The inverse spatial Laplacian of spherically symmetric spacetimes

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Abstract

In this paper we derive the inverse spatial Laplacian for static, spherically symmetric backgrounds by solving Poisson's equation for a point source. This is different from the electrostatic Green function, which is defined on the four dimensional static spacetime, while the equation we consider is defined on the spatial hypersurface of such spacetimes. This Green function is relevant in the Hamiltonian dynamics of theories defined on spherically symmetric backgrounds, and closed form expressions for the solutions we find are absent in the literature. We derive an expression in terms of elementary functions for the Schwarzschild spacetime, and comment on the relation of this solution with the known Green function of the spacetime Laplacian operator. We also find an expression for the Green function on the static pure de Sitter space in terms of hypergeometric functions.

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I. INTRODUCTION

Let us consider a static four-dimensional spacetime with metric

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = -\lambda^2 dt^2 + h_{\alpha\beta}dx^\alpha dx^\beta\,. \tag{1.1}$$

By static, we mean that the spacetime admits a hypersurface orthogonal timelike Killing vector ξ^μ , such that $\xi^\mu\xi_\mu = -\lambda^2$. The induced metric on the spacelike hypersurface Σ is $h_{\alpha\beta} = g_{\alpha\beta} + \lambda^{-2}\xi_\alpha\xi_\beta$. Staticity also implies spherical symmetry by Birkhoff's theorem, so we will write $\lambda = \lambda(r)$. We will often use Greek indices $\alpha, \beta, \cdots$ for spacetime indices, while Latin indices $i, j, \cdots$ will run over spatial indices. So the spatial metric will be written as h_{ij} .

The object of our interest in this paper is the Green function $G(\vec{x}, \vec{y})$ for the three-dimensional spatial Laplacian operator, which formally satisfies the equation

$$\mathcal{D}_i\mathcal{D}^i G(\vec{x}, \vec{y}) = -4\pi\delta\left(\vec{x}, \vec{y}\right)\,, \tag{1.2}$$

where $\mathcal{D}_i$ is the connection compatible with h_{ij} , $h = \det h_{ij}$, and the 3-dimensional covariant delta function $\delta\left(\vec{x}, \vec{y}\right)$ is defined by

$$\int_\sigma dV_x f(\vec{x})\delta\left(\vec{x}, \vec{y}\right) = f(\vec{y})\,, \tag{1.3}$$

if $\sigma \subseteq \Sigma$ includes the point $\vec{y}$, and zero otherwise. We will consider this Green function in two specific backgrounds, the asymptotically flat Schwarzschild black hole and the empty de Sitter spacetime. Both of these contain a horizon, defined by $\lambda = 0$. This Green function is relevant for the Hamiltonian dynamics of fields, as we discuss below.

On the other hand, a different Green function appears in solving for the Coulomb potential in static spherically symmetric spacetimes, and a closed form expression for it is well known. For Maxwell's equation

$$\nabla_\alpha F^{\alpha\mu} = -4\pi J^\mu\,. \tag{1.4}$$

By defining

$$e^\mu := \lambda^{-1}\xi_\alpha F^{\mu\alpha}\,, \quad \phi := \lambda^{-1}\xi^\alpha A_\alpha\,, \tag{1.5}$$

we find that the contraction of Eq. (1.4) with $\lambda^{-1}\xi_\mu$, which is the equivalent of setting $\mu = 0$, leads to

$$D_\mu e^\mu = D_\mu\left(\lambda^{-1}D^\mu\left(\lambda\phi\right) - \lambda^{-1}\pounds_\xi a^\mu\right) = -4\pi J^0\,, \tag{1.6}$$

where $J^0 = \lambda^{-1}\xi_\mu J^\mu$ and $\pounds_\xi$ is the Lie derivative with respect to ξ^α. If we also set $\pounds_\xi \phi = 0 = \pounds_\xi a_\mu$, and take a point charge by setting $J^0 = \delta(\vec{x}, \vec{y})$, Eq. (1.6) reduces to that for the Green function for the electrostatic potential,

$$D^x_\mu \left(\lambda^{-1}(\vec{x}) D^\mu_x \left(\lambda(\vec{x})\phi(\vec{x})\right)\right) = -4\pi\delta(\vec{x}, \vec{y})\,. \tag{1.7}$$

We will call the Green function corresponding to this equation as the 4d static scalar Green function, as opposed to the 3d spatial Green function which corresponds to Eq. (1.2). It is also clear that this equation agrees with Eq. (1.2) in flat space, or whenever λ is a constant.

On curved spacetimes however, Eq. (1.2) and Eq. (1.7) differ. The left hand side of Eq. (1.7) can be written in coordinates as

$$\frac{1}{\lambda\sqrt{h}}\partial_i \left(\lambda^{-1}\sqrt{h} h^{ij}\partial_j \Phi(\vec{x})\right) = \lambda^{-2}\mathcal{D}_i\mathcal{D}^i\Phi(\vec{x}) + \lambda^{-1}h^{ij}\left(\partial_i\lambda^{-1}\right)\partial_j\Phi(\vec{x})\,, \tag{1.8}$$

where we have written $\Phi = \lambda\phi$. The second term is non-zero in general for curved spacetimes.

For the Schwarzschild background, the 4d Green function is known in closed form. It can be derived by direct construction of the Hadamard elementary solution [5] and also using the method of multipole expansion [6, 7]. A closed form expression was given in [8], which included an additional term missed in [5]. This term accounts for the induced charge behind the horizon of the black hole on the Schwarzschild background, and corresponds to the zero mode contribution in the multipole expansion result. The closed form expression for the static, scalar Green function for the spacetime Laplacian on curved backgrounds has found numerous applications [9–13] predominantly in its use in determining the self force acting on the particle placed on such backgrounds [14–20]. Such closed form expressions have additionally been determined for the Reissner-Nördstrom [21], and more recently for Kerr backgrounds [22].

