta_0,F_0)$ is a function of x. Therefore, by equation (6.8) and assumption (T2), there is no $a\in R^d$ such that $a^T\tilde{\ell}_{\theta,F}(1,z)$ is constant in y for almost all x. By Theorem 1.4 in Seber and Lee [15], $E(\tilde{\ell}_{\theta_0,F_0}\tilde{\ell}_{\theta_0,F_0}^T)$ is nonsingular with the bounded inverse.

Conditions (R3). Since verification of condition (R3) require more assumptions and it does not add anything new, we simply assume:

(T3) Let $\mathcal{F}$ be the set of c.d.f. functions and for some $\rho>0$ define $\mathcal{C}_\rho=\{F\in\mathcal{F}\colon\ \|F-F_0\|_\infty\le\rho\}$. The class of function

$$\{\tilde{\ell}_{\theta,F}(s,z)\colon\ (\theta,F)\in\Theta\times\mathcal{C}_\rho\}$$

is P_{θ_0,g_0}-Donsker with square integrable envelope function and the class

$$\left\{\frac{\partial}{\partial\theta^T}\tilde{\ell}_{\theta,F}(s,z)\colon\ (\theta,F)\in\Theta\times\mathcal{C}_\rho\right\}$$

is P_{θ_0,g_0}-Glivenko–Cantelli with integrable envelope function.

7. Discussion

In Theorem 3.1, we have shown the differentiability of implicitly defined function which we encounter in the maximum likelihood estimation in semi-parametric models. In the theorem, we assumed the implicitly defined function is the solution to the operator equation (1.4) and we obtained the derivatives of the (implicitly defined) function. In application of the theorem, we need to verify condition (A3) in the theorem (that is $\|\mathrm{d}_\eta\Psi_{\theta_0,F_0}(\eta_0)\| < 1$). This required additional conditions in the examples ((4.1) in Example 1 and (5.4) in Example 2). The future work is to relax the condition to $\|\mathrm{d}_\eta\Psi_{\theta_0,F_0}(\eta_0)\| < \infty$ so that the additional conditions can be weaken. Once the differentiability of the implicitly defined function has been established, the results in Hirose [6] (we summarized in Section 6, Theorem 6.1) are applicable.

Appendix A: Verification of efficient score function

To verify condition (R0) in Theorem 6.1, the following theorem may be useful. This is a modification of the proof in Breslow, McNeney and Wellner [4] which was originally adapted from Newey [12].

Theorem A.1. *We assume the general semi-parametric model given in the Introduction with the density $p_{\theta,\eta}(x) = p(x;\theta,\eta)$ is differentiable with respect to θ and Hadamard differentiable with respect to η. Suppose g_t is an arbitrary path such that $g_{t=0} = g_0$ and let $\alpha_t = g_t - g_0$. If $g_{\theta,F}$ is a function of (θ, F) such that*

$$g_{\theta_0,F_0} = g_0 \tag{A.1}$$

and, for each $\theta \in \Theta$,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} E_0[\log p(x;\theta, g_{\theta,F_0} + \alpha_t)] = 0, \tag{A.2}$$

then the function $\tilde{\ell}_{\theta_0,F_0}(x) = \frac{\partial}{\partial\theta}|_{\theta=\theta_0} \log p(x;\theta, g_{\theta,F_0})$ is the efficient score function.

Proof. Condition (A.2) implies that

$$\begin{aligned} 0 &= \left.\frac{\partial}{\partial\theta}\right|_{\theta=\theta_0} \left.\frac{\partial}{\partial t}\right|_{t=0} E_0[\log p(x;\theta, g_{\theta,F_0} + \alpha_t)] \\ &= \left.\frac{\partial}{\partial t}\right|_{t=0} E_0\left[\left.\frac{\partial}{\partial\theta}\right|_{\theta=\theta_0} \log p(x;\theta, g_{\theta,F_0} + \alpha_t)\right]. \end{aligned} \tag{A.3}$$

By differentiating the identity

$$\int \left(\frac{\partial}{\partial\theta} \log p(x;\theta, g_{\beta,F_0} + \alpha_t) \right) p(x;\theta, g_{\beta,F_0} + \alpha_t)\,\mathrm{d}x = 0$$

with respect to t at $t = 0$ and $\theta = \theta_0$, we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\bigg|_{t=0,\theta=\theta_0} \int \left(\frac{\partial}{\partial\theta} \log p(x;\theta, g_{\theta,F_0} + \alpha_t) \right) p(x;\theta, g_{\theta,F_0} + \alpha_t)\,\mathrm{d}x \\ &= E_0\left[\tilde{\ell}_{\theta_0,F_0}(x) \left(\frac{\partial}{\partial t}\bigg|_{t=0} \log p(x;\theta_0, g_t) \right) \right] \qquad \text{(by (A.1))} \\ &\quad + \frac{\partial}{\partial t}\bigg|_{t=0} E_0\left[\frac{\partial}{\partial\theta}\bigg|_{\theta=\theta_0} \log p(x;\theta, g_{\theta,F_0} + \alpha_t) \right] \\ &= E_0\left[\tilde{\ell}_{\theta_0,F_0}(x) \left(\frac{\partial}{\partial t}\bigg|_{t=0} \log p(x;\theta_0, g_t) \right) \right] \qquad \text{(by (A.3))}. \end{aligned} \tag{A.4}$$

