Well-posedness for the Navier-Stokes equations with datum in Sobolev-Fourier-Lorentz spaces

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Abstract: In this note, for $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we introduce and study Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$. In the family spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$, the critical invariant spaces for the Navier-Stokes equations correspond to the value $s = \frac{d}{p} - 1$. When the initial datum belongs to the critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d \geq 2, 1 \leq p < \infty$, and $1 \leq r < \infty$, we establish the existence of local mild solutions to the Cauchy problem for the Navier-Stokes equations in spaces $L^\infty([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d))$ with arbitrary initial value, and existence of global mild solutions in spaces $L^\infty([0,\infty); \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d))$ when the norm of the initial value in the Besov spaces $\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d)$ is small enough, where $\tilde{p}$ may take some suitable values.

§1. INTRODUCTION

We consider the Navier-Stokes equations (NSE) in d dimensions in special setting of a viscous, homogeneous, incompressible fluid which fills the entire space and is not submitted to external forces. Thus, the equations we consider are the system:

$$\begin{cases} \partial_t u = \Delta u - \nabla.(u \otimes u) - \nabla p, \\ \mathrm{div}(u) = 0, \\ u(0,x) = u_0, \end{cases}$$

which is a condensed writing for

$$\begin{cases} 1 \leq k \leq d, \quad \partial_t u_k = \Delta u_k - \sum_{l=1}^d \partial_l (u_l u_k) - \partial_k p, \\ \sum_{l=1}^d \partial_l u_l = 0, \\ 1 \leq k \leq d, \quad u_k(0,x) = u_{0k}. \end{cases}$$

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The unknown quantities are the velocity $u(t,x) = (u_1(t,x), \ldots, u_d(t,x))$ of the fluid element at time t and position x and the pressure $p(t,x)$.
A translation invariant Banach space of tempered distributions $\mathcal{E}$ is called a critical space for NSE if its norm is invariant under the action of the scaling $f(.) \longrightarrow \lambda f(\lambda.)$. One can take, for example, $\mathcal{E} = L^d(\mathbb{R}^d)$ or the smaller space $\mathcal{E} = \dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$. In fact, one has the chain of critical spaces given by the continuous imbedding

$$\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d) \hookrightarrow L^d(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d)_{(p<\infty)} \hookrightarrow BMO^{-1}(\mathbb{R}^d) \hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^d). \quad (1)$$

It is remarkable feature that the NSE are well-posed in the sense of Hadarmard (existence, uniqueness and continuous dependence on data) when the initial datum is divergence-free and belongs to the critical function spaces (except $\dot{B}^{-1}_{\infty,\infty}$) listed in (1) (see [4] for $\dot{H}^{\frac{d}{2}-1}(\mathbb{R}^d)$, $L^d(\mathbb{R}^d)$, and $\dot{B}^{\frac{d}{p}-1}_{p,\infty}(\mathbb{R}^d)$, see [23] for $BMO^{-1}(\mathbb{R}^d)$, and the recent ill-posedness result [3] for $\dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^d)$).
In the 1960s, mild solutions were first constructed by Kato and Fujita ([17], [18]) that are continuous in time and take values in the Sobolev space $H^s(\mathbb{R}^d)$, $(s \geq \frac{d}{2}-1)$, say $u \in C([0,T]; H^s(\mathbb{R}^d))$. In 1992, a modern treatment for mild solutions in $H^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$ was given by Chemin [8]. In 1995, using the simplified version of the bilinear operator, Cannone proved the existence for mild solutions in $\dot{H}^s(\mathbb{R}^d), (s \geq \frac{d}{2}-1)$, see [4]. Results on the existence of mild solutions with value in $L^p(\mathbb{R}^d), (p > d)$ were established in the papers of Fabes, Jones and Rivière [9] and of Giga [10]. Concerning the initial datum in the space L^∞, the existence of a mild solution was obtained by Cannone and Meyer in ([4], [7]). Moreover, in ([4], [7]), they also obtained theorems on the existence of mild solutions with value in Morrey-Campanato space $M_2^p(\mathbb{R}^d), (p > d)$ and Sobolev space $H_p^s(\mathbb{R}^d), (p < d, \frac{1}{p} - \frac{s}{d} < \frac{1}{d})$, and in general in the case of a so-called well-suited sapce $\mathcal{W}$ for NSE. The NSE in the Morrey-Campanato spaces were also treated by Kato [21] and Taylor [27].
In 1981, Weissler [29] gave the first existence result of mild solutions in the half space $L^3(\mathbb{R}^3_+)$. Then Giga and Miyakawa [11] generalized the result to $L^3(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^3$. Finally, in 1984, Kato [20] obtained, by means of a purely analytical tool (involving only Hölder and Young inequalities and without using any estimate of fractional powers of the Stokes operator), an existence theorem in the whole space $L^3(\mathbb{R}^3)$. In ([4], [5], [6]), Cannone showed how to simplify Kato's proof. The idea is to take advantage of the structure of the bilinear operator in its scalar form. In particular, the divergence ∇ and heat $e^{t\Delta}$ operators can be treated as a single convolution operator. In 1994, Kato and Ponce [22] showed that the NSE are well-posed when the initial datum belongs to homogeneous Sobolev spaces $\dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d), (d \leq p < \infty)$. Recently, the authors of this article have

considered NSE in mixed-norm Sobolev-Lorentz spaces, see [13]. In [15], we showed that NSE are well-posed when the initial datum belongs to Sobolev spaces $\dot{H}^s_p(\mathbb{R}^d)$ with non-positive-regular indexes $(p \geq d, \frac{d}{p} - 1 \leq s \leq 0)$. In [14], we showed that the bilinear operator

$$B(u,v)(t) = \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\mathrm{d}\tau \qquad (2)$$

is bicontinuous in $L^\infty([0,T];\dot{H}^s_p(\mathbb{R}^d))$ with super-critical and non-negative-regular indexes $(0 \leq s < d, p > 1, \text{ and } \frac{s}{d} < \frac{1}{p} < \frac{s+1}{d})$, and we established the inequality

$$\big\|B(u,v)\big\|_{L^\infty([0,T];\dot{H}^s_p)} \leq C_{s,p,d}T^{\frac{1}{2}(1+s-\frac{d}{p})}\big\|u\big\|_{L^\infty([0,T];\dot{H}^s_p)}\big\|v\big\|_{L^\infty([0,T];\dot{H}^s_p)}.$$

In this case existence and uniqueness theorems of local mild solutions can therefore be easily deduced. In [16] we prove that NSE are well-posed when the initial datum belongs to the Sobolev spaces $\dot{H}_p^{\frac{d}{p}-1}(\mathbb{R}^d)$ with $(1 < p \leq d)$. In this paper, for $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, we first recall the notion of the Fourier-Lebesgue spaces $\mathcal{L}^p(\mathbb{R}^d)$, introduced and investigated in [12]; then we introduce and study Sobolev-Fourier-Lebesgue spaces $\dot{H}^s_{\mathcal{L}^p}(\mathbb{R}^d)$, and Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$. After that we show that the Navier-Stokes equations are well-posed when the initial datum belongs to the critical Sobolev-Fourier-Lorentz spaces $\dot{H}_{\mathcal{L}^{p,r}}^{\frac{d}{p}-1}(\mathbb{R}^d)$ with $d \geq 2, 1 \leq p < \infty$, and $1 \leq r < \infty$. The spaces $\dot{H}_{\mathcal{L}^{p,r}}^{\frac{d}{p}-1}(\mathbb{R}^d)$ are more general than the spaces $\dot{H}_{\mathcal{L}^{p}}^{\frac{d}{p}-1}(\mathbb{R}^d)$. In particular, $\dot{H}_{\mathcal{L}^{p,r}}^{\frac{d}{p}-1}(\mathbb{R}^d) = \dot{H}_{\mathcal{L}^{p}}^{\frac{d}{p}-1}(\mathbb{R}^d)$ when $\frac{1}{p} + \frac{1}{r} = 1$.

In 1997, Le Jan and Sznitman [26] considered a very simple space convenient to the study of NSE, which is the space E of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ so that $\hat{f}(\xi)$ is a locally integrable function on $\mathbb{R}^d$ and $\sup_{\xi}|\xi|^{d-1}|\hat{f}(\xi)| < \infty$, with $\hat{}$ standing for the Fourier transform. This space may be defined as a Besov space based on the spaces PM of pseudomeasures (PM is the space of the image of the Fourier transforms of essentially bounded functions: $PM = \mathcal{F}L^\infty$). More precisely, $E = \dot{B}^{d-1,\infty}_{PM}(\mathbb{R}^d)$. They showed that the bilinear operator B is bicontinuous in $L^\infty([0,T];\dot{B}^{d-1,\infty}_{PM})$ for all $0 < T \leq \infty$. Therefore they can easily deduce the existence of global mild solutions in spaces $L^\infty([0,\infty);\dot{B}^{d-1,\infty}_{PM})$ when norm of the initial value in the spaces $\dot{B}^{d-1,\infty}_{PM}(\mathbb{R}^d)$ is small enough. From Definitions 1 and 2 in Section 2, we have

$$PM = \mathcal{L}^1, \dot{B}^{d-1,\infty}_{PM}(\mathbb{R}^d) = \dot{H}^{d-1}_{\mathcal{L}^1}(\mathbb{R}^d).$$

In 2011, Lei and Lin [25] showed that NSE are well-posed when the initial datum belongs to the spaces $\mathcal{X}^{-1}(\mathbb{R}^d)$, which is defined by

$$f \in \mathcal{X}^{-1}(\mathbb{R}^d) \text{ if and only if } \big\|(-\Delta)^{-\frac{1}{2}} f\big\|_{\mathcal{X}} < \infty, \text{where } \big\|f\big\|_{\mathcal{X}} = \big\|\hat{f}\big\|_{L^1}.$$

They established the existence of global mild solutions in the space $L^\infty([0,\infty);\mathcal{X}^{-1})$ when norm of the initial value in the spaces $\mathcal{X}^{-1}(\mathbb{R}^d)$ is small enough. From Definitions 1 and 2 in Section 2, we see that

$$\mathcal{X}^{-1}(\mathbb{R}^d) = \dot{H}^{-1}_{\mathcal{L}^\infty}(\mathbb{R}^d).$$

Thus, the spaces $\dot{B}^{d-1,\infty}_{PM}$ and $\mathcal{X}^{-1}$, studied in [26] and [25], are particular cases of the critical Sobolev-Fourier-Lebesgue spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^p}$ with $p=1$ and $p=\infty$, respectively. Note that estimates in the Lorentz spaces were also studied in [1], [19] (see also the refererences therein). Very recently, ill-poseness of NSE in critical Besov spaces $\dot{B}^{-1}_{\infty,q}$ was investigated in [28].

The paper is organized as follows. In Section 2 we introduce and investigate the Sobolev-Fourier-Lorentz spaces and some auxiliary lemmas. In Section 3 we present the main results of the paper. Due to some technical difficulties we will consider three cases $1 < p \le d, d \le q < \infty$, and $p = 1$ separately. In subsection 3.1 we treat the case $1 < p \le d$. In subsection 3.2 we consider the case $d \le q < \infty$. Finally, in subsection 3.3 we study the case $p = 1$. In the sequence, for a space of functions defined on $\mathbb{R}^d$, say $E(\mathbb{R}^d)$, we will abbreviate it as E. Throughout the paper, we sometimes use the notation $A \lesssim B$ as an equivalent to $A \le CB$ with a uniform constant C. The notation $A \simeq B$ means that $A \lesssim B$ and $B \lesssim A$

§2. SOBOLEV-FOURIER-LORENTZ SPACES

Definition 1. (Fourier-Lebesgue spaces). (See [12].)
For $1 \le p \le \infty$, the Fourier-Lebesgue spaces $\mathcal{L}^p(\mathbb{R}^d)$ is defined as the space $\mathcal{F}^{-1}(L^{p'}(\mathbb{R}^d)), (\frac{1}{p'} + \frac{1}{p} = 1)$, equipped with the norm

$$\big\|f\big\|_{\mathcal{L}^p(\mathbb{R}^d)} := \big\|\mathcal{F}(f)\big\|_{L^{p'}(\mathbb{R}^d)},$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse.