In contrast, a closed form expression for the Green function $G(\vec{x}, \vec{y})$ of Eq. (1.2) seems to be absent from the literature. One place where this Green function is relevant is the Hamiltonian dynamics of fields. The specific context we have in mind is the constrained dynamics of gauge field theories, where this function appears for gravitational [1] and electromagnetic [2–4] fields. For example, the Maxwell field has the first class Gauss law constraint $\mathcal{D}_i\pi^i \approx 0$, which implies the existence of redundant or gauge degrees of freedom, which can be eliminated by fixing the gauge and then applying Dirac's procedure. A useful choice of gauge fixing is the radiation gauge, $\mathcal{D}_i A^i \approx 0$. After gauge fixing the constraints become second class, and the matrix of their Poisson brackets has to be inverted in order to define the Dirac brackets of the theory. The Poisson bracket of the Gauss constraint with the radiation gauge condition produces the spatial Laplacian $\mathcal{D}^i\mathcal{D}_i$, and inversion of the matrix requires the Green function noted in Eq. (1.2). Though the need for this Green function is known, an explicit expression seems to be absent from the literature, which motivated us to find one.

The organization of our paper is as follows. In Sec. II, we review the derivation of the static, scalar Green function for the spacetime Laplacian defined on the Schwarzschild background. In Sec. III, after considering the geometric framework which will help describe the problem we wish to solve generally, we derive the specific solutions for the Schwarzschild and static pure de Sitter backgrounds. While we were able to determine the closed form expression for the Schwarzschild case in terms of elementary functions, we have as yet been unable to find a similar expression for the pure de Sitter background. Finally, in Sec. (IV), we discuss the consequences of our result in the quantization of the Maxwell field on spherically symmetric backgrounds.

II. DERIVATION OF GREEN FUNCTION

The Green function corresponding to Eq. (1.7) on a Schwarzschild background is relevant for Coulomb's law, as we have seen. Let us briefly review its derivation following [6], as we will follow a similar procedure for deriving the Green function for Eq. (1.2).

We take $\lambda^2 = g^{rr} = 1 - \frac{2m}{r}$, and place a point charge e at $r = r'$ and $\theta = 0$, thus reducing the problem to an axisymmetric one. With this choice Eq. (1.7) becomes, in explicit coordinates,

$$\sin\theta\partial_r\left(r^2\partial_r\Phi\right) + \frac{1}{\left(1-\frac{2m}{r}\right)}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right) = -4e\pi\delta(r-r')\delta(\theta)\,, \tag{2.1}$$

where we have written $\lambda\phi = \Phi = \Phi(r,\theta)$, utilizing the symmetry, and the delta functions are normalized according to

$$\int_{2m}^{\infty} dr\delta(r-r') = 1\,, \qquad\qquad \int_0^{\pi} d\theta\,\delta(\theta-\theta') = 1\,. \tag{2.2}$$

Away from the point charge, the right hand side of Eq. (2.1) vanishes, and we can expand Φ as

$$\Phi(r,\theta) = \sum_{l=0}^{\infty} R_l(r) P_l(\cos\theta)\,. \tag{2.3}$$

Substituting this expression in Eq. (2.1) for vanishing right hand side, we find that $R_l(r)$ is a linear combination of two independent solutions,

$$R_l(r) = A_l g_l(r) + B_l f_l(r)\,, \tag{2.4}$$

where $g_l(r)$ and $f_l(r)$ are given by [6, 23]

$$g_l(r) = \begin{cases} 1 & (\text{for}\, l = 0)\,, \\ \frac{2^l l!\,(l-1)!\,m^l}{(2l)!}(r-2m)\frac{d}{dr}P_l\left(\frac{r}{m}-1\right) & (\text{for}\, l \neq 0)\,, \end{cases} \tag{2.5}$$

$$f_l(r) = -\frac{(2l+1)!}{2^l(l+1)!\,l!\,m^{l+1}}(r-2m)\frac{d}{dr}Q_l\left(\frac{r}{m}-1\right)\,. \tag{2.6}$$

Here P_l and Q_l are the Legendre functions of the first and second kind, respectively. With the exception of $g_0(r) = 1$, the leading term of $g_l(r)$ is proportional to r^l and diverges as $r \to \infty$. Thus this solution is ruled out for large values of r. Both $g_l(r)$ and $f_l(r)$ are well behaved at the horizon $r = 2m$. However, $\frac{d}{dr}f_l(r)$ diverges logarithmically as $r \to 2m$, except when $l = 0$. On the other hand, the leading behaviour of $f_l(r)$ for large r is proportional to r^{-l-1} [24].

This information is sufficient to now solve Eq. (2.1). Given the point (r', θ'), we know that the solution consists of $f_l(r)$ for $r > r'$, and of $g_l(r)$ for $r < r'$, as these are the functions that are well behaved in those regions. The continuity of the solution Φ requires matching the solutions at $(r', \theta', 0)$, with which the following general solution can be determined for Eq. (2.1)

$$\begin{aligned} \Phi(r,\theta) &= \sum_{l=0}^{\infty} C_l g_l(r') f_l(r) P_l(\cos\theta) & r > r' \\ &= \sum_{l=0}^{\infty} C_l f_l(r') g_l(r) P_l(\cos\theta) & r < r'\,, \end{aligned} \tag{2.7}$$

We can now simply integrate Eq. (2.1) to determine the constants C_l, and hence the solution of Eq. (2.1).

$$\Phi(r,\theta) = e\sum_{l=0}^{\infty} g_l(r')f_l(r)P_l(\cos\theta) \qquad\qquad r > r'$$
$$= e\sum_{l=0}^{\infty} f_l(r')g_l(r)P_l(\cos\theta) \qquad\qquad r < r'\,, \tag{2.8}$$

A bit of algebra shows that these can be rewritten in the form

$$\Phi(r,\theta) = \frac{e}{rr'}\left[\frac{(r-m)(r'-m) - m^2\cos\theta}{\sqrt{(r-m)^2 + (r'-m)^2 - 2(r-m)(r'-m)\cos\theta - m^2\sin^2\theta}} + m\right]. \tag{2.9}$$

This expression, found in [8], differs from a solution provided many years earlier [5] because of the term $\frac{em}{rr'}$, which accounts for the zero-mode contribution in Eq. (2.8). It should also be noted that the general solution for a point charge located at (r', θ', ϕ') can be found from the expressions given in Eq. (2.9) and Eq. (2.8) by substituting $\cos\theta$ with $\cos\gamma$, given by

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')\ . \tag{2.10}$$

Eq. (2.8) has been derived using a similar method in [7], while that of Eq. (2.9) has been derived more recently using the heat kernel method and bi-conformal symmetry in [25].