Let $c \in R^m$ be arbitrary. Then it follows from equation (A.4) that the product $c'\tilde{\ell}_{\theta_0,F_0}(x)$ is orthogonal to the nuisance tangent space $\dot{\mathcal{P}}_g$ which is the closed linear span of score functions of the form $\frac{\partial}{\partial t}|_{t=0} \log p(x;\beta_0, g_t)$.

Using condition (A.1), we have

$$\begin{aligned} \tilde{\ell}_{\theta_0,F_0}(x) &= \frac{\partial}{\partial\theta}\bigg|_{\theta=\theta_0} \log p(x;\theta, g_0) + \frac{\partial}{\partial\beta}\bigg|_{\theta=\theta_0} \log p(x;\theta_0, g_{\theta,F_0}) \\ &= \dot{\ell}_{\theta_0,g_0}(x) - \psi_{\theta_0,g_0}(x), \end{aligned}$$

where $\dot{\ell}_{\theta_0,g_0}(x) = \frac{\partial}{\partial\theta}|_{\theta=\theta_0} \log p(x;\theta, g_0)$ is the score function for θ and $\psi_{\theta_0,g_0}(x) = -\frac{\partial}{\partial\theta}|_{\theta=\theta_0} \log p(x;\theta_0, g_{\theta,F_0})$. Finally, $c'\tilde{\ell}_{\theta_0,F_0}(x) = c'\dot{\ell}_{\theta_0,g_0}(x) - c'\psi_{\theta_0,g_0}(x)$ is orthogonal to the nuisance tangent space $\dot{\mathcal{P}}_g$ and $c'\psi_{\theta_0,g_0}(x) \in \dot{\mathcal{P}}_g$ implies that $c'\psi_{\theta_0,g_0}(x)$ is the orthogonal projection of $c'\dot{\ell}_{\theta_0,g_0}(x)$ onto the nuisance tangent space $\dot{\mathcal{P}}_g$. Since $c \in R^m$ is arbitrary, $\tilde{\ell}_{\theta_0,F_0}(x)$ is the efficient score function. □

Appendix B: Verification of (A1) in Example 1: Continued from Section 4

In verification of (A1) in Example 1, Section 4, we gave proof the Hadamard differentiability of functions with additional condition for the derivative of $\Psi_{\beta,F}(A)$ with respect to F. For the rest the derivatives, we give them without proofs.

(*The derivative of* $\Psi_{\beta,F}(A)$ *with respect to* A) Let $h_1 = h_1(U)$ be a function of U.

$$\mathrm{d}_A\Psi_{\beta,F}(A)h_1 = -E_F \int_0^u \frac{E_F\,\mathrm{d}_A W(s;\beta,A)h_1}{\{E_F W(s;\beta,A)\}^2}\,\mathrm{d}N(s), \tag{B.1}$$

where

$$\mathrm{d}_A W(s;\beta,A)h_1 = \frac{-(1+\delta)\mathrm{e}^{2\beta' Z}Y(s)h_1(U)}{\{1+\mathrm{e}^{\beta' Z}A(U)\}^2}. \tag{B.2}$$

(*The second derivative of* $\Psi_{\beta,F}(A)$ *with respect to* A) If $h_1(U)$, $h_2(U)$ are functions,

$$\begin{aligned}\mathrm{d}_A^2\Psi_{\beta,F}(A)h_1h_2 &= E_F\int_0^u \frac{E_F\,\mathrm{d}_A^2 W(s;\beta,A)h_1h_2}{\{E_F W(s;\beta,A)\}^2}\,\mathrm{d}N(s)\\ &\quad + E_F\int_0^u \frac{2\{E_F\,\mathrm{d}_A W(s;\beta,A)h_1\}\{E_F\,\mathrm{d}_A W(s;\beta,A)h_2\}}{\{E_F W(s;\beta,A)\}^3}\,\mathrm{d}N(s),\end{aligned}$$

where

$$\mathrm{d}_A^2 W(s;\beta,A_t)h_1h_2 = \frac{2(1+\delta)\mathrm{e}^{3\beta' Z}Y(s)h_1(U)h_2(U)}{\{1+\mathrm{e}^{\beta' Z}A(U)\}^3}.$$

(The expression of $\mathrm{d}_A W(s;\beta,A)h_1$ is given in (B.2).)