Definition 2. (Sobolev-Fourier-Lebesgue spaces).
For $s \in \mathbb{R}$, and $1 \le p \le \infty$, the Sobolev-Fourier-Lebesgue spaces $\dot{H}^s_{\mathcal{L}^p}(\mathbb{R}^d)$ is defined as the space $\dot{\Lambda}^{-s}\mathcal{L}^p(\mathbb{R}^d)$, equipped with the norm

$$\big\|u\big\|_{\dot{H}^s_{\mathcal{L}^p}} := \big\|\dot{\Lambda}^s u\big\|_{\mathcal{L}^p}.$$

where $\dot{\Lambda} = \sqrt{-\Delta}$ is the homogeneous Calderon pseudo-differential operator defined as

$$\widehat{\dot{\Lambda} g}(\xi) = |\xi|\hat{g}(\xi).$$

Definition 3. (Lorentz spaces). (See [2].)
For $1 \leq p, r \leq \infty$, the Lorentz space $L^{p,r}(\mathbb{R}^d)$ is defined as follows. A measurable function $f \in L^{p,r}(\mathbb{R}^d)$ if and only if
$\big\|f\big\|_{L^{p,r}}(\mathbb{R}^d) := \big(\int_0^\infty (t^{\frac{1}{p}} f^*(t))^r \frac{\mathrm{d}t}{t}\big)^{\frac{1}{r}} < \infty$ when $1 \leq r < \infty$,
$\big\|f\big\|_{L^{p,\infty}}(\mathbb{R}^d) := \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) < \infty$ when $r = \infty$,
where $f^*(t) = \inf\big\{\tau : \mathcal{M}^d(\{x : |f(x)| > \tau\}) \leq t\big\}$, with $\mathcal{M}^d$ being the Lebesgue measure in $\mathbb{R}^d$.

Definition 4. (Fourier-Lorentz spaces).
For $1 \leq p, r \leq \infty$, the Fourier-Lorentz spaces $\mathcal{L}^{p,r}(\mathbb{R}^d)$ is defined as the space $\mathcal{F}^{-1}(L^{p',r}(\mathbb{R}^d))$, $(\frac{1}{p'} + \frac{1}{p} = 1)$, equipped with the norm

$$\big\|f\big\|_{\mathcal{L}^{p,r}(\mathbb{R}^d)} := \big\|\mathcal{F}(f)\big\|_{L^{p',r}(\mathbb{R}^d)}.$$

Definition 5. (Sobolev-Fourier-Lorentz spaces).
For $s \in \mathbb{R}$ and $1 \leq r, p \leq \infty$, the Sobolev-Fourier-Lorentz spaces $\dot{H}^s_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ is defined as the space $\dot{\Lambda}^{-s}\mathcal{L}^{p,r}(\mathbb{R}^d)$, equipped with the norm

$$\big\|u\big\|_{\dot{H}^s_{\mathcal{L}^{p,r}}} := \big\|\dot{\Lambda}^s u\big\|_{\mathcal{L}^{p,r}}.$$

Theorem 1. (*Holder's inequality in Fourier-Lorentz spaces*).
Let $1 < r, q, \tilde{q} < \infty$ and $1 \leq h, \tilde{h}, \hat{h} \leq +\infty$ satisfy the relations

$$\frac{1}{r} = \frac{1}{q} + \frac{1}{\tilde{q}} \text{ and } \frac{1}{h} = \frac{1}{\tilde{h}} + \frac{1}{\hat{h}}.$$

Suppose that $u \in \mathcal{L}^{q,\tilde{h}}$ and $v \in \mathcal{L}^{\tilde{q},\hat{h}}$. Then $uv \in \mathcal{L}^{r,h}$ and we have the inequality

$$\big\|uv\big\|_{\mathcal{L}^{r,h}} \lesssim \big\|u\big\|_{\mathcal{L}^{q,\tilde{h}}}\big\|v\big\|_{\mathcal{L}^{\tilde{q},\hat{h}}}. \tag{3}$$

Proof. Let r', q', and $\tilde{q}'$ be such that

$$\frac{1}{r} + \frac{1}{r'} = 1, \frac{1}{q} + \frac{1}{q'} = 1, \text{ and } \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1.$$

It is easily checked that the following conditions are satisfied

$$1 < r', q', \tilde{q}' < +\infty \text{ and } \frac{1}{r'} + 1 = \frac{1}{q'} + \frac{1}{\tilde{q}'}.$$

We have
$$\|uv\|_{\mathcal{L}^{r,h}} = \|\widehat{uv}\|_{L^{r',h}} = \frac{1}{(2\pi)^{d/2}}\|\hat{u} * \hat{v}\|_{L^{r',h}}. \tag{4}$$
Applying Proposition 2.4 (c) in ([24], p. 20), we have
$$\|\hat{u} * \hat{v}\|_{L^{r',h}} \lesssim \|\hat{u}\|_{L^{q',\tilde{h}}}\|\hat{v}\|_{L^{\tilde{q}',\hat{h}}} = \|u\|_{\mathcal{L}^{q,\tilde{h}}}\|v\|_{\mathcal{L}^{\tilde{q},\hat{h}}}. \tag{5}$$
Now, the estimate (3) follows from the equality (4) and the inequality (5). ☐

Theorem 2. (*Young's inequality for convolution in Fourier-Lorentz spaces*). *Let $1 < r, q, \tilde{q} < \infty$, and $1 \le h, \tilde{h}, \hat{h} \le \infty$ satisfy the relations*
$$\frac{1}{r} + 1 = \frac{1}{q} + \frac{1}{\tilde{q}} \text{ and } \frac{1}{h} = \frac{1}{\tilde{h}} + \frac{1}{\hat{h}}.$$
Suppose that $u \in L^{q,\tilde{h}}$ and $v \in L^{\tilde{q},\hat{h}}$. Then $u * v \in L^{r,h}$ and the following inequality holds
$$\|u * v\|_{\mathcal{L}^{r,h}} \lesssim \|u\|_{\mathcal{L}^{q,\tilde{h}}}\|v\|_{\mathcal{L}^{\tilde{q},\hat{h}}}. \tag{6}$$

Proof. Let r', q', and $\tilde{q}'$ be such that
$$\frac{1}{r} + \frac{1}{r'} = 1, \frac{1}{q} + \frac{1}{q'} = 1, \text{ and } \frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1.$$
By definition
$$\|u * v\|_{\mathcal{L}^{r,h}} = \|\widehat{u * v}\|_{L^{r',h}} = (2\pi)^{d/2}\|\hat{u}\hat{v}\|_{L^{r',h}}. \tag{7}$$
We can check that the following conditions are satisfied
$$1 < r', q', \tilde{q}' < +\infty \text{ and } \frac{1}{r'} = \frac{1}{q'} + \frac{1}{\tilde{q}'}.$$
Applying Proposition 2.3 (c) in ([24], p. 19), we have
$$\|\hat{u}\hat{v}\|_{L^{r',h}} \lesssim \|\hat{u}\|_{L^{q',\tilde{h}}}\|\hat{v}\|_{L^{\tilde{q}',\hat{h}}} = \|u\|_{\mathcal{L}^{q,\tilde{h}}}\|v\|_{\mathcal{L}^{\tilde{q},\hat{h}}}. \tag{8}$$
Now, the estimate (6) follows from the equality (7) and the inequality (8). ☐

Theorem 3. (*Sobolev inequality for Sobolev-Fourier-Lorentz spaces*). *Let $1 < q \le \tilde{q} < \infty, s, \tilde{s} \in \mathbb{R}, s - \frac{d}{q} = \tilde{s} - \frac{d}{\tilde{q}}$, and $1 \le r \le \infty$. Then*
$$\|u\|_{\dot{H}^{\tilde{s}}_{\mathcal{L}^{\tilde{q},r}}} \lesssim \|u\|_{\dot{H}^{s}_{\mathcal{L}^{q,r}}}, \forall u \in \dot{H}^{s}_{\mathcal{L}^{q,r}}. \tag{9}$$

Proof. We have

$$\|u\|_{\dot H^{\tilde s}_{\mathcal{L}^{\tilde q,r}}} = \|\dot\Lambda^{\tilde s - s}\dot\Lambda^s u\|_{\mathcal{L}^{\tilde q,r}} = \||\xi|^{\tilde s - s}\widehat{\dot\Lambda^s u}(\xi)\|_{L^{\tilde q',r}}, \qquad (10)$$

where

$$\frac{1}{\tilde q} + \frac{1}{\tilde q'} = 1.$$

Note that

$$|\xi|^{-r} \in L^{\frac{d}{r},\infty}(\mathbb{R}^d) \text{ for all } r \text{ satisfying } 0 < r \le d.$$

Applying Proposition 2.3 (c) in ([24], p. 19), we have

$$\||\xi|^{\tilde s - s}\widehat{\dot\Lambda^s u}(\xi)\|_{L^{\tilde q',r}} \lesssim \||\xi|^{\tilde s - s}\|_{L^{\frac{d}{s-\tilde s},\infty}}.\|\widehat{\dot\Lambda^s u}(\xi)\|_{L^{q',r}} \simeq \|u\|_{\dot H^s_{\mathcal{L}^{q,r}}}. \qquad (11)$$

The estimate (9) follows from the equality (10) and the inequality (11). □

Lemma 1. *Let $s \in \mathbb{R}, 1 \le p \le \infty,$ and $1 \le r \le \tilde r \le \infty$.*
(a) *We have the following imbedding maps*

$$\mathcal{L}^{p,1} \hookrightarrow \mathcal{L}^{p,r} \hookrightarrow \mathcal{L}^{p,\tilde r} \hookrightarrow \mathcal{L}^{p,\infty},$$
$$\dot H^s_{\mathcal{L}^{p,1}} \hookrightarrow \dot H^s_{\mathcal{L}^{p,r}} \hookrightarrow \dot H^s_{\mathcal{L}^{p,\tilde r}} \hookrightarrow \dot H^s_{\mathcal{L}^{p,\infty}}.$$

(b) $\dot H^s_{\mathcal{L}^p} = \dot H^s_{\mathcal{L}^{p,p'}}$ *(equality of the norm), where* $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. It is easily deduced from the properties of the standard Lorentz spaces. □

Lemma 2. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. We have*
(a) *If $1 < q \le 2$ then $\dot H^s_q \hookrightarrow \dot H^s_{\mathcal{L}^q}$.*
(b) *If $2 \le q < \infty$ then $\dot H^s_{\mathcal{L}^q} \hookrightarrow \dot H^s_q$.*

Proof. It is deduced from Theorem 1.2.1 ([2], p. 6). □

Lemma 3. *Assume that $1 \le r, p \le \infty$ and $k \in \mathbb{N}$, then the two quantities*

$$\|u\|_{\dot H^k_{\mathcal{L}^{p,r}}} \text{ and } \sum_{|\alpha|=k} \|\partial^\alpha u\|_{\mathcal{L}^{p,r}}$$

are equivalent.