III. SPHERICALLY SYMMETRIC SPACETIMES

We are interested in solving Eq. (1.2) for spherically symmetric backgrounds, specifically the Schwarzschild and de Sitter backgrounds. For these spacetimes, the metric takes the form

$$ds^2 = -\lambda(r)^2 dt^2 + \frac{1}{\lambda(r)^2}dr^2 + r^2 d\Omega^2\,. \tag{3.1}$$

We will consider a point source at the point (r', θ', ϕ'), but will not reduce the problem to an axisymmetric one as in the previous section. Then we can write Eq. (1.2) as

$$\sin\theta\partial_r\left(r^2\lambda(r)\partial_r\Phi\right) + \frac{1}{\lambda(r)}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right) + \frac{1}{\lambda(r)\sin\theta}\partial^2_\phi\Phi = -4\pi\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')\,. \tag{3.2}$$

A. Schwarzschild background

For the Schwarzschild background, $\lambda(r)^2 = 1 - \frac{2m}{r}$, and Eq. (3.2) takes the form

$$\sin\theta\partial_r\left(r^2\sqrt{1-\frac{2m}{r}}\partial_r\Phi\right)+\frac{1}{\sqrt{1-\frac{2m}{r}}}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right)+\frac{1}{\sqrt{1-\frac{2m}{r}}\sin\theta}\partial_\phi^2\Phi = -4\pi\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')\,. \tag{3.3}$$

It will be convenient to make a change of variables from r to $y = \frac{r}{m} - 1$. After we find the solution, we can change variables again to express the Green function in terms of the original coordinates.

In terms of y, Eq. (3.3) takes the form

$$\sin\theta\left[\partial_y\left((y+1)^2\sqrt{\frac{y-1}{y+1}}\partial_y\Phi\right)+\sqrt{\frac{y+1}{y-1}}\left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right)+\frac{1}{\sin^2\theta}\partial_\phi^2\Phi\right)\right] = -4\pi\frac{\delta(y-y')}{m}\delta(\theta-\theta')\delta(\phi-\phi')\,, \tag{3.4}$$

where $\Phi = \Phi(y,\theta,\phi)$, and the point source is located at (y',θ',ϕ') in the new coordinates. The delta functions now satisfy

$$\int_1^\infty dy\,\delta(y-y') = 1\,,\qquad \int_0^\pi d\theta\,\delta(\theta-\theta') = 1\,,\qquad \int_0^{2\pi} d\phi\,\delta(\phi-\phi') = 1\,. \tag{3.5}$$

The first step in our derivation is to consider Eq. (3.4) without the source, and to solve for the homogeneous equation that results. Thus, we need to solve the following equation

$$0 = \sqrt{\frac{y-1}{y+1}}\partial_y\left((y+1)^2\sqrt{\frac{y-1}{y+1}}\partial_y\Phi\right)+\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right)+\frac{1}{\sin^2\theta}\partial_\phi^2\Phi\,, \tag{3.6}$$

for which we make the following choice for the separation of variables

$$\Phi(y,\theta,\phi) = \sum_{l=0}^\infty R_l(y)P_l(\cos\gamma)\,, \tag{3.7}$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$. We note that $P_l(\cos\gamma)$ satisfies

$$\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta P_l(\cos\gamma)\right)+\frac{1}{\sin^2\theta}\partial_\phi^2 P_l(\cos\gamma) = -l(l+1)P_l(\cos\gamma)\,, \tag{3.8}$$

and the normalization and orthogonality conditions

$$\int_0^\pi d\theta\int_0^{2\phi} d\phi\delta(\phi-\phi')\delta(\theta-\theta')P_{l'}(\cos\gamma) = 1\,, \tag{3.9}$$

$$\int_{-1}^{1} d\cos\theta \int_{0}^{2\pi} d\phi P_{l'}(\cos\gamma) P_l(\cos\gamma) = \delta_{ll'} \frac{4\pi}{2l+1} \,. \tag{3.10}$$

On account of this, the substitution of Eq. (3.7) in Eq. (3.6) gives the following differential equation

$$\sqrt{\frac{y-1}{y+1}} \frac{d}{dy} \left((y+1)^2 \sqrt{\frac{y-1}{y+1}} \frac{d}{dy} R_l(y) \right) - l(l+1) R_l(y) = 0 \,. \tag{3.11}$$

We have described the solution of Eq. (3.11) in Appendix A. The general solution is given in Eq. (A8), and it is of the form

$$R_l(y) = A_l g_l(y) + B_l f_l(y) \,, \tag{3.12}$$

where the functions $g_l(y)$ and $f_l(y)$ involve Legendre polynomials of fractional degree, with the argument $y > 1$. Legendre polynomials of fractional degree can be described in terms of hypergeometric functions, for which there exist several representations. A particular representation which we will use is [24]

$$\begin{aligned} P_\nu^\mu(y) &= \frac{\Gamma\left(-\nu - \frac{1}{2}\right)}{2^{\nu+1}\sqrt{\pi}\Gamma\left(-\nu-\mu\right)} y^{-\nu+\mu-1} (y^2-1)^{-\frac{\mu}{2}} {}_2F_1\left(\frac{1+\nu-\mu}{2}, \frac{2+\nu-\mu}{2}; \nu + \frac{3}{2}; \frac{1}{y^2}\right) \\ &\quad + \frac{2^\nu \Gamma\left(\nu+\frac{1}{2}\right)}{\sqrt{\pi}\Gamma\left(1+\nu-\mu\right)} y^{\nu+\mu} (y^2-1)^{-\frac{\mu}{2}} {}_2F_1\left(\frac{-\nu-\mu}{2}, \frac{1-\nu-\mu}{2}; -\nu + \frac{1}{2}; \frac{1}{y^2}\right) \,, \\ e^{-i\pi\mu} Q_\nu^\mu(y) &= \frac{\sqrt{\pi}\Gamma\left(1+\nu+\mu\right)}{2^{\nu+1}\Gamma\left(\frac{3}{2}+\nu\right)} y^{-\nu-\mu-1} (y^2-1)^{\frac{\mu}{2}} {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\nu+\mu+1}{2}; \nu + \frac{3}{2}; \frac{1}{y^2}\right) \,, \end{aligned} \tag{3.13}$$