(*The first and second derivative of* $\Psi_{\beta,F}(A)$ *with respect to* β) Let us denote the first and second derivatives (with respect to β) by $\dot\Psi_{\beta,F}(A)$ and $\ddot\Psi_{\beta,F}(A)$, respectively. Then they are given by, for $a,b\in R^d$,

$$\begin{aligned}a^T\dot\Psi_{\beta,F}(A) &= a^T\left\{\frac{\partial}{\partial\beta}\Psi_{\beta,F}(A)\right\}\\ &= -E_F\int_0^u \frac{E_F a^T\dot W(s;\beta,A)}{\{E_F W(s;\beta,A)\}^2}\,\mathrm{d}N(s),\\ a^T\ddot\Psi_{\beta,A}(g)b &= a^T\left\{\frac{\partial^2}{\partial\beta\,\partial\beta^T}\Psi_{\beta,F}(A)\right\}b\\ &= E_F\int_0^u \frac{E_F a^T\ddot W(s;\beta,A)b}{\{E_F W(s;\beta,A)\}^2}\,\mathrm{d}N(s)\\ &\quad + E_F\int_0^u \frac{2\{E_F a^T\dot W(s;\beta,A)h_1\}\{E_F\dot W^T(s;\beta,A)b\}}{\{E_F W(s;\beta,A)\}^3}\,\mathrm{d}N(s).\end{aligned}$$

Here,

$$a^T\dot W(s;\beta,A) = a^T\left\{\frac{\partial}{\partial\beta}W(s;\beta,A)\right\} = \frac{(1+\delta)a^T\beta\mathrm{e}^{\beta^T Z}Y(s)}{\{1+\mathrm{e}^{\beta^T Z}A(U)\}^2}$$

and

$$\begin{aligned}a^T\ddot W(s;\beta,A)b &= a^T\left\{\frac{\partial^2}{\partial\beta\,\partial\beta^T}W(s;\beta,A)\right\}b\\ &= \frac{(1+\delta)\{(a^Tb)\mathrm{e}^{\beta^T Z}+(a^T\beta)(\beta^Tb)\mathrm{e}^{\beta^T Z}\}Y(s)}{\{1+\mathrm{e}^{\beta^T Z}A(U)\}^2}\end{aligned}$$

$$- \frac{2(1+\delta)(a^T\beta)(\beta^T b)\mathrm{e}^{2\beta^T Z} Y(s) A(U)}{\{1+\mathrm{e}^{\beta^T Z} A(U)\}^3}.$$

(*The derivative of* $\Psi_{\beta,F}(A)$ *with respect to* β *and* A) For given function $h_1(U)$ and $a \in R^d$,

$$\begin{aligned} & a^T \,\mathrm{d}_A \dot{\Psi}_{\beta,F}(A) h_1 \\ & \quad = -E_F \int_0^u \left\{ \frac{E_F a^T \,\mathrm{d}_A \dot{W}(s;\beta,A) h_1}{\{E_F W(s;\beta,A)\}^2} - 2 \frac{E_F a^T \dot{W}(s;\beta,A) E_F \,\mathrm{d}_A W(s;\beta,A) h_1}{\{E_F W(s;\beta,A)\}^3} \right\} \mathrm{d}N(s), \end{aligned}$$

here $a^T \dot{W}(s;\beta,A)$ is given above, $\mathrm{d}_A W(s;\beta,A)h_1$ is given in (B.2) and

$$a^T \,\mathrm{d}_A \dot{W}(s;\beta,A) h_1 = \frac{-2(1+\delta) a^T \beta \mathrm{e}^{2\beta^T Z} Y(s) h_1(U)}{\{1+\mathrm{e}^{\beta^T Z} A(U)\}^3}.$$

Appendix C: Verification of (A1) in Example 2: Continued from Section 5

We proved the Hadamard differentiability of functions and additional condition for the derivative of $\Psi_{\theta,F}(g)$ with respect to F in Section 5, verification of (A1) in Example 2. The rest of the derivatives are listed here.