Proof. First, we prove that

$$\sum_{|\alpha|=k} \|\partial^\alpha u\|_{\mathcal{L}^{p,r}} \lesssim \|u\|_{\dot H^k_{\mathcal{L}^{p,r}}}.$$

We have

$$\sum_{|\alpha|=k} \left\|\partial^\alpha u\right\|_{\mathcal{L}^{p,r}} = \sum_{|\alpha|=k} \left\|i^k \xi^\alpha \hat{u}(\xi)\right\|_{L^{p',r}} = \sum_{|\alpha|=k} \left\|\frac{\xi^\alpha}{|\xi|^k} |\xi|^k \hat{u}(\xi)\right\|_{L^{p',r}}$$
$$\leq \sum_{|\alpha|=k} \left\||\xi|^k \hat{u}(\xi)\right\|_{L^{p',r}} \lesssim \left\|\widehat{\dot{\Lambda}^k u}(\xi)\right\|_{L^{p',r}} = \left\|u\right\|_{\dot{H}^k_{\mathcal{L}^{p,r}}}.$$

Next, we prove that

$$\left\|u\right\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} \lesssim \sum_{|\alpha|=k} \left\|\partial^\alpha u\right\|_{\mathcal{L}^{p,r}}.$$

It is easy to see that for all $\xi \in \mathbb{R}^d$, we have

$$|\xi|^k \leq d^{\frac{k}{2}} \sum_{|\alpha|=k} |\xi^\alpha|.$$

This gives the desired result

$$\left\|u\right\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} = \left\||\xi|^k \hat{u}(\xi)\right\|_{L^{p',r}} \leq d^{\frac{k}{2}} \left\| \sum_{|\alpha|=k} |\xi^\alpha| \hat{u}(\xi)\right\|_{L^{p',r}}$$
$$\leq d^{\frac{k}{2}} \sum_{|\alpha|=k} \left\|\xi^\alpha \hat{u}(\xi)\right\|_{L^{p',r}} = d^{\frac{k}{2}} \sum_{|\alpha|=k} \left\|\partial^\alpha u\right\|_{\mathcal{L}^{p,r}}. \quad \square$$

Lemma 4. *Let $k \in \mathbb{N}, p \in \mathbb{R}$, and $r \in \mathbb{R}$ be such that*

$$0 \leq k \leq d-1, \frac{k}{d} < \frac{1}{p} < \frac{1}{2} + \frac{k}{2d}, \text{ and } 1 \leq r \leq \infty.$$

Then the following inequality holds

$$\left\|uv\right\|_{\dot{H}^k_{\mathcal{L}^{q,r}}} \lesssim \left\|u\right\|_{\dot{H}^k_{\mathcal{L}^{p,r}}} \left\|v\right\|_{\dot{H}^k_{\mathcal{L}^{p,r}}}, \ \forall u, v \in \dot{H}^k_{\mathcal{L}^{p,r}},$$

where

$$\frac{1}{q} = \frac{2}{p} - \frac{k}{d}.$$

Proof. First, we estimate $\left\|\partial^\alpha (uv)\right\|_{\mathcal{L}^{q,r}}$, where

$$\alpha = (\alpha_1, \alpha_2, ..., \alpha_d) \in \mathbb{N}^d, \ |\alpha| = \sum_{i=1}^d \alpha_i = k.$$

By the general Leibniz rule, we have

$$\partial^\alpha (uv) = \sum_{\gamma+\beta=\alpha} \binom{\alpha}{\gamma} (\partial^\gamma u)(\partial^\beta v).$$

Set

$$\frac{1}{q_1} = \frac{1}{p} - \frac{k-|\gamma|}{d}, \frac{1}{q_2} = \frac{1}{p} - \frac{k-|\beta|}{d}.$$

We have

$$\frac{1}{q_1} + \frac{1}{q_2} = \frac{2}{p} - \frac{2k}{d} + \frac{|\gamma|+|\beta|}{d} = \frac{2}{p} - \frac{k}{d} = \frac{1}{q}.$$

Therefore applying Theorems 1, 3, and Lemma 1 (a) in order to obtain

$$\begin{aligned}\big\|(\partial^\gamma u)(\partial^\beta v)\big\|_{\mathcal{L}^{q,r}} &\lesssim \big\|\partial^\gamma u\big\|_{\mathcal{L}^{q_1,r}}\big\|\partial^\beta v\big\|_{\mathcal{L}^{q_2,\infty}} \lesssim \big\|\partial^\gamma u\big\|_{\dot{H}^{k-|\gamma|}_{\mathcal{L}^{p,r}}}\big\|\partial^\beta v\big\|_{\dot{H}^{k-|\beta|}_{\mathcal{L}^{p,\infty}}} \\ &\lesssim \big\|\partial^\gamma u\big\|_{\dot{H}^{k-|\gamma|}_{\mathcal{L}^{p,r}}}\big\|\partial^\beta v\big\|_{\dot{H}^{k-|\beta|}_{\mathcal{L}^{p,r}}} \lesssim \big\|u\big\|_{\dot{H}^{k}_{\mathcal{L}^{p,r}}}\big\|v\big\|_{\dot{H}^{k}_{\mathcal{L}^{p,r}}}.\end{aligned}$$

Thus, for all $\alpha \in \mathbb{N}^d$ with $|\alpha| = k$, we have

$$\big\|\partial^\alpha(uv)\big\|_{\mathcal{L}^{q,r}} \lesssim \big\|u\big\|_{\dot{H}^{k}_{\mathcal{L}^{p,r}}}\big\|v\big\|_{\dot{H}^{k}_{\mathcal{L}^{p,r}}}.$$

Applying Lemma 3, we have

$$\big\|uv\big\|_{\dot{H}^{k}_{\mathcal{L}^{q,r}}} \lesssim \big\|u\big\|_{\dot{H}^{k}_{\mathcal{L}^{p,r}}}\big\|v\big\|_{\dot{H}^{k}_{\mathcal{L}^{p,r}}},\ \forall u, v \in \dot{H}^{k}_{\mathcal{L}^{p,r}}. \quad \square$$

Lemma 5. *Assume that* $1 \le p, r \le \infty$ *and* $s \in \mathbb{R}$. *If* $u_0 \in \dot{H}^{s}_{\mathcal{L}^{p,r}}$ *then* $e^{t\Delta}u_0 \in L^\infty([0,\infty); \dot{H}^{s}_{\mathcal{L}^{p,r}})$ *and*

$$\big\|e^{t\Delta}u_0\big\|_{L^\infty([0,\infty);\dot{H}^{s}_{\mathcal{L}^{p,r}})} \le \big\|u_0\big\|_{\dot{H}^{s}_{\mathcal{L}^{p,r}}}.$$

Proof. For $t \ge 0$, we have

$$\begin{aligned}\big\|e^{t\Delta}u_0\big\|_{\dot{H}^{s}_{\mathcal{L}^{p,r}}} &= \big\|e^{t\Delta}\dot{\Lambda}^s u_0\big\|_{\mathcal{L}^{p,r}} = \big\|e^{-t|\xi|^2}|\xi|^s\hat{u}_0\big\|_{L^{p',r}} \le \\ \big\||\xi|^s\hat{u}_0\big\|_{L^{p',r}} &= \big\|\widehat{\dot{\Lambda}^s u_0}(\xi)\big\|_{L^{p',r}} = \big\|\dot{\Lambda}^s u_0(\xi)\big\|_{\mathcal{L}^{p,r}} = \big\|u_0\big\|_{\dot{H}^{s}_{\mathcal{L}^{p,r}}}. \quad \square\end{aligned}$$

Finally, let us recall the following result on solutions of a quadratic equation in Banach spaces (Theorem 22.4, [24], p. 227).

Theorem 4. *Let* E *be a Banach space, and* $B : E \times E \to E$ *be a continuous bilinear form such that there exists* $\eta > 0$ *so that*

$$\|B(x,y)\| \le \eta\|x\|\|y\|,$$

for all x *and* y *in* E. *Then for any fixed* $y \in E$ *such that* $\|y\| \le \frac{1}{4\eta}$, *the equation* $x = y - B(x,x)$ *has a unique solution* $\overline{x} \in E$ *satisfying* $\|\overline{x}\| \le \frac{1}{2\eta}$.

§3. MAIN RESULTS

For $T>0$, we say that u is a mild solution of NSE on $[0,T]$ corresponding to a divergence-free initial data u_0 when u satisfies the integral equation

$$u = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes u(\tau)\big)\mathrm{d}\tau.$$

Above we have used the following notation: For a tensor $F=(F_{ij})$ we define the vector $\nabla.F$ by $(\nabla.F)_i = \sum_{i=1}^d \partial_j F_{ij}$ and for vectors u and v, we define their tensor product $(u\otimes v)_{ij} = u_i v_j$. The operator $\mathbb{P}$ is the Leray projection onto the divergence-free fields

$$(\mathbb{P}f)_j = f_j + \sum_{1\leq k\leq d} R_j R_k f_k,$$

where R_j is the Riesz transforms defined as

$$R_j = \frac{\partial_j}{\dot{\Lambda}}, \quad \text{i.e.} \quad \widehat{R_j g}(\xi) = \frac{i\xi_j}{|\xi|}\hat{g}(\xi).$$

The heat kernel $e^{t\Delta}$ is defined as

$$e^{t\Delta}u(x) = ((4\pi t)^{-d/2}e^{-|.|^2/4t} * u)(x).$$

If X is a normed space and $u=(u_1,u_2,...,u_d), u_i\in X, 1\leq i\leq d$, then we write

$$u\in X, \|u\|_X = \Big(\sum_{i=1}^d \big\|u_i\big\|_X^2\Big)^{1/2}.$$

In this main section we investigate mild solutions to NSE when the initial datum belongs to critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $1\leq p<\infty$ and $1\leq r<\infty$. We consider three cases $1<p\leq d, d\leq q<\infty$, and $p=1$ separately.

3.1. Solutions to the Navier-Stokes equations with the initial value in the critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $1<p\leq d$ and $1\leq r<\infty$.

We define an auxiliary space $\mathcal{K}^{\tilde{p}}_{p,r,T}$ which is made up by the functions $u(t,x)$ such that

$$\big\|u\big\|_{\mathcal{K}^{\tilde{p}}_{p,r,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}}\Big\|u(t,x)\Big\|_{\dot{H}^{\frac{d}{\tilde{p}}-1}_{\mathcal{L}^{\tilde{p},r}}} < \infty,$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\| u(t,x) \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},r}}} = 0, \tag{12}$$

with

$$1 < p \leq \tilde{p} < \infty, \frac{1}{p} - \frac{1}{d} < \frac{1}{\tilde{p}}, 1 \leq r \leq \infty, T > 0,$$

and

$$\alpha = \alpha(p, \tilde{p}) = d\Big(\frac{1}{p} - \frac{1}{\tilde{p}}\Big).$$

In the case $\tilde{p} = p$, it is also convenient to define the space $\mathcal{K}^{p}_{p,r,T}$ as the natural space $L^{\infty}\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big)$ with the additional condition that its elements $u(t,x)$ satisfy

$$\lim_{t\to 0} \Big\| u(t,x) \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}} = 0. \tag{13}$$

Lemma 6. *Let $1 \leq r \leq \tilde{r} \leq \infty$. Then we have the following imbedding*

$$\mathcal{K}^{\tilde{p}}_{p,1,T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,r,T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,\tilde{r},T} \hookrightarrow \mathcal{K}^{\tilde{p}}_{p,\infty,T}.$$

Proof. It is easily deduced from Lemma 1 (a) and the definition of $\mathcal{K}^{\tilde{p}}_{p,r,T}$. □

Lemma 7. *Suppose that $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $1 < p \leq d$ and $1 \leq r < \infty$, then $e^{t\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{p,1,\infty}$ with $\frac{1}{p} - \frac{1}{d} < \frac{1}{\tilde{p}} < \frac{1}{p}$.*

Proof. Before proving this lemma, we need to prove the following lemma.