The solutions $g_l(y)$ and $f_l(y)$ make use of these solutions for the case of $\mu = \frac{1}{2}$ and $\nu = l$ as shown in Eq. (A8), and can be written as

$$\begin{aligned} g_l(y) &= \frac{1}{\sqrt{y+1}} \left[\frac{1}{2^{l+1}} y^{-l-\frac{1}{2}} {}_2F_1\left(\frac{l+\frac{1}{2}}{2}, \frac{l+\frac{3}{2}}{2}; l + \frac{3}{2}; \frac{1}{y^2}\right) \right. \\ &\quad \left. + 2^l y^{l+\frac{1}{2}} {}_2F_1\left(\frac{-l-\frac{1}{2}}{2}, \frac{-l+\frac{1}{2}}{2}; -l + \frac{1}{2}; \frac{1}{y^2}\right) \right] \,, \\ f_l(y) &= \sqrt{y-1} \left[\frac{1}{2^l} y^{-l-\frac{3}{2}} {}_2F_1\left(\frac{l+\frac{5}{2}}{2}, \frac{l+\frac{3}{2}}{2}; l + \frac{3}{2}; \frac{1}{y^2}\right) \right] \,. \end{aligned} \tag{3.14}$$

It turns out that the functions given in Eq. (3.14) admit expressions in terms of more elementary functions, which we will now describe. These expressions will be relevant in determining the final form of the Green function for the spatial Laplacian. The hypergeometric functions contained in $g_l(y)$ in Eq. (3.14) are both of the following generic form, with the known representation

$$ {}_2F_1\left(a, a+\frac{1}{2}, 2a+1, \frac{1}{y^2}\right) = 2^{2a}\left(\frac{y+\sqrt{y^2-1}}{y}\right)^{-2a}, \tag{3.15} $$

where a stands for both $\frac{l+\frac{1}{2}}{2}$ and $\frac{-l-\frac{1}{2}}{2}$ in the above expression. We can thus write the expression for $g_l(y)$ as

$$ g_l(y) = \frac{1}{\sqrt{2}\sqrt{y+1}}\left[\left(y+\sqrt{y^2-1}\right)^{-l-\frac{1}{2}} + \left(y+\sqrt{y^2-1}\right)^{l+\frac{1}{2}}\right] \tag{3.16} $$

Likewise, the hypergeometric function given in $f_l(y)$ has the following expression in terms of elementary functions,

$$ {}_2F_1\left(b, b+\frac{1}{2}, 2b, \frac{1}{y^2}\right) = \frac{2^{2b-1}y^{2b}}{\sqrt{y^2-1}}\left(y+\sqrt{y^2-1}\right)^{-2b+1}, \tag{3.17} $$

where $b = \frac{l+\frac{3}{2}}{2}$. We can thus write $f_l(y)$ as

$$ f_l(y) = \sqrt{2}\,\frac{\left(y+\sqrt{y^2-1}\right)^{-l-\frac{1}{2}}}{\sqrt{y+1}}. \tag{3.18} $$

The calculation which follows will also require us to determine the Wronskian of the solutions given in Eq. (3.14). Using the above expressions, we readily find that the Wronskian $W(g_l(y), f_l(y), y)$ is given by

$$ W(g_l(y), f_l(y), y) = -\frac{(2l+1)}{(1+y)^{\frac{3}{2}}\sqrt{y-1}}. \tag{3.19} $$

For the differential equation $u''(r) + p(r)u'(r) + q(r)u(r) = 0$ (where primes denote differentiation with respect to r), the Wronskian of the two independent solutions $u_1(r)$ and $u_2(r)$ satisfies

$$ W(u_1(r), u_2(r), r) = W(u_1(r_0), u_2(r_0), r_0)\,\exp\left[-\int_{r_0}^{r} p(x)dx\right]. \tag{3.20} $$

Applying this formula to Eq. (3.11) tells us that

$$ W(g_l(y), f_l(y), y)(1+y)^{\frac{3}{2}}\sqrt{y-1} $$

must be a constant. Thus the Wronskian calculated in Eq. (3.19) is as it should be, and shows that $f_l(y)$ and $g_l(y)$ are the two real, independent solutions of Eq. (3.11) for each value of l.

There are two limits to consider of the solutions given in Eq. (3.18) and Eq. (3.16), and their derivatives. These are the $y \to 1$ and $y \to \infty$ limits, which correspond to the $r \to 2m$ and $r \to \infty$ respectively. Before describing these, we note that $g_0(y)$ is a special case in that it is a constant, $g_0(y) = 1$ for all values of y.

For all the other terms we find the following. As $y \to 1$, both $g_l(y) \to 1$ and $f_l(y) \to 1$ for all values of l, i.e. they are both finite. However, all derivatives of $f_l(y)$ diverge as $y \to 1$, while $\frac{d}{dy} g_l(y) \to l(l+1)$ as $y \to 1$. Thus the near horizon solution must only contain $g_l(y)$, and we must set $B_l = 0$ in Eq. (3.12) in the region between (y', θ', ϕ') and the event horizon of the black hole.