(*The derivative of* $\Psi_{\theta,F}(g)$ *with respect to* g) For a function $h^*(x)$ of x,

$$\mathrm{d}_g \Psi_{\theta,F}(g) h^* = \frac{-(\partial_x \int \pi_1(\mathrm{d}F))\{\mathrm{d}_g A(x;\theta,g,F) h^*\}}{\{A(x;\theta,g,F)\}^2}, \tag{C.1}$$

where

$$\mathrm{d}_g A(x;\theta,g,F) h^* = \int f(y|x;\theta) \frac{\int f(y|x;\theta) h^*(x)\,\mathrm{d}x}{\{f_Y(y;\theta,g)\}^2} \pi_2(\mathrm{d}F). \tag{C.2}$$

(*The second derivative of* $\Psi_{\theta,F}(g)$ *with respect to* g) For functions $h_1(x)$ and $h_2(x)$ of x,

$$\begin{aligned} & \mathrm{d}_g^2 \Psi_{\theta,F}(g) h_1 h_2 \\ & \quad = \left(\partial_x \int \pi_1(\mathrm{d}F)\right) \left[- \frac{\mathrm{d}_g^2 A(x;\theta,g,F) h_1 h_2}{\{A(x;\theta,g_t,F)\}^2} + \frac{2\{\mathrm{d}_g A(x;\theta,g,F) h_1\}\{\mathrm{d}_g A(x;\theta,g,F) h_2\}}{\{A(x;\theta,g,F)\}^3} \right], \end{aligned}$$

where

$$\mathrm{d}_g^2 A(x;\theta,g_t,F) h_1 h_2 = -2 \int f(y|x;\theta) \frac{\{\int f(y|x;\theta) h_1(x)\,\mathrm{d}x\}\{\int f(y|x;\theta) h_2(x)\,\mathrm{d}x\}}{\{f_Y(y;\theta,g)\}^3} \pi_2(\mathrm{d}F).$$

(*The first and second derivative of $\Psi_{\theta,F}(g)$ with respect to θ*) Let us denote the first and second derivatives with respect to θ by $\dot{\Psi}_{\theta,F}(g)$ and $\ddot{\Psi}_{\theta,F}(g)$, respectively. They are given by, for $a, b \in R^d$,

$$a^T\dot{\Psi}_{\theta,F}(g) = a^T\left\{\frac{\partial}{\partial\theta}\Psi_{\theta,F}(g)\right\} = -\frac{(\partial_x\int\pi_1(\mathrm{d}F))a^T\dot{A}}{A^2},$$

$$a^T\ddot{\Psi}_{\theta,F}(g)b = a^T\left\{\frac{\partial^2}{\partial\theta\,\partial\theta^T}\Psi_{\theta,F}(g)\right\}b = -\frac{(\partial_x\int\pi_1(\mathrm{d}F))\{A(a^T\ddot{A}b) - 2(a^T\dot{A})(\dot{A}^Tb)\}}{A^3},$$

where

$$a^T\dot{A} = a^T\left\{\frac{\partial}{\partial\theta}A(x;\theta,g,F)\right\} = -\int\frac{f_Y(a^T\dot{f}) - f(a^T\dot{f}_Y)}{f_Y^2}\pi_2(\mathrm{d}F)$$

and

$$\begin{aligned}a^T\ddot{A}b &= a^T\left\{\frac{\partial^2}{\partial\theta\,\partial\theta^T}A(x;\theta,g,F)\right\}b\\ &= -\int(f_Y^2(a^T\ddot{f}b) - ff_Y(a^T\ddot{f}_Yb) + 2f(a^T\dot{f}_Y)(\dot{f}_Y^Tb)\\ &\qquad - f_Y(a^T\dot{f})(\dot{f}_Y^Tb) - f_Y(a^T\dot{f}_Y)(\dot{f}^Tb))/f_Y^3\pi_2(\mathrm{d}F).\end{aligned}$$

(*The derivative of $\Psi_{\theta,F}(g)$ with respect to θ and g*) For $a \in R^d$ and function $h^*(x)$ of x,

$$\begin{aligned}&a^T\,\mathrm{d}_g\dot{\Psi}_{\theta,F}(g)h^*\\ &\quad = -\left(\partial_x\int\pi_1(\mathrm{d}F)\right)\left[\frac{a^T\,\mathrm{d}_g\dot{A}(x;\theta,g,F)h^*}{\{A(x;\theta,g,F)\}^2} - \frac{2a^T\dot{A}(x;\theta,g,F)\,\mathrm{d}_gA(x;\theta,g,F)h^*}{\{A(x;\theta,g,F)\}^3}\right],\end{aligned}$$

where

$$\begin{aligned}&a^T\,\mathrm{d}_g\dot{A}(x;\theta,g,F)h^*\\ &\quad = \int(a^T\dot{f})\frac{\int fh^*\,\mathrm{d}x}{f_Y^2}\pi_2(\mathrm{d}F) + \int f\frac{\int(a^T\dot{f})h^*\,\mathrm{d}x}{f_Y^2}\pi_2(\mathrm{d}F) - 2\int f(a^T\dot{f}_Y)\frac{\int fh^*\,\mathrm{d}x}{f_Y^3}\pi_2(\mathrm{d}F).\end{aligned}$$

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