Lemma 8. *Suppose that $u_0 \in L^{q,r}(\mathbb{R}^d)$ with $1 \leq q \leq \infty$ and $1 \leq r < \infty$. Then $\lim_{n\to\infty} \big\| 1_{B_n^c} u_0 \big\|_{L^{q,r}} = 0$, where $n \in \mathbb{N}, B_n = \{x \in \mathbb{R}^d : |x| < n\}, B_n^c = \mathbb{R}^d \backslash B_n$, and $1_{B_n^c}$ is the indicator function of the set B_n^c on $\mathbb{R}^d$: $1_{B_n^c}(x) = 1$ for $x \in B_n^c$ and $1_{B_n^c}(x) = 0$ otherwise.*

Proof. With $\delta > 0$ being fixed, we have

$$\big\{x : |1_{B_n^c} u_0(x)| > \delta\big\} \supseteq \big\{x : |1_{B_{n+1}^c} u_0(x)| > \delta\big\}, \tag{14}$$

and

$$\bigcap_{n=0}^{\infty} \{x : |1_{B_n^c} u_0(x)| > \delta\} = \emptyset. \tag{15}$$

Note that

$$\mathcal{M}^d\big(\{x : |1_{B_0^c} u_0(x)| > \delta\}\big) = \mathcal{M}^d\big(\{x : |u_0(x)| > \delta\}\big).$$

We prove that

$$\mathcal{M}^d\big(\{x : |u_0(x)| > \delta\}\big) < \infty, \tag{16}$$

assuming on the contrary

$$\mathcal{M}^d\big(\{x : |u_0(x)| > \delta\}\big) = \infty.$$

Set

$$u_0^*(t) = \inf\big\{\tau : \mathcal{M}^d\big(\{x : |u_0(x)| > \tau\}\big) \le t\big\}.$$

We have $u_0^*(t) \ge \delta$ for all $t > 0$, from the definition of the Lorentz space, we get

$$\big\|u_0\big\|_{L^{q,r}} = \Big(\int_0^\infty (t^{\frac{1}{q}} u_0^*(t))^r \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} \ge \Big(\int_0^\infty (t^{\frac{1}{q}} \delta)^r \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} = \delta\Big(\int_0^\infty t^{\frac{r}{q}-1}\mathrm{d}t\Big)^{\frac{1}{r}} = \infty,$$

a contradiction.
From (14), (15), and (16), we have

$$\lim_{n\to\infty} \mathcal{M}^d\big(\{x : |1_{B_n^c} u_0(x)| > \delta\}\big) = 0. \tag{17}$$

Set

$$u_n^*(t) = \inf\big\{\tau : \mathcal{M}^d\big(\{x : |1_{B_n^c} u_0(x)| > \tau\}\big) \le t\big\}.$$

We have

$$u_n^*(t) \ge u_{n+1}^*(t). \tag{18}$$

Fixed $t > 0$. For any $\epsilon > 0$, from (17) it follows that there exist $n_0 = n_0(t, \epsilon)$ is large enough such that

$$\mathcal{M}^d\big(\{x : |1_{B_n^c} u_0(x)| > \epsilon\}\big) \le t, \forall n \ge n_0.$$

From this we deduce that

$$u_n^*(t) \le \epsilon, \forall n \ge n_0,$$

therefore

$$\lim_{n\to\infty} u_n^*(t) = 0. \tag{19}$$

From (18) and (19), we apply Lebesgue's monotone convergence theorem to get

$$\lim_{n\to\infty} \big\|1_{B_n^c} u_0\big\|_{L^{q,r}} = \lim_{n\to\infty} \Big(\int_0^\infty (t^{\frac{1}{q}} u_n^*(t))^r \frac{\mathrm{d}t}{t}\Big)^{\frac{1}{r}} = 0. \quad \square$$

Now we return to prove Lemma 7. We prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}} \Big\|e^{t\Delta} u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} \lesssim \Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}. \tag{20}$$

Let p' and $\tilde{p}'$ be such that

$$\frac{1}{p} + \frac{1}{p'} = 1 \text{ and } \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1.$$

We have

$$\Big\| e^{t\Delta} u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} = \big\| e^{-t|\xi|^2} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{\tilde{p}',1}_{\xi}}. \tag{21}$$

Applying Holder's inequality in the Lorentz spaces (see Proposition 2.3 (c) in [24], p. 19), we have

$$\begin{aligned} &\big\| e^{-t|\xi|^2} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{\tilde{p}',1}_{\xi}} \lesssim \big\| e^{-t|\xi|^2} \big\|_{L_{\xi}^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',\infty}} = \\ &t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{\tilde{p}})} \big\| e^{-|\xi|^2} \big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',\infty}} \lesssim t^{-\frac{\alpha}{2}} \big\| |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',r}} \\ &= t^{-\frac{\alpha}{2}} \Big\| u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}. \end{aligned} \tag{22}$$

The estimate (20) follows from the equality (21) and the estimate (22).
We claim now that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\| e^{t\Delta} u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} = 0. \tag{23}$$

From the equality (21), we have

$$\begin{aligned} t^{\frac{\alpha}{2}} \Big\| e^{t\Delta} u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} &\le t^{\frac{\alpha}{2}} \big\| e^{-t|\xi|^2} 1_{B_n^c} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{\tilde{p}',1}_{\xi}} \\ &+ t^{\frac{\alpha}{2}} \big\| e^{-t|\xi|^2} 1_{B_n} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{\tilde{p}',1}_{\xi}}. \end{aligned}$$

For any $\epsilon > 0$. Applying Holder's inequality in the Lorentz spaces and using Lemma 8, we have

$$\begin{aligned} &t^{\frac{\alpha}{2}} \big\| e^{-t|\xi|^2} 1_{B_n^c} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{\tilde{p}',1}_{\xi}} \le C t^{\frac{\alpha}{2}} \big\| e^{-t|\xi|^2} \big\|_{L_{\xi}^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| 1_{B_n^c} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',\infty}} = \\ &C \big\| e^{-|\xi|^2} \big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| 1_{B_n^c} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',\infty}} \le C' \big\| 1_{B_n^c} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',r}} < \frac{\epsilon}{2} \end{aligned} \tag{24}$$

for large enough n. Fixed one of such n and applying Holder's inequality in the Lorentz spaces, we have

$$\begin{aligned} &t^{\frac{\alpha}{2}} \big\| e^{-t|\xi|^2} 1_{B_n} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{\tilde{p}',1}_{\xi}} \le C t^{\frac{\alpha}{2}} \big\| 1_{B_n} e^{-t|\xi|^2} \big\|_{L_{\xi}^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',\infty}} \\ &\le C t^{\frac{\alpha}{2}} \big\| 1_{B_n} \big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',\infty}} \le C''(n) t^{\frac{\alpha}{2}} \big\| |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',r}} \\ &= C''(n) t^{\frac{\alpha}{2}} \Big\| u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}} < \frac{\epsilon}{2} \end{aligned} \tag{25}$$

for small enough $t = t(n) > 0$. From estimates (24) and (25), we have,

$$t^{\frac{\alpha}{2}}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{\tilde{p},1}}} \leq C'\big\|1_{B_n^c}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi)\big\|_{L^{p',r}} + C''(n)t^{\frac{\alpha}{2}}\Big\|u_0\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}} < \epsilon. \quad \square$$

In the following lemmas a particular attention will be devoted to the study of the bilinear operator $B(u,v)(t)$ defined by (2).

In the following lemmas, denote by $[x]$ the integer part of x and by $\{x\}$ the fraction part of x.

Lemma 9. *Let $1 < p \leq d$. Then for all $\tilde{p}$ be such that*

$$\frac{1}{2p} + \frac{[\frac{d}{p}]-1}{2d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}, \tag{26}$$

the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T} \times \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$ into $\mathcal{K}^{p}_{p,1,T}$ and the following inequality holds

$$\big\|B(u,v)\big\|_{\mathcal{K}^p_{p,1,T}} \leq C\Big\|u\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}\Big\|v\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}, \tag{27}$$

where C is a positive constant and independent of T.

Proof. We have

$$\begin{aligned}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} &\leq \int_0^t \Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.(u(\tau)\otimes v(\tau))\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}}\mathrm{d}\tau \\ &= \int_0^t \Big\|\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla.(u(\tau)\otimes v(\tau))\Big\|_{\mathcal{L}^{p,1}}\mathrm{d}\tau. \end{aligned}\tag{28}$$

Note that

$$\begin{aligned}&\Big(\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\Big)^{\wedge}_{j}(\xi) \\ &= \Big(\dot{\Lambda}^{\{\frac{d}{p}\}}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\Big)^{\wedge}_{j}(\xi) \\ &= |\xi|^{\{\frac{d}{p}\}}e^{-(t-\tau)|\xi|^2}\sum_{l,k=1}^{d}\Big(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l)\Big(\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u_l(\tau)v_k(\tau)\big)\Big)^{\wedge}(\xi).\end{aligned}$$

Thus

$$\begin{aligned}&\Big(\dot{\Lambda}^{\frac{d}{p}-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\Big)_{j} \\ &= \frac{1}{(t-\tau)^{\frac{\{\frac{d}{p}\}+d+1}{2}}}\sum_{l,k=1}^{d}K_{l,k,j}\Big(\frac{.}{\sqrt{t-\tau}}\Big) * \Big(\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u_l(\tau)v_k(\tau)\big)\Big),\end{aligned}\tag{29}$$

where

$$\widehat{K_{l,k,j}}(\xi) = \frac{1}{(2\pi)^{d/2}} |\xi|^{\{\frac{d}{p}\}} e^{-|\xi|^2} \Big(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}\Big)(i\xi_l). \tag{30}$$

Setting the tensor $K(x) = \{K_{l,k,j}(x)\}$, we can rewrite the equality (29) in the tensor form

$$\dot{\Lambda}^{\frac{d}{p}-1} e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u(\tau) \otimes v(\tau)\big)$$
$$= \frac{1}{(t-\tau)^{\frac{\{\frac{d}{p}\}+d+1}{2}}} K\Big(\frac{.}{\sqrt{t-\tau}}\Big) * \Big(\dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau) \otimes v(\tau)\big)\Big).$$

Applying Theorem 2 for convolution in the Fourier-Lorentz spaces, we have

$$\Big\| \dot{\Lambda}^{\frac{d}{p}-1} e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u(\tau) \otimes v(\tau)\big)\Big\|_{\mathcal{L}^{p,1}} \lesssim$$
$$\frac{1}{(t-\tau)^{\frac{\{\frac{d}{p}\}+d+1}{2}}} \Big\| K\Big(\frac{.}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}} \big\| \dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau) \otimes v(\tau)\big)\big\|_{\mathcal{L}^{q,\infty}}, \tag{31}$$

where

$$\frac{1}{q} = \frac{2}{\tilde{p}} - \frac{[\frac{d}{p}]-1}{d} \text{ and } \frac{1}{r} = 1 + \frac{1}{p} - \frac{2}{\tilde{p}} + \frac{[\frac{d}{p}]-1}{d}. \tag{32}$$

Note that from the inequality (26), we can check that r and q satisfy the relations

$$1 < r, q < \infty, \frac{1}{p} + 1 = \frac{1}{q} + \frac{1}{r}.$$

Applying Lemma 4, we have

$$\big\| \dot{\Lambda}^{[\frac{d}{p}]-1}\big(u(\tau) \otimes v(\tau)\big)\big\|_{\mathcal{L}^{q,\infty}} \lesssim \big\| u(\tau)\big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} \big\| v(\tau)\big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}. \tag{33}$$

From the equalities (30) and (32), we obtain

$$\Big\| K\Big(\frac{.}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}} = (t-\tau)^{\frac{d}{2}} \Big\| \hat{K}\big(\sqrt{t-\tau}\ (.)\big)\Big\|_{L^{r',1}} =$$
$$(t-\tau)^{\frac{d}{2}-\frac{d}{2.r'}} \big\| \hat{K}\big\|_{L^{r',1}} = (t-\tau)^{\frac{d}{2.r}} \big\| \hat{K}\big\|_{L^{r',1}} \simeq (t-\tau)^{\frac{d}{2}\big(1+\frac{1}{p}-\frac{2}{\tilde{p}}+\frac{[\frac{d}{p}]-1}{d}\big)}. \tag{34}$$

From the estimates (31), (33), and (34), we deduce that

$$\Big\| \dot{\Lambda}^{\frac{d}{p}-1} e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u(\tau) \otimes v(\tau)\big)\Big\|_{\mathcal{L}^{p,1}} \lesssim (t-\tau)^{[\frac{d}{p}]-\frac{d}{\tilde{p}}-1} \Big\| u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} \Big\| v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}$$
$$= (t-\tau)^{\alpha-1} \Big\| u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} \Big\| v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}},$$

where

$$\alpha = \alpha\Big(\frac{d}{[\frac{d}{p}]}, \tilde{p}\Big) = \Big[\frac{d}{p}\Big] - \frac{d}{\tilde{p}}\ ,$$

this gives the desired result

$$\Big\|B(u,v)(t)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} \lesssim \int_0^t (t-\tau)^{\alpha-1}\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\mathrm{d}\tau$$

$$\lesssim \int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\mathrm{d}\tau$$

$$= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\mathrm{d}\tau$$

$$\simeq \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}. \tag{35}$$

Let us now check the validity of the condition (13) for the bilinear term $B(u,v)(t)$. Indeed, from (35)

$$\lim_{t\to 0}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|u(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|v(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} = 0.$$

The estimate (27) is deduced from the inequality (35). □

Lemma 10. *Let $1 < p \leq d$. Then for all $\tilde{p}$ be such that*

$$\frac{[\frac{d}{p}]-1}{d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\}, \tag{36}$$

the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T} \times \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$ into $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}$ and the following inequality holds

$$\Big\|B(u,v)\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}} \leq C\Big\|u\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}\Big\|v\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}, \tag{37}$$

where C is a positive constant and independent of T.