On the other hand, as $y \to \infty$, we find that $f_l(y) \to 0$ for all values of l, and the derivatives of $f_l(y)$ also well behaved, but $g_l(y)$ diverges for $l \neq 0$. We must thus set $A_l = 0$ in Eq. (3.12) to describe the region from (y', θ', ϕ') to ∞.

We can therefore write the solution in the following way in the two regions,

$$\begin{aligned} \Phi\left(y,\theta,\phi\right) &= \sum_{l=0}^{\infty} A_l g_l(y) P_l(\cos\gamma) & \qquad (y < y') \\ &= \sum_{l=0}^{\infty} B_l f_l(y) P_l(\cos\gamma) & \qquad (y > y')\,. \end{aligned} \tag{3.21}$$

Continuity of Φ at $y = y'$ implies that $A_l g_l(y') = B_l f_l(y')$. Then we can define a constant C_l such that

$$C_l = \frac{A_l}{f_l(y')} = \frac{B_l}{g_l(y')}\,, \tag{3.22}$$

using which we can write the solution in the form

$$\begin{aligned} \Phi\left(y,\theta,\phi\right) &= \sum_{l=0}^{\infty} C_l f_l(y') g_l(y) P_l(\cos\gamma)\,, & \qquad (y < y') \\ &= \sum_{l=0}^{\infty} C_l g_l(y') f_l(y) P_l(\cos\gamma)\,. & \qquad (y > y') \end{aligned} \tag{3.23}$$

We can now determine the constants C_l by appropriately integrating Eq. (3.4). To begin with, we insert Eq. (3.7) into Eq. (3.4), multiply both sides with $P_{l'}(\cos\gamma)$ and integrate with respect to θ and ϕ to find

$$\frac{1}{2l+1}\left[\frac{d}{dy}\left((y+1)^2\sqrt{\frac{y-1}{y+1}}\frac{d}{dy}R_l(y)\right) - l(l+1)\sqrt{\frac{y+1}{y-1}}R_l(y)\right] = -\frac{\delta(y-y')}{m}\,. \tag{3.24}$$

Integrating Eq. (3.24) over an infinitesimal region from $y' - \epsilon$ to $y' + \epsilon$, we get

$$\begin{aligned} -\frac{1}{m} &= \frac{1}{2l+1} C_l (y'+1)^2 \sqrt{\frac{y'-1}{y'+1}} \left[g_l(y') \left(\frac{d}{dy} f_l(y) \right) |_{y'+\epsilon} - f_l(y') \left(\frac{d}{dy} g_l(y) \right) |_{y'-\epsilon} \right] \\ &= \frac{1}{2l+1} C_l (y'+1)^{\frac{3}{2}} \sqrt{y'-1} W(g_l(y'), f_l(y'), y') \\ &= -C_l \,, \end{aligned} \tag{3.25}$$

where in going from the second to the third equality in Eq. (3.25), we made use of Eq. (3.19). Thus we have determined that C_l is independent of l,

$$C_l = \frac{1}{m} \,. \tag{3.26}$$

Thus we can write the solution of Eq. (3.4) as

$$\Phi\left(\vec{y}_<\,, \vec{y}_>\right) = \frac{1}{m} \sum_{l=0}^{\infty} g_l(y_<)\, f_l(y_>) P_l(\cos\gamma) \,, \tag{3.27}$$

where $y_< = \min(y\,, y')$ and $y_> = \max(y\,, y')$. Using Eq. (3.18) and Eq. (3.16), we find that the product $g_l(y_<)\, f_l(y_>)$ is given by

$$g_l(y_<) f_l(y_>) = \frac{1}{\sqrt{y_< + 1}\sqrt{y_> + 1}} \left[\left(\frac{y_< + \sqrt{y_<^2 - 1}}{y_> + \sqrt{y_>^2 - 1}} \right)^{\frac{1}{2}+l} + \left(\left(y_< + \sqrt{y_<^2 - 1} \right) \left(y_> + \sqrt{y_>^2 - 1} \right) \right)^{-l-\frac{1}{2}} \right] \tag{3.28}$$

For the sake of notational convenience, let us define

$$A = y_> + \sqrt{y_>^2 - 1} \qquad \text{and} \qquad B = y_< + \sqrt{y_<^2 - 1} \,. \tag{3.29}$$

Using Eq. (3.28), and the standard expression for the generating function for Legendre polynomials

$$\sum_{l=0}^{\infty} t^l P_l(x) = \frac{1}{\sqrt{1 - 2xt + t^2}} \,, \tag{3.30}$$

we find that Eq. (3.27) takes the form

$$\Phi\left(\vec{y}_<\,, \vec{y}_>\right) = \frac{1}{m} \frac{1}{\sqrt{y_< + 1}\sqrt{y_> + 1}} \left[\frac{\sqrt{AB}}{\sqrt{A^2 + B^2 - 2\,A\,B\,\cos\gamma}} + \frac{\sqrt{AB}}{\sqrt{A^2\,B^2 + 1 - 2\,A\,B\,\cos\gamma}} \right] . \tag{3.31}$$

To write the solution in terms of Schwarzschild coordinates, we simply make the substitution for y, and write

$$\Phi\left(\vec{r},\vec{r}'\right)=\frac{1}{\sqrt{rr'}}\left[\frac{\sqrt{\left(\kappa(r)r-m\right)\left(\kappa(r')r'-m\right)}}{\sqrt{\left(\kappa(r)r-m\right)^2+\left(\kappa(r')r'-m\right)^2-2\,\left(\kappa(r)r-m\right)\,\left(\kappa(r')r'-m\right)\,\cos\gamma}}\right.$$
$$\left.+\frac{m\sqrt{\left(\kappa(r)r-m\right)\left(\kappa(r')r'-m\right)}}{\sqrt{\left(\kappa(r)r-m\right)^2\,\left(\kappa(r')r'-m\right)^2+m^4-2\,m^2\left(\kappa(r)r-m\right)\,\left(\kappa(r')r'-m\right)\,\cos\gamma}}\right], \tag{3.32}$$

where we have defined $\kappa(r)=1+\lambda(r)=1+\sqrt{1-\frac{2m}{r}}$. As noted earlier, we see that as we take the flat space limit ($m\to 0$), this solution as well as that of Eq. (2.9) reduce to the Green function of flat space. We also note that just as in the Green function result given in the previous section, this solution is regular at the horizon.

B. de Sitter background

We now turn our attention to writing a closed form expression for the Green function on a de Sitter background. The scalar de Sitter Green function for cosmological spacetimes has been derived in [26–28]. In static coordinates, the thermal Green function for the massless scalar field equation [29], as well as the Green function for the massive scalar field equation [30, 31] are known in the literature. These Green functions correspond to the de Sitter generalization of Eq. (1.8), whereas we will be concerned with the derivation of the solution of the inverse spatial Laplacian, i.e. of Eq. (1.2), but the procedure described in this subsection can be used for finding the solution of Eq. (1.8) as well.

For pure de Sitter space with cosmological constant Λ, we have $\lambda(r)^2=1-\frac{r^2}{L^2}$, where $L=\sqrt{\frac{3}{\Lambda}}$. In order to solve Eq. (3.2), let us again make a change of coordinates and write $y=\frac{r}{L}$. For this choice, Eq. (3.2) takes the form

$$\sin\theta\left[\partial_y\left(y^2\sqrt{1-y^2}\partial_y\Phi\right)+\frac{1}{\sqrt{1-y^2}}\left(\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right)\;+\;\frac{1}{\sin^2\theta}\partial_\phi^2\Phi\right)\right]$$
$$=-4\pi\frac{\delta(y-y')}{L}\delta(\theta-\theta')\delta(\phi-\phi')\,. \tag{3.33}$$

The delta functions for the angular variables satisfy the same relations given in Eq. (3.5), but the y delta function now satisfies

$$\int_0^1 dy\delta(y-y') = 1\,.$$

As in the Schwarzschild case, we begin by solving the source free equation

$$\sqrt{1-y^2}\partial_y\left(y^2\sqrt{1-y^2}\partial_y\Phi\right) + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Phi\right) + \frac{1}{\sin^2\theta}\partial^2_\phi\Phi = 0\,, \tag{3.34}$$

with

$$\Phi(y,\theta,\phi) = \sum_{l=0}^{\infty} R_l(y)P_l(\cos\gamma)\,. \tag{3.35}$$

Substituting Eq. (3.35) in Eq. (3.34), and using Eq. (3.8), we get the equation

$$\sqrt{1-y^2}\frac{d}{dy}\left(y^2\sqrt{1-y^2}\frac{d}{dy}R_l(y)\right) - l(l+1)R_l(y) = 0\,. \tag{3.36}$$

To find the general solution in this case, it will be convenient to express Eq. (3.36) in terms of $t=\sqrt{1-y^2}$, which results in

$$\sqrt{1-t^2}\frac{d}{dt}\left((1-t^2)^{\frac{3}{2}}\frac{d}{dt}R_l(t)\right) - l(l+1)R_l(t) = 0\,. \tag{3.37}$$

Using the ansatz $R_l(t) = P^\nu_\mu(t)A(t)$ as before (see Appendix A) , we find the following general solution

$$R_l(t) = A'_l(1-t^2)^{-\frac{1}{4}}P^{l+\frac{1}{2}}_{\frac{1}{2}}(t) + B'_l(1-t^2)^{-\frac{1}{4}}Q^{l+\frac{1}{2}}_{\frac{1}{2}}(t)\,. \tag{3.38}$$

The Legendre polynomials described in Eq. (3.38) can be described in terms of hypergeometric functions. For Legendre polynomials defined in the region between -1 and $+1$, we have [24]

$$\begin{aligned}\Gamma(1-\mu)P^\mu_\nu(x) &= 2^\mu(1-x^2)^{-\frac{\mu}{2}}\,{}_2F_1\left(\frac{1}{2}+\frac{\nu}{2}-\frac{\mu}{2}, -\frac{\nu}{2}-\frac{\mu}{2}; 1-\mu; 1-x^2\right)\,,\\ \frac{2}{\pi}\sin(\pi\mu)Q^\mu_\nu(x) &= \cos(\pi\mu)P^\mu_\nu(x) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P^{-\mu}_\nu(x)\,.\end{aligned} \tag{3.39}$$

By using the expressions in Eq. (3.39), and writing the results in terms of the variables y by substituting $1-t^2=y^2$, one can find the following general solution

$$R_l(y) = A_l g_l(y) + B_l f_l(y)\,, \tag{3.40}$$

where $g_l(y)$ and $f_l(y)$ are now given by

$$\begin{aligned} g_l(y) &= y^l\,{}_2F_1\left(\frac{l}{2},\frac{l}{2}+1;\frac{3}{2}+l;y^2\right) \\ f_l(y) &= \frac{1}{y^{l+1}}\,{}_2F_1\left(\frac{-l-1}{2},\frac{-l+1}{2};\frac{1}{2}-l;y^2\right)\,. \end{aligned} \tag{3.41}$$

Here, A_l and B_l are real coefficients, and the solutions themselves are positive and real in the region between 0 and +1. The Wronskian of the two solutions given in Eq. (3.41 can be found to satisfy the following relation

$$W(g_l(y),f_l(y),y) = -\frac{2l+1}{y^2\sqrt{1-y^2}}\,, \tag{3.42}$$

which is again the expected form of the Wronskian given two linearly independent solutions of Eq. (3.36).

Unlike in the Schwarzschild case, we were unable to determine an exact representation of the solutions for arbitrary l. The solutions for specific choices of l however can be easily determined. Using the derivative relations satisfied by the hypergeometric functions, we have derived in Appendix B the following general form of the $f_l(y)$ solutions

$$\begin{aligned} f_l(y) &= \sum_{n=0}^{\frac{l-1}{2}} \frac{c_n}{y^{2n+2}} && (l\,\text{odd})\,, \\ &= \frac{\sqrt{1-y^2}}{y} && (l=0)\,, \\ &= \sqrt{1-y^2}\sum_{n=1}^{\frac{l}{2}} \frac{c_n}{y^{2n+1}} && (l\,\text{even}\,;l\neq 0)\,, \end{aligned} \tag{3.43}$$

where $c_{\frac{l-1}{2}} = 1$ for the odd l case, and $c_{\frac{l}{2}} = 1$ for the even l case.