Proof. First, arguing as in Lemma 9, we derive

$$\dot\Lambda^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)$$
$$=\frac{1}{(t-\tau)^{\frac{d+1}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)*\Big(\dot\Lambda^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\Big),$$

where

$$\widehat{K_{l,k,j}}(\xi)=\frac{1}{(2\pi)^{d/2}}e^{-|\xi|^2}\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l). \tag{38}$$

Applying Theorem 2 for the convolution in the Fourier-Lorentz spaces, we have

$$\Big\|\dot\Lambda^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{\tilde p,1}}$$
$$\lesssim\frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}\big\|\dot\Lambda^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{q,\infty}}, \tag{39}$$

where

$$\frac{1}{q}=\frac{2}{\tilde p}-\frac{[\frac{d}{p}]-1}{d}\ \text{and}\ \frac{1}{r}=1-\frac{1}{\tilde p}+\frac{[\frac{d}{p}]-1}{d}. \tag{40}$$

Note that from the inequality (36), we can check that r and q satisfy the relations

$$1<r,q<\infty,\frac{1}{p}+1=\frac{1}{q}+\frac{1}{r}.$$

Applying Lemma 4, we have

$$\big\|\dot\Lambda^{[\frac{d}{p}]-1}\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{q,\infty}}\lesssim\Big\|u(\tau)\Big\|_{\dot H^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde p,\infty}}}\Big\|v(\tau)\Big\|_{\dot H^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde p,\infty}}}. \tag{41}$$

From the equalities (38) and (40), we obtain

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}=(t-\tau)^{\frac{d}{2.r}}\big\|\hat K\big\|_{L^{r',1}}\simeq(t-\tau)^{\frac{d}{2}\big(1-\frac{1}{\tilde p}+\frac{[\frac{d}{p}]-1}{d}\big)}. \tag{42}$$

From the estimates (39), (41), and (42), we deduce that

$$\Big\|\dot\Lambda^{[\frac{d}{p}]-1}e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{\tilde p,1}}$$
$$\lesssim(t-\tau)^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde p})-1}\Big\|u(\tau)\Big\|_{\dot H^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde p,\infty}}}\Big\|v(\tau)\Big\|_{\dot H^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde p,\infty}}}$$
$$=(t-\tau)^{\frac{\alpha}{2}-1}\Big\|u(\tau)\Big\|_{\dot H^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde p,\infty}}}\Big\|v(\tau)\Big\|_{\dot H^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde p,\infty}}},$$

where

$$\alpha = \alpha\Big(\frac{d}{[\frac{d}{p}]}, \tilde{p}\Big) = \Big[\frac{d}{p}\Big] - \frac{d}{\tilde{p}} \ ,$$

this gives the desired result

$$\begin{aligned}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},1}}} &\lesssim \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\Big\|u(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\Big\|v(\tau)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\mathrm{d}\tau \\ &\leq \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\mathrm{d}\tau \\ &= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\mathrm{d}\tau \\ &\simeq t^{-\frac{\alpha}{2}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}. \qquad (43)\end{aligned}$$

Now we check the validity of condition (12) for the bilinear term $B(u,v)(t)$. From (43) we infer that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},1}}} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|u(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|v(t)\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} = 0.$$

Finally, the estimate (37) can be deduced from the inequality (43). □

Theorem 5. *Let $1 < p \leq d$ and $1 \leq r < \infty$. Then for all $\tilde{p}$ be such that*

$$\frac{1}{2p} + \frac{[\frac{d}{p}]-1}{2d} < \frac{1}{\tilde{p}} < \min\Big\{\frac{[\frac{d}{p}]}{d}, \frac{1}{2} + \frac{[\frac{d}{p}]-1}{2d}\Big\},$$

there exists a positive constant $\delta_{p,\tilde{p},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $\mathrm{div}(u_0) = 0$ satisfying

$$\sup_{0<t<T} t^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde{p}})}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} \leq \delta_{p,\tilde{p},d}, \qquad (44)$$

NSE has a unique mild solution $u \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T} \cap L^\infty\big([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big)$.

In particular, the inequality (44) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{p,\tilde{p},d}$ such that we can take $T = \infty$ whenever $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \leq \sigma_{p,\tilde{p},d}$.

Proof. From Lemmas 6 and 10, the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T} \times \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$ into $\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}$ and we have the inequality

$$\Big\|B(u,v)\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}} \leq \Big\|B(u,v)\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}} \leq C_{p,\tilde{p},d}\Big\|u\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}}\Big\|v\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}},$$

where $C_{p,\tilde{p},d}$ is positive constant independent of T. From Theorem 4 and the above inequality, we deduce that for any $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$ such that

$$\Big\|e^{t\Delta}u_0\Big\|_{\mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}} = \sup_{0<t<T} t^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde{p}})}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}} \leq \frac{1}{4C_{p,\tilde{p},d}},$$

the Navier-Stokes equations has a solution u on the interval $(0,T)$ so that

$$u \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}. \tag{45}$$

From Lemmas 6 and 9, and (45), we have

$$B(u,u) \in \mathcal{K}^{p}_{p,1,T} \subseteq \mathcal{K}^{p}_{p,r,T} \subseteq L^\infty\big([0,T];\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big).$$

From Lemma 5, we also have $e^{t\Delta}u_0 \in L^\infty\big([0,T];\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big)$. Therefore

$$u = e^{t\Delta}u_0 - B(u,u) \in L^\infty\big([0,T];\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}\big).$$

For all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$, applying Theorem 3, we deduce that

$$u_0 \in \dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{d/[\frac{d}{p}],r}}. \tag{46}$$

From (46), applying Lemma 7, we get $e^{t\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},\infty,T}$. From the definition of $\mathcal{K}^{\tilde{p}}_{p,r,T}$, we deduce that the left-hand side of the inequality (44) converges to 0 when T tends to 0. Therefore the inequality (44) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$ when $T(u_0)$ is small enough. Applying Lemmas 7 and 10, we conclude that $u \in \mathcal{K}^{\tilde{p}}_{\frac{d}{[\frac{d}{p}]},1,T}$.
Next, applying Theorem 5.4 ([24], p. 45), we deduce that the two quantities $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}}$ and $\sup_{0<t<\infty} t^{\frac{1}{2}([\frac{d}{p}]-\frac{d}{\tilde{p}})}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{\tilde{p},\infty}}}$ are equivalent, then there exists a positive constant $\sigma_{p,\tilde{p},d}$ such that $T=\infty$ and (44) holds whenever $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \leq \sigma_{p,\tilde{p},d}$. $\square$

Remark 1. From Theorem 3 and the proof of Lemma 7, and Theorem 5.4 ([24], p. 45), we have the following imbedding maps

$$\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d) \hookrightarrow \dot{H}^{[\frac{d}{p}]-1}_{\mathcal{L}^{d/[\frac{d}{p}],r}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},1}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d).$$

On the other hand, a function in $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ norm but small in the $\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d)$ norm.

3.2. Solutions to the Navier-Stokes equations with the initial value in the critical spaces $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $d \le p < \infty$ and $1 \le r < \infty$.

Lemma 11. *Suppose that $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$ with $d \le p < \infty$ and $1 \le r < \infty$. Then $e^{t\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{d,1,\infty}$ for all $\tilde{p} > p$.*

Proof. We prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}} \big\| e^{t\Delta}u_0 \big\|_{\mathcal{L}^{\tilde{p},1}} \lesssim \Big\| u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}},$$

where

$$\alpha = \alpha(d,\tilde{p}) = 1 - \frac{d}{\tilde{p}}.$$

Let p' and $\tilde{p}'$ be such that

$$\frac{1}{p} + \frac{1}{p'} = 1, \frac{1}{\tilde{p}} + \frac{1}{\tilde{p}'} = 1.$$

We have

$$\big\| e^{t\Delta}u_0 \big\|_{\mathcal{L}^{\tilde{p},1}} = \big\| e^{-t|\xi|^2}\hat{u}_0(\xi) \big\|_{L^{\tilde{p}',1}_{\xi}} = \big\| e^{-t|\xi|^2}|\xi|^{1-\frac{d}{p}}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi) \big\|_{L^{\tilde{p}',1}_{\xi}}.$$

Applying Holder's inequality in the Lorentz spaces to obtain

$$\Big\| e^{-t|\xi|^2}|\xi|^{1-\frac{d}{p}}|\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi) \Big\|_{L^{\tilde{p}',1}_{\xi}} = \Big\| e^{-t|\xi|^2}|\xi|^{1-\frac{d}{p}} \Big\|_{L_{\xi}^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| |\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi) \big\|_{L^{p',\infty}}$$
$$= t^{-\frac{1}{2}(1-\frac{d}{\tilde{p}})} \Big\| e^{-|\xi|^2}|\xi|^{1-\frac{d}{p}} \Big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| |\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi) \big\|_{L^{p',\infty}}$$
$$\simeq t^{-\frac{\alpha}{2}} \big\| |\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi) \big\|_{L^{p',\infty}} \lesssim t^{-\frac{\alpha}{2}} \big\| |\xi|^{\frac{d}{p}-1}\hat{u}_0(\xi) \big\|_{L^{p',r}} = t^{-\frac{\alpha}{2}} \Big\| u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}.$$

Therefore this gives the desired result

$$\big\| e^{t\Delta}u_0 \big\|_{\mathcal{L}^{\tilde{p},1}} \lesssim t^{-\frac{\alpha}{2}} \Big\| u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}}.$$

We claim now that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\| e^{t\Delta} u_0 \big\|_{\mathcal{L}^{\tilde{p},1}} = 0.$$

For any $\epsilon > 0$. Applying Lemma 8 and from the above proof we deduce that

$$t^{\frac{\alpha}{2}} \big\| e^{t\Delta} u_0 \big\|_{\mathcal{L}^{\tilde{p},1}} \leq$$

$$t^{\frac{\alpha}{2}} \Big\| e^{-t|\xi|^2} |\xi|^{1-\frac{d}{p}} 1_{B_n^c} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \Big\|_{L_\xi^{\tilde{p}',1}} + t^{\frac{\alpha}{2}} \Big\| e^{-t|\xi|^2} |\xi|^{1-\frac{d}{p}} 1_{B_n} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \Big\|_{L_\xi^{\tilde{p}',1}} \leq$$

$$\begin{aligned} & C_1 \Big\| e^{-|\xi|^2} |\xi|^{1-\frac{d}{p}} \Big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \Big\| 1_{B_n^c} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \Big\|_{L^{p',\infty}} \\ & + C_2 t^{\frac{\alpha}{2}} \Big\| 1_{B_n} |\xi|^{1-\frac{d}{p}} \Big\|_{L^{\frac{p\tilde{p}}{\tilde{p}-p},1}} \big\| |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \big\|_{L^{p',\infty}} \\ \leq & C_3 \Big\| 1_{B_n^c} |\xi|^{\frac{d}{p}-1} \hat{u}_0(\xi) \Big\|_{L^{p',r}} + C_4(n) t^{\frac{\alpha}{2}} \Big\| u_0 \Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}} < \epsilon \end{aligned}$$

for large enough n and small enough $t = t(n) > 0$. □

Lemma 12. *Let*

$$p \geq d \text{ and } d < \tilde{p} < 2p. \tag{47}$$

Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}}_{d,\infty,T} \times \mathcal{K}^{\tilde{p}}_{d,\infty,T}$ into $\mathcal{K}^{p}_{p,1,T}$, and we have the inequality

$$\big\| B(u,v) \big\|_{\mathcal{K}^p_{p,1,T}} \leq C \big\| u \big\|_{\mathcal{K}^{\tilde{p}}_{d,\infty,T}} \big\| v \big\|_{\mathcal{K}^{\tilde{p}}_{d,\infty,T}}, \tag{48}$$

where C is a positive constant and independent of T.