To proceed further, we need to determine the behaviour of the solution in the limit $y \to 0$ and $y \to 1$. As before, $g_0(y) = 1$, which follows from ${}_2F_1\left(0,1;\frac{3}{2};y^2\right) = 1$, and will not be considered in the following. As $y \to 0$, we can make use of the following derivative relation satisfied by the hypergeometric functions

$$\frac{d}{dx}\,{}_2F_1\left(a,b,c,x\right) = \frac{a\,b}{c}\,{}_2F_1\left(a+1,b+1,c+1,x\right)\,, \tag{3.44}$$

as well as ${}_2F_1\left(a,b,c,0\right) = 1$, to determine the behaviour of the solutions. We see that $g_l(y)$ and its first derivative vanish, while $f_l(y)$ and its first derivative diverge for all values of values of l . We must thus set $B_l = 0$ in Eq. (3.41) in the region where y can vanish, in order to have regular solutions.

As $y \to 1$, we need to consider the integral representation of the hypergeometric function to demonstrate that $g_l(y)$ is finite while its first derivative diverges, for $l \neq 0$. This is shown in Appendix B. The solutions provided in Eq. (3.43) tell us the following about the $f_l(y)$ solutions in the limit $y \to 1$. While $f_l(y)$ remains finite for all l, and $f_l(1) = 0$ when l is even, the behaviours of the first derivatives differ for even and odd l. The first derivative of $f_l(y)$ diverges when l is even, and is finite when l is odd. Regularity of the solutions requires that in the region where $y \to 1$, we not only set $A_l = 0$ for all $l \neq 0$, but also set $B_l = 0$ for even l.

We can now determine the general solution $\Phi(y, \theta, \phi)$ for the point source located at (y', θ', ϕ'). Away from the source the solution is given by Eq. (3.35). As explained above, in the region $y < y'$ we simply set $B_l = 0$ and sum over all l. In the region $y > y'$ we set $A_l = 0$ for all $l \neq 0$ and sum over all odd l, but we in addition have the $g_0(y) = 1$ term which contributes a constant term. Then we can write

$$\begin{aligned} \Phi(y, \theta, \phi) &= \sum_{l=0}^{\infty} A_l g_l(y) P_l(\cos\gamma) \qquad (y < y')\,, \\ &= A'_0 + \sum_{l=0}^{\infty} B_{2l+1} f_{2l+1}(y) P_{2l+1}(\cos\gamma) \qquad (y > y')\,. \end{aligned} \tag{3.45}$$

Finally, we need to match these solutions at $y = y'$. This leads us to define the constant $C_{2l+1} = \frac{A_{2l+1}}{f_{2l+1}(y')} = \frac{B_{2l+1}}{g_{2l+1}(y')}$, and we also find that A_k vanishes for even $k(\neq 0)$. Then we can write

$$\begin{aligned} \Phi(y, \theta, \phi) &= C_0 + \sum_{l=0}^{\infty} C_{2l+1} g_{2l+1}(y) f_{2l+1}(y') P_{2l+1}(\cos\gamma) \qquad (y < y')\,, \\ &= C_0 + \sum_{l=0}^{\infty} C_{2l+1} f_{2l+1}(y) g_{2l+1}(y') P_{2l+1}(\cos\gamma) \qquad (y > y')\,, \end{aligned} \tag{3.46}$$

where we have denoted the $g_0(y)$ contribution as the constant C_0. Multiplying both sides of Eq. (3.33) with $P_{2l'+1}(\cos\gamma)$ and integrating with respect to θ and ϕ, we get

$$-\frac{\delta(y - y')}{L} = \frac{1}{4l+3}\left[\frac{d}{dy}\left(y^2\sqrt{1-y^2}\frac{d}{dy}R_{2l+1}(y)\right) - \frac{(2l+1)(2l+3)}{\sqrt{1-y^2}}R_{2l+1}(y)\right], \tag{3.47}$$

where we have used the normalization and orthogonality conditions Eq. (3.9) and Eq. (3.10). Using Eq. (3.46) we can write $R_{2l+1}(y)$ as

$$\begin{aligned} R_{2l+1}(y) &= C_{2l+1} g_{2l+1}(y) f_{2l+1}(y') \qquad (y < y')\,, \\ &= C_{2l+1} f_{2l+1}(y) g_{2l+1}(y') \qquad (y > y')\,. \end{aligned} \tag{3.48}$$

We next integrate over y from $y' - \epsilon$ to $y' + \epsilon$, i.e. over an infinitesimal region about the point source, for which we find

$$
\begin{aligned}
-\frac{1}{L} &= \frac{1}{4l+3} C_{2l+1} {y'}^2 \sqrt{1 - y'^2} \left[g_{2l+1}(y') \left(\frac{d}{dy} f_{2l+1}(y) \right) \Big|_{y'+\epsilon} - f_{2l+1}(y') \left(\frac{d}{dy} g_{2l+1}(y) \right) \Big|_{y'-\epsilon} \right] \\
&= \frac{1}{4l+3} C_{2l+1} {y'}^2 \sqrt{1 - y'^2} W(g_{2l+1}(y'), f_{2l+1}(y'), y') \\
&= -C_{2l+1} \,,
\end{aligned} \tag{3.49}
$$

where we made use of Eq. (3.42) in going from the second to the third equality given in Eq. (3.49). Using this, we can write the Green function in the de Sitter case as

$$
\Phi\left(\vec{y}_<, \vec{y}_>\right) = \frac{1}{L} \sum_{l=0}^{\infty} g_{2l+1}(y_<) f_{2l+1}(y_>) P_{2l+1}(\cos\gamma) \,, \tag{3.50}
$$