Proof. First, arguing as in Lemma 9, we derive

$$\dot{\Lambda}^{\frac{d}{p}-1} e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u(\tau) \otimes v(\tau)\big) = \frac{1}{(t-\tau)^{\frac{d}{2}(\frac{1}{p}+1)}} K\Big(\frac{.}{\sqrt{t-\tau}}\Big) * \big(u(\tau) \otimes v(\tau)\big),$$

where the tensor $K(x) = \{K_{l,k,j}(x)\}$ is given by the formula

$$\widehat{K_{l,k,j}}(\xi) = \frac{1}{(2\pi)^{d/2}} |\xi|^{\frac{d}{p}-1} e^{-|\xi|^2} \Big(\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2}\Big)(i\xi_l). \tag{49}$$

Applying Theorem 2 for the convolution in the Fourier-Lorentz spaces, we have

$$\begin{aligned} & \Big\| \dot{\Lambda}^{\frac{d}{p}-1} e^{(t-\tau)\Delta} \mathbb{P}\nabla.\big(u(\tau) \otimes v(\tau)\big) \Big\|_{\mathcal{L}^{p,1}} \\ \lesssim & \frac{1}{(t-\tau)^{\frac{d}{2}(\frac{1}{p}+1)}} \Big\| K\Big(\frac{.}{\sqrt{t-\tau}}\Big) \Big\|_{\mathcal{L}^{r,1}} \big\| \big(u(\tau) \otimes v(\tau)\big) \big\|_{\mathcal{L}^{\frac{\tilde{p}}{2},\infty}}, \end{aligned} \tag{50}$$

where

$$\frac{1}{r} = 1 + \frac{1}{p} - \frac{2}{\tilde{p}}. \tag{51}$$

Note that from the inequality (47), we can check that $1 < r < \infty$. Applying Theorem 1, we have

$$\big\|u(\tau) \otimes v(\tau)\big\|_{\mathcal{L}^{\frac{\tilde{p}}{2},\infty}} \lesssim \big\|u(\tau)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \big\|v(\tau)\big\|_{\mathcal{L}^{\tilde{p},\infty}}. \tag{52}$$

From the equalities (49) and (51) it follows that

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}} = (t-\tau)^{\frac{d}{2r}} \big\|\hat{K}\big\|_{L^{r',1}} \simeq (t-\tau)^{\frac{d}{2}(1+\frac{1}{p}-\frac{2}{\tilde{p}})}. \tag{53}$$

From the estimates (50), (52), and (53), we deduce that

$$\begin{aligned}\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} &\lesssim (t-\tau)^{-\frac{d}{\tilde{p}}} \big\|u(\tau)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \big\|v(\tau)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \\ &= (t-\tau)^{\alpha-1} \big\|u(\tau)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \big\|v(\tau)\big\|_{\mathcal{L}^{\tilde{p},\infty}},\end{aligned}$$

where

$$\alpha = \alpha(d,\tilde{p}) = 1 - \frac{d}{\tilde{p}}.$$

This gives the desired result

$$\begin{aligned}\Big\|B(u,v)(t)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} &\lesssim \int_0^t (t-\tau)^{\alpha-1} \big\|u(\tau)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \big\|v(\tau)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \mathrm{d}\tau \\ &\leq \int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha} \sup_{0<\eta<t} \eta^{\frac{\alpha}{2}} \big\|u(\eta)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \sup_{0<\eta<t} \eta^{\frac{\alpha}{2}} \big\|v(\eta)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \mathrm{d}\tau \\ &= \sup_{0<\eta<t} \eta^{\frac{\alpha}{2}} \big\|u(\eta)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \sup_{0<\eta<t} \eta^{\frac{\alpha}{2}} \big\|v(\eta)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha} \mathrm{d}\tau \\ &\simeq \sup_{0<\eta<t} \eta^{\frac{\alpha}{2}} \big\|u(\eta)\big\|_{\mathcal{L}^{\tilde{p},\infty}} \sup_{0<\eta<t} \eta^{\frac{\alpha}{2}} \big\|v(\eta)\big\|_{\mathcal{L}^{\tilde{p},\infty}}.\end{aligned} \tag{54}$$

From (54) it follows the validity of (13) since

$$\lim_{t\to 0} \Big\|B(u,v)(t)\Big\|_{\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,1}}} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|u(t)\big\|_{\mathcal{L}^{\tilde{p},\infty}} = \lim_{t\to 0} t^{\frac{\alpha}{2}} \big\|v(t)\big\|_{\mathcal{L}^{\tilde{p},\infty}} = 0.$$

The estimate (48) can be deduced from the inequality (54). □

Lemma 13. *Let $\tilde p > d$, then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde p}_{d,\infty,T} \times \mathcal{K}^{\tilde p}_{d,\infty,T}$ into $\mathcal{K}^{\tilde p}_{d,1,T}$, and we have the inequality*

$$\big\|B(u,v)\big\|_{\mathcal{K}^{\tilde p}_{d,1,T}} \le C\big\|u\big\|_{\mathcal{K}^{\tilde p}_{d,\infty,T}}\big\|v\big\|_{\mathcal{K}^{\tilde p}_{d,\infty,T}}, \tag{55}$$

where C is a positive constant and independent of T.

Proof. First, arguing as in Lemma 9, we derive

$$e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big) = \frac{1}{(t-\tau)^{\frac{d+1}{2}}}K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big) * \big(u(\tau)\otimes v(\tau)\big),$$

where the tensor $K(x) = \{K_{l,k,j}(x)\}$ is given by the formula

$$\widehat{K_{l,k,j}}(\xi) = \frac{1}{(2\pi)^{d/2}}e^{-|\xi|^2}\Big(\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l). \tag{56}$$

Applying Theorem 2 for the convolution in the Fourier-Lorentz spaces, we have

$$\begin{aligned}&\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{\tilde p,1}}\\ &\lesssim \frac{1}{(t-\tau)^{\frac{d+1}{2}}}\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}}\big\|\big(u(\tau)\otimes v(\tau)\big)\big\|_{\mathcal{L}^{\frac{\tilde p}{2},\infty}},\end{aligned} \tag{57}$$

where

$$\frac{1}{r} = 1 - \frac{1}{\tilde p}. \tag{58}$$

Applying Theorem 1, we have

$$\big\|u(\tau)\otimes v(\tau)\big\|_{\mathcal{L}^{\frac{\tilde p}{2},\infty}} \lesssim \big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}. \tag{59}$$

From the equalities (56) and (58) it follows that

$$\Big\|K\Big(\frac{\cdot}{\sqrt{t-\tau}}\Big)\Big\|_{\mathcal{L}^{r,1}} = (t-\tau)^{\frac{d}{2r}}\big\|\hat K\big\|_{L^{r',1}} \simeq (t-\tau)^{\frac{d}{2}(1-\frac{1}{\tilde p})}. \tag{60}$$

From the estimates (57), (59), and (60), we deduce that

$$\begin{aligned}\Big\|e^{(t-\tau)\Delta}\mathbb{P}\nabla.\big(u(\tau)\otimes v(\tau)\big)\Big\|_{\mathcal{L}^{\tilde p,1}} &\lesssim (t-\tau)^{-\frac{1}{2}(\frac{d}{\tilde p}+1)}\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\\ &= (t-\tau)^{\frac{\alpha}{2}-1}\big\|u(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}}\big\|v(\tau)\big\|_{\mathcal{L}^{\tilde p,\infty}},\end{aligned}$$

where

$$\alpha = \alpha(d, \tilde{p}) = 1 - \frac{d}{\tilde{p}}.$$

This gives the desired result

$$\begin{aligned}
&\|B(u,v)(t)\|_{\mathcal{L}^{\tilde{p},1}} \lesssim \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\|u(\tau)\|_{\mathcal{L}^{\tilde{p},\infty}}\|v(\tau)\|_{\mathcal{L}^{\tilde{p},\infty}}\mathrm{d}\tau \\
&\leq \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\|u(\eta)\|_{\mathcal{L}^{\tilde{p},\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\|v(\eta)\|_{\mathcal{L}^{\tilde{p},\infty}}\mathrm{d}\tau \\
&= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\|u(\eta)\|_{\mathcal{L}^{\tilde{p},\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\|v(\eta)\|_{\mathcal{L}^{\tilde{p},\infty}}\int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\mathrm{d}\tau \\
&\simeq t^{-\frac{\alpha}{2}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\|u(\eta)\|_{\mathcal{L}^{\tilde{p},\infty}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\|v(\eta)\|_{\mathcal{L}^{\tilde{p},\infty}}.
\end{aligned} \tag{61}$$

From (61) it follows the validity (12) since

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\|B(u,v)(t)\|_{\mathcal{L}^{\tilde{p},1}} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\|u(t)\|_{\mathcal{L}^{\tilde{p},\infty}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\|v(t)\|_{\mathcal{L}^{\tilde{p},\infty}} = 0.$$

Finally, the estimate (55) can be deduced from the inequality (61). □
The following lemma is a generalization of Lemma 13.

Lemma 14. *Let $d < \tilde{p}_1 < \infty$ and $d \leq \tilde{p}_2 < \infty$ be such that one of the following conditions is satisfied*

$$d < \tilde{p}_1 < 2d, d \leq \tilde{p}_2 < \frac{d\tilde{p}_1}{2d - \tilde{p}_1},$$

or

$$\tilde{p}_1 = 2d, d \leq \tilde{p}_2 < \infty,$$

or

$$2d < \tilde{p}_1 < \infty, \frac{\tilde{p}_1}{2} < \tilde{p}_2 < \infty.$$

Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}^{\tilde{p}_1}_{d,\infty,T} \times \mathcal{K}^{\tilde{p}_1}_{d,\infty,T}$ into $\mathcal{K}^{\tilde{p}_2}_{d,1,T}$, and we have the inequality

$$\|B(u,v)\|_{\mathcal{K}^{\tilde{p}_2}_{d,1,T}} \leq C\|u\|_{\mathcal{K}^{\tilde{p}_1}_{d,\infty,T}}\|v\|_{\mathcal{K}^{\tilde{p}_1}_{d,\infty,T}},$$

where C is a positive constant and independent of T.

Theorem 6. *Let $p \geq d$ and $1 \leq r < \infty$. Then for any $\tilde{p}$ such that*

$$\tilde{p} > p, \tag{62}$$

there exists a positive constant $\delta_{\tilde{p},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$, with $\mathrm{div}(u_0) = 0$ satisfying

$$\sup_{0<t<T} t^{\frac{1}{2}(1-\frac{d}{\tilde{p}})} \big\| e^{t\Delta} u_0 \big\|_{\mathcal{L}^{\tilde{p},\infty}} \leq \delta_{\tilde{p},d}, \tag{63}$$

NSE has a unique mild solution $u \in \bigcap\limits_{q>p} \mathcal{K}^q_{d,1,T} \cap L^\infty([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}})$.

In particular, the inequality (63) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ with $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{\tilde{p},d}$ such that we can take $T = \infty$ whenever $\Big\| u_0 \Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \leq \sigma_{\tilde{p},d}$.

Proof. Applying Lemma 13 and Theorem 4, we deduce that there exists a positive constant $\delta_{\tilde{p},d}$ such that for all $T > 0$ and for all $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$, with $\mathrm{div}(u_0) = 0$ satisfying the inequality (63) then NSE has a unique mild solution $u \in \mathcal{K}^{\tilde{p}}_{d,1,T}$. Next, we prove that $u \in \bigcap\limits_{q>p} \mathcal{K}^q_{d,1,T}$.
Consider two cases $d < \tilde{p} < 2d$ and $2d \leq \tilde{p} < \infty$ separately.
First, we consider the case $d < \tilde{p} < 2d$. We consider two possibilities $\tilde{p} > \frac{4d}{3}$ and $\tilde{p} \leq \frac{4d}{3}$. In the case $\tilde{p} > \frac{4d}{3}$, we apply Lemmas 11 and 14 to obtained $u \in \mathcal{K}^q_{d,1,T}$ for all q satisfying $p < q < \tilde{p}_1$ where $\tilde{p}_1 = \frac{d\tilde{p}}{2d-\tilde{p}} > 2d$. Thus, $u \in \mathcal{K}^{2d}_{d,1,T}$. Applying again Lemmas 11 and 14, we deduce that $u \in \mathcal{K}^q_{d,1,T}$ for all $q > p$. In the case $\tilde{p} \leq \frac{4d}{3}$, we set up the following series of numbers $\{\tilde{p}_i\}_{0\leq i\leq N}$ by inductive. Set $\tilde{p}_0 = \tilde{p}$ and $\tilde{p}_1 = \frac{d\tilde{p}_0}{2d-\tilde{p}_0}$. We have $\tilde{p}_1 > \tilde{p}_0$. If $\tilde{p}_1 > \frac{4d}{3}$ then set $N = 1$ and stop here. In the case $\tilde{p}_1 \leq \frac{4d}{3}$ set $\tilde{p}_2 = \frac{d\tilde{p}_1}{2d-\tilde{p}_1}$. We have $\tilde{p}_2 > \tilde{p}_1$. If $\tilde{p}_2 > \frac{4d}{3}$ then set $N = 2$ and stop here. In the case $\tilde{p}_2 \leq \frac{4d}{3}$, set $\tilde{p}_3 = \frac{d\tilde{p}_2}{2d-\tilde{p}_2}$. We have $\tilde{p}_3 > \tilde{p}_2$, and so on, there exists $k \geq 0$ such that $\tilde{p}_k \leq \frac{4d}{3}, \tilde{p}_{k+1} = \frac{d\tilde{p}_k}{2d-\tilde{p}_k} > \frac{4d}{3}$. We set $N = k+1$ and stop here, and we have

$$\tilde{p}_0 = \tilde{p}, \tilde{p}_i = \frac{d\tilde{p}_{i-1}}{2d-\tilde{p}_{i-1}}, \tilde{p}_i > \tilde{p}_{i-1} \text{ for } i = 1,2,3,..,N,$$
$$2d \geq \tilde{p}_N > \frac{4d}{3} \geq \tilde{p}_{N-1}.$$

From $u \in \mathcal{K}^{\tilde{p}_0}_{d,1,T}$, applying Lemmas 11 and 14 to obtained $u \in \mathcal{K}^q_{d,1,T}$ for all q satisfying $p < q < \tilde{p}_1$. Then applying again Lemmas 11 and 14 to obtained $u \in \mathcal{K}^q_{d,1,T}$ for all q satisfying $p < q < \tilde{p}_2$, and so on, finishing we have $u \in \mathcal{K}^q_{d,1,T}$ for all q satisfying $p < q < \tilde{p}_N$. Therefore $u \in \mathcal{K}^q_{d,1,T}$ for all q satisfying $\frac{4d}{3} < q < \tilde{p}_N$. From the proof of the case $\tilde{p} > \frac{4d}{3}$, we have $u \in \mathcal{K}^q_{d,1,T}$ for all $q > p$.

Next, we consider the case $2d \le \tilde{p} < \infty$. Let $i \in \mathbb{N}$ be such that

$$\frac{\tilde{p}}{2^{i-1}} \ge \max\{2d, p\} > \frac{\tilde{p}}{2^i}.$$

From $\tilde{p} \ge \max\{2d, p\}$, we have $i \ge 1$. Applying the Lemmas 11 and 14 to obtained $u \in \mathcal{K}^q_{d,1,T}$ for all $q > \frac{\tilde{p}}{2}$. Applying again Lemmas 11 and 14 to obtained $u \in \mathcal{K}^q_{d,1,T}$ for all $q > \frac{\tilde{p}}{2^2}$, and so on, finishing we have $u \in \mathcal{K}^q_{d,1,T}$ for all $q > \frac{\tilde{p}}{2^{i-1}}$. Applying again Lemmas 11 and 14 to obtained $u \in \mathcal{K}^q_{d,1,T}$ for all $q > \max\{p, \frac{\tilde{p}}{2^i}\}$. If $p \ge \frac{\tilde{p}}{2^i}$ then we have $u \in \mathcal{K}^q_{d,1,T}$ for all $q > p$. If $p < \frac{\tilde{p}}{2^i}$ then $2d > \frac{\tilde{p}}{2^i}$. Thus $u \in \mathcal{K}^q_{d,1,T}$ for all q satisfying $\frac{\tilde{p}}{2^i} < q < 2d$. Therefore, from the proof of the case $d < \tilde{p} < 2d$, we have $u \in \mathcal{K}^q_{d,1,T}$ for all $q > p$.

The fact that $u \in L^\infty([0,T]; \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}})$ can be deduced from Lemmas 5 and 12. Applying Lemma 11, we get $e^{t\Delta}u_0 \in \mathcal{K}^{\tilde{p}}_{d,\infty,T}$. From the definition of $\mathcal{K}^{\tilde{p}}_{p,r,T}$, we deduce that the left-hand side of the inequality (63) converges to 0 when T tends to 0. Therefore the inequality (63) holds for arbitrary $u_0 \in \dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}$ when $T(u_0)$ is small enough.

Next, applying Theorem 5.4 ([24], p. 45), we deduce that the two quantities $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}}$ and $\sup\limits_{0<t<\infty} t^{\frac{1}{2}(1-\frac{d}{\tilde{p}})}\big\|e^{t\Delta}u_0\big\|_{\mathcal{L}^{\tilde{p},\infty}}$ are equivalent, then there exists a positive constant $\sigma_{\tilde{p},d}$ such that $T = \infty$ and (63) holds whenever $\Big\|u_0\Big\|_{\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}} \le \sigma_{\tilde{p},d}$. □

Remark 2. From the proof of Lemma 11 and Theorem 5.4 ([24], p. 45), we have the following imbedding maps

$$\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},1}}(\mathbb{R}^d) \hookrightarrow \dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d).$$

On the other hand, a function in $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}^{\frac{d}{p}-1}_{\mathcal{L}^{p,r}}(\mathbb{R}^d)$ norm but small in the $\dot{B}^{\frac{d}{\tilde{p}}-1,\infty}_{\mathcal{L}^{\tilde{p},\infty}}(\mathbb{R}^d)$ norm.

3.3. Solutions to the Navier-Stokes equations with initial value in the critical spaces $\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ with $1 \le r < \infty$.

We define an auxiliary space $\mathcal{K}_{s,r,T}$ which is made up by the functions $u(t,x)$ such that

$$\big\|u\big\|_{\mathcal{K}_{s,r,T}} := \sup_{0<t<T} t^{\frac{\alpha}{2}} \Big\|u(t,x)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} < \infty,$$

and

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\|u(t,x)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} = 0, \tag{64}$$

with

$$d-1 \le s < d, 1 \le r \le \infty, T > 0,$$

and

$$\alpha = \alpha(s) = s + 1 - d.$$

In the case $s = d-1$, it is also convenient to define the space $\mathcal{K}_{d-1,r,T}$ as the natural space $L^\infty\big([0,T]; \dot{H}^{d-1}_{\mathcal{L}^{1,r}}\big)$ with the additional condition that its elements $u(t,x)$ satisfy

$$\lim_{t\to 0} \big\|u(t,x)\big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}} = 0. \tag{65}$$

Lemma 15. *Let $1 \le r \le \tilde{r} \le \infty$. Then we have the following imbedding*

$$\mathcal{K}_{s,1,T} \hookrightarrow \mathcal{K}_{s,r,T} \hookrightarrow \mathcal{K}_{s,\tilde{r},T} \hookrightarrow \mathcal{K}_{s,\infty,T}.$$

Proof. It is deduced from Lemma 1 (a) and the definition of $\mathcal{K}_{s,r,T}$. □

Lemma 16. *Suppose that $u_0 \in \dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ with $1 \le r < \infty$, then $e^{t\Delta}u_0 \in \mathcal{K}_{s,r,\infty}$ with $d-1 < s < d$.*

Proof. We prove that

$$\sup_{0<t<\infty} t^{\frac{\alpha}{2}} \Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} \lesssim \Big\|u_0\Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}} \text{ for } 1 \le r \le \infty. \tag{66}$$

We have

$$\begin{aligned} \Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} &= \big\|e^{-t|\xi|^2}|\xi|^s\hat{u}_0(\xi)\big\|_{L^{\infty,r}_\xi} = \big\||\xi|^{s+1-d}e^{-t|\xi|^2}|\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L^{\infty,r}_\xi} \\ &\le t^{-\frac{s+1-d}{2}}\big\||\xi|^{s+1-d}e^{-|\xi|^2}\big\|_{L^\infty}\big\||\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L^{\infty,r}} \simeq t^{-\frac{\alpha}{2}}\Big\|u_0\Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}}. \end{aligned} \tag{67}$$

We claim now that

$$\lim_{t\to 0} t^{\frac{\alpha}{2}} \Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}} = 0 \text{ for } 1 \le r < \infty.$$

From the inequality (67), we have

$$t^{\frac{\alpha}{2}}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}}\leq$$
$$t^{\frac{\alpha}{2}}\big\||\xi|^{s+1-d}e^{-t|\xi|^2}1_{B_n^c}|\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\infty,r}}+t^{\frac{\alpha}{2}}\big\||\xi|^{s+1-d}e^{-t|\xi|^2}1_{B_n}|\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\infty,r}}.$$

For any $\epsilon>0$, applying Lemma 8, we have

$$\begin{aligned} t^{\frac{\alpha}{2}}\big\||\xi|^{s+1-d}e^{-t|\xi|^2}1_{B_n^c}|\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\infty,r}}&\leq\big\||\xi|^{s+1-d}e^{-|\xi|^2}\big\|_{L^\infty}\big\|1_{B_n^c}|\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L^{\infty,r}}\\ &=C\big\|1_{B_n^c}|\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L^{\infty,r}}<\frac{\epsilon}{2}, \qquad (68)\end{aligned}$$

for large enough n. Fixed one of such n, we have the following estimates

$$\begin{aligned} &t^{\frac{\alpha}{2}}\big\||\xi|^{s+1-d}e^{-t|\xi|^2}1_{B_n}|\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L_\xi^{\infty,r}}\\ &\leq t^{\frac{\alpha}{2}}\big\|1_{B_n}|\xi|^{s+1-d}e^{-t|\xi|^2}\big\|_{L^\infty}\big\||\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L^{\infty,r}}\\ &\leq t^{\frac{\alpha}{2}}\big\|1_{B_n}|\xi|^{s+1-d}\big\|_{L^\infty}\big\||\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L^{\infty,r}}=t^{\frac{\alpha}{2}}n^{s+1-d}\big\||\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L^{\infty,r}}\\ &=t^{\frac{\alpha}{2}}n^{s+1-d}\Big\|u_0\Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}}<\frac{\epsilon}{2} \qquad (69)\end{aligned}$$

for small enough $t=t(n)>0$. From the estimates (68) and (69), we have,

$$t^{\frac{\alpha}{2}}\Big\|e^{t\Delta}u_0\Big\|_{\dot{H}^s_{\mathcal{L}^{1,r}}}\leq C\big\|1_{B_n^c}|\xi|^{d-1}\hat{u}_0(\xi)\big\|_{L^{\infty,r}}+t^{\frac{\alpha}{2}}n^{s+1-d}\Big\|u_0\Big\|_{\dot{H}^{d-1}_{\mathcal{L}^{1,r}}}<\epsilon.\quad\square$$

Lemma 17. *Let $d-1<s<d$. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}_{s,\infty,T}\times\mathcal{K}_{s,\infty,T}$ into $\mathcal{K}_{s,1,T}$ and we have the inequality*

$$\big\|B(u,v)\big\|_{\mathcal{K}_{s,1,T}}\leq C\Big\|u\Big\|_{\mathcal{K}_{s,\infty,T}}\Big\|v\Big\|_{\mathcal{K}_{s,\infty,T}}, \qquad (70)$$

where C is a positive constant and independent of T.

Proof. Using the Fourier transform we get

$$\mathcal{F}\big(B(u,v)_j(t)\big)(\xi)=$$
$$\frac{1}{(2\pi)^{\frac{d}{2}}}\int_0^t e^{-(t-\tau)|\xi|^2}\sum_{l,k=1}^d\Big(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2}\Big)(i\xi_l)\big(\widehat{u_l(\tau)}*\widehat{v_k(\tau)}\big)(\xi)\mathrm{d}\tau.$$

Thus

$$\big||\xi|^s\mathcal{F}\big(B(u,v)(t)\big)(\xi)\big|\lesssim\int_0^t|\xi|^s e^{-(t-\tau)|\xi|^2}|\xi|\big(|\widehat{u(\tau)}|*|\widehat{v(\tau)}|\big)(\xi)\mathrm{d}\tau.$$

We have

$$|\xi|^s|\widehat{u(\tau)}(\xi)| \le \sup_{\xi\in\mathbb{R}^d}\big||\xi|^s\widehat{u(\tau)}(\xi)\big| = \big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}} \text{ and } |\xi|^s|\widehat{v(\tau)}(\xi)| \le \big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}},$$

therefore

$$|\widehat{u(\tau)}(\xi)| \le \frac{\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^s},\ |\widehat{v(\tau)}(\xi)| \le \frac{\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^s}.$$

A standard argument shows that

$$\frac{1}{|\xi|^s} * \frac{1}{|\xi|^s} = \frac{C}{|\xi|^{2s-d}}.$$

From the above estimates and Lemma 1 (b), we have

$$\big(|\widehat{u(\tau)}| * |\widehat{v(\tau)}|\big)(\xi) \le \frac{\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^s} * \frac{\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^s} \simeq \frac{\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^1}}}{|\xi|^{2s-d}} = \frac{\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}}{|\xi|^{2s-d}},$$

this gives the desired result

$$\int_0^t |\xi|^s e^{-(t-\tau)|\xi|^2}|\xi|\big(|\widehat{u(\tau)}| * |\widehat{v(\tau)}|\big)(\xi)\mathrm{d}\tau \lesssim \int_0^t |\xi|^{d+1-s}e^{-(t-\tau)|\xi|^2}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau.$$

Thus

$$\begin{aligned}
&\Big\||\xi|^s\mathcal{F}\big(B(u,v)(t)\big)(\xi)\Big\|_{L^{\infty,1}_\xi} \lesssim \\
&\int_0^t \Big\||\xi|^{d+1-s}e^{-(t-\tau)|\xi|^2}\Big\|_{L^{\infty,1}_\xi}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&= \int_0^t (t-s)^{\frac{s-d-1}{2}}\Big\||\xi|^{d+1-s}e^{-|\xi|^2}\Big\|_{L^{\infty,1}}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&\lesssim \int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\int_0^t (t-\tau)^{\frac{\alpha}{2}-1}\tau^{-\alpha}\mathrm{d}\tau \\
&\simeq t^{-\frac{\alpha}{2}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}. \qquad (71)
\end{aligned}$$

Let us now check the validity of the condition (64) for the bilinear term $B(u,v)(t)$. Indeed, from (71)

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|B(u,v)(t)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,1}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\||\xi|^s\mathcal{F}\big(B(u,v)(t)\big)(\xi)\Big\|_{L^{\infty,1}_\xi} = 0,$$

whenever

$$\lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|u(t)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}} = \lim_{t\to 0} t^{\frac{\alpha}{2}}\Big\|v(t)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}} = 0.$$

The estimate (70) is deduced from the inequality (71). □

Lemma 18. *Let $d-1 < s < d$. Then the bilinear operator $B(u,v)(t)$ is continuous from $\mathcal{K}_{s,\infty,T}\times\mathcal{K}_{s,\infty,T}$ into $\mathcal{K}_{d-1,1,T}$ and we have the inequality*

$$\big\|B(u,v)\big\|_{\mathcal{K}_{d-1,1,T}} \le C\Big\|u\Big\|_{\mathcal{K}_{s,\infty,T}}\Big\|v\Big\|_{\mathcal{K}_{s,\infty,T}}, \tag{72}$$

where C is a positive constant and independent of T.

Proof. First, arguing as in Lemma 17, we have the following estimates

$$\begin{aligned}
&\big||\xi|^{d-1}\mathcal{F}\big(B(u,v)(t)\big)(\xi)\big| \\
&\lesssim \int_0^t |\xi|^{d-1}e^{-(t-\tau)|\xi|^2}|\xi|\big(|\widehat{u(\tau)}|*|\widehat{v(\tau)}|\big)(\xi)\mathrm{d}\tau \\
&\lesssim \int_0^t |\xi|^{2d-2s}e^{-(t-\tau)|\xi|^2}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau,
\end{aligned}$$

this gives the desired result

$$\begin{aligned}
&\Big\||\xi|^{d-1}\mathcal{F}\big(B(u,v)(t)\big)(\xi)\Big\|_{L^{\infty,1}_\xi} \\
&\lesssim \int_0^t \Big\||\xi|^{2d-2s}e^{-(t-\tau)|\xi|^2}\Big\|_{L^{\infty,1}_\xi}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&= \int_0^t (t-s)^{s-d}\Big\||\xi|^{2d-2s}e^{-|\xi|^2}\Big\|_{L^{\infty,1}}\big\|u(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\big\|v(\tau)\big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&\lesssim \int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\mathrm{d}\tau \\
&= \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}\mathrm{d}\tau \\
&\simeq \sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|u(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}\sup_{0<\eta<t}\eta^{\frac{\alpha}{2}}\Big\|v(\eta)\Big\|_{\dot{H}^s_{\mathcal{L}^{1,\infty}}}.
\end{aligned} \tag{73}$$

From (73) it follows (65) since

$$\lim_{t\to 0}\left\|B(u,v)(t)\right\|_{\dot H^{d-1}_{\mathcal{L}^{1,1}}}=\lim_{t\to 0}\left\||\xi|^{d-1}\mathcal{F}\big(B(u,v)(t)\big)(\xi)\right\|_{L^{\infty,1}_\xi}=0,$$

whenever

$$\lim_{t\to 0}t^{\frac{\alpha}{2}}\left\|u(t)\right\|_{\dot H^s_{\mathcal{L}^{1,\infty}}}=\lim_{t\to 0}t^{\frac{\alpha}{2}}\left\|v(t)\right\|_{\dot H^s_{\mathcal{L}^{1,\infty}}}=0.$$

The estimate (72) can be deduced from the inequality (73). □

Theorem 7. *Let $d-1<s<d$ and $1\le r<\infty$. Then there exists a positive constant $\delta_{s,d}$ such that for all $T>0$ and for all $u_0\in \dot H^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$, with* $\operatorname{div}(u_0)=0$ *satisfying*

$$\sup_{0<t<T}t^{\frac12(s+1-d)}\left\|e^{t\Delta}u_0\right\|_{\dot H^s_{\mathcal{L}^1}}\le \delta_{s,d}, \tag{74}$$

NSE has a unique mild solution $u\in\mathcal{K}_{s,r,T}\cap L^\infty([0,T];\dot H^{d-1}_{\mathcal{L}^{1,r}})$.
In particular, the inequality (74) holds for arbitrary $u_0\in\dot H^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough, and there exists a positive constant $\sigma_{s,d}$ such that we can take $T=\infty$ whenever $\left\|u_0\right\|_{\dot H^{d-1}_{\mathcal{L}^1}}\le\sigma_{s,d}$.

Proof. The proof of Theorem 7 is similar to that of Theorem 5. Applying Lemma 17 and Theorem 4, we deduce that there exists a positive constant $\delta_{s,d}$ such that for any $u_0\in\dot H^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ with $\operatorname{div}(u_0)=0$ such that

$$\sup_{0<t<T}t^{\frac12(s+1-d)}\left\|e^{t\Delta}u_0\right\|_{\dot H^s_{\mathcal{L}^{1,\infty}}}=\sup_{0<t<T}t^{\frac12(s+1-d)}\left\|e^{t\Delta}u_0\right\|_{\dot H^s_{\mathcal{L}^1}}\le\delta_{s,d},$$

the Navier-Stokes equations has a solution $u\in\mathcal{K}_{s,\infty,T}$. Applying Lemmas 5 and 18 we deduce that $u\in L^\infty([0,T];\dot H^{d-1}_{\mathcal{L}^{1,r}})$. Applying Lemma 16, we get $e^{t\Delta}u_0\in\mathcal{K}_{s,r,T}$. From the definition of $\mathcal{K}_{s,r,T}$, we deduce that the left-hand side of the inequality (74) converges to 0 when T tends to 0. Therefore the inequality (74) holds for arbitrary $u_0\in\dot H^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ when $T(u_0)$ is small enough.
Next, from the inequality (66) with $r=\infty$, we deduce that

$$\sup_{0<t<\infty}t^{\frac12(s+1-d)}\left\|e^{t\Delta}u_0\right\|_{\dot H^s_{\mathcal{L}^1}}\lesssim\left\|u_0\right\|_{\dot H^{d-1}_{\mathcal{L}^1}},$$

then there exists a positive constant $\sigma_{s,d}$ such that $T=\infty$ and (74) holds whenever $\left\|u_0\right\|_{\dot H^{d-1}_{\mathcal{L}^1}}\le\sigma_{s,d}$. □

Remark 3. The case $r = \infty$ was studied by Le Jan and Sznitman in [26]. They showed that NSE are well-posed when the initial datum belongs to the space $\dot{H}^{d-1}_{\mathcal{L}^{1,\infty}}$. For $1 \leq r < \infty$ we have the following imbedding map

$$\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d) \hookrightarrow \dot{H}^{d-1}_{\mathcal{L}^{1,\infty}}(\mathbb{R}^d) = \dot{H}^{d-1}_{\mathcal{L}^{1}}(\mathbb{R}^d).$$

However, note that for $1 \leq r < \infty$ a function in $\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ can be arbitrarily large in the $\dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d)$ norm but small in the $\dot{H}^{d-1}_{\mathcal{L}^{1}}(\mathbb{R}^d)$ norm. Theorem 7 shows the existence of global mild solutions in the spaces $L^\infty([0,\infty); \dot{H}^{d-1}_{\mathcal{L}^{1,r}}(\mathbb{R}^d))$ (with $1 \leq r < \infty$) when the norm of the initial value in the spaces $\dot{H}^{d-1}_{\mathcal{L}^{1}}(\mathbb{R}^d)$ is small enough.

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