where $y_< = \min(y, y')$ and $y_> = \max(y, y')$ as before. Unlike in the Schwarzschild case, we have not been able to write this in a simpler form. We can nonetheless substitute for y in Eq. (3.41) and use this in Eq. (3.50), by writing $y_< = \frac{r_<}{L}$ and $y_> = \frac{r_>}{L}$, to find the solution in terms of r ,

$$
\begin{aligned}
\Phi\left(\vec{r}_<, \vec{r}_>\right) = \frac{1}{r_>^2} \sum_{l=0}^{\infty} \left(\frac{r_<}{r_>} \right)^{2l+1} {}_2F_1\left(l + \frac{1}{2}, l + \frac{3}{2}, 2l + \frac{5}{2}, \frac{3r_<^2}{\Lambda} \right) \\
{}_2F_1\left(-l - 1, -l, -2l - \frac{1}{2}, \frac{3r_>^2}{\Lambda} \right) P_{2l+1}(\cos\gamma) \,.
\end{aligned} \tag{3.51}
$$

IV. CONCLUSION

In this paper, we have discussed a new class of static, scalar Green functions on spherically symmetric spacetimes, those corresponding to the inverse spatial Laplacian defined exclusively on the spatial hypersurface of the spherically symmetric spacetime. Specifically, we have derived the inverse spatial Laplacian in the form of mode solutions for the Schwarzschild and pure de Sitter backgrounds. We have determined the closed form expression for Green function on Schwarzschild spacetime in terms of elementary functions, and on the pure de Sitter space in terms of hypergeometric functions. Since both the de Sitter and Schwarzschild cases admit a mode expansion, where the functions depending on r are ultimately associated Legendre polynomials, it seems plausible to presume that a similar result would hold for the Schwarzschild-de Sitter background. Unfortunately, we have been unable to find a simple transformation for this case since the cubic dependence on r in the lapse function λ poses a significant obstacle to the procedure. From the nature of the equation to solve for the Schwarzschild-de Sitter background, the solution for the corresponding Green function will require a different approach than what was considered here. We look forward to addressing the Green function for this background in future work.

Appendix A: Derivation of the general solution of the homogeneous equations

We seek to solve Eq. (3.11) and Eq. (3.37), which take the general form

$$(1-y^2)\frac{d^2}{dy^2}R(y) + f(y)\frac{d}{dy}R(y) + g(y)R(y) - l(l+1)R(y) = 0\,. \tag{A1}$$

We will solve this equation, for the cases of Eq. (3.11) and Eq. (3.37) by making use of the ansatz $R(y) = P_\nu^\mu(y)A(y)$. We first recall that the Legendre polynomial is a solution of the following differential equation

$$(1-y^2)\frac{d^2}{dy^2}P_\nu^\mu(y) - 2y\frac{d}{dy}P_\nu^\mu(y) + \left[\nu(\nu+1) - \frac{\mu^2}{1-y^2}\right]P_\nu^\mu(y) = 0\,. \tag{A2}$$

Expanding Eq. (3.11), we find

$$(1-y^2)\frac{d^2}{dy^2}R(y) - (2y-1)\frac{d}{dy}R(y) + l(l+1)R(y) = 0\,. \tag{A3}$$

Using $R(y) = P_\nu^\mu(y)A(y)$, and substituting for $(1-y^2)\frac{d^2}{dy^2}P_\nu^\mu(y)$ by making use of Eq. (A2), we get

$$A(y)\left[-\left(\nu(\nu+1) - \frac{\mu^2}{1-y^2}\right)P_\nu^\mu(y) + \frac{d}{dy}P_\nu^\mu(y)\right] + P_\nu^\mu(y)\left[(1-y^2)\frac{d^2}{dy^2}A(y) - (2y-1)\frac{d}{dy}A(y)\right]$$
$$+ 2(1-y^2)\frac{d}{dy}P_\nu^\mu(y)\frac{d}{dy}A(y) + l(l+1)P_\nu^\mu(y)A(y) = 0\,. \tag{A4}$$

Collecting terms, we have

$$\frac{d}{dy}P_\nu^\mu(y)\left[2(1-y^2)\frac{d}{dy}A(y) + A(y)\right]$$
$$+ P_\nu^\mu(y)\left[(1-y^2)\frac{d^2}{dy^2}A(y) - (2y-1)\frac{d}{dy}A(y) - \left(\nu(\nu+1) - \frac{\mu^2}{1-y^2} - l(l+1)\right)A(y)\right] = 0\,. \tag{A5}$$

We can now explore the simplest possibility which makes Eq. (A5) true, namely, that the coefficients of $\frac{d}{dy}P^\mu_\nu(y)$ and $P^\mu_\nu(y)$ individually vanish. The coefficient is of $\frac{d}{dy}P^\mu_\nu(y)$ can be trivially solved to give the following solution for $A(y)$

$$A(y) = \left(\frac{y-1}{y+1}\right)^{\frac{1}{4}} . \tag{A6}$$

Substituting this solution back in Eq. (A5) gives us the following expression

$$\left(\frac{1}{4} - \mu^2\right)(y-1)^{-\frac{3}{4}}(y+1)^{-\frac{5}{4}} - (\nu(\nu+1) - l(l+1))\,(y-1)^{\frac{1}{4}}(y+1)^{-\frac{1}{4}} = 0 . \tag{A7}$$

Eq. (A7) holds, provided $\mu = \frac{1}{2}$ and $\nu = l$.

One solution of Eq. (A4) is thus $\left(\frac{y-1}{y+1}\right)^{\frac{1}{4}} P_l^{\frac{1}{2}}(y)$. Since our procedure made use of the Legendre polynomials, we would get another solution by simply using $R(y) = Q^\mu_\nu(y)A(y)$, with the same solution for $A(y)$. The general solution is thus found to be

$$R_l(y) = A_l \left(\frac{y-1}{y+1}\right)^{\frac{1}{4}} P_l^{\frac{1}{2}}(y) + B_l \left(\frac{y-1}{y+1}\right)^{\frac{1}{4}} \left(iQ_l^{\frac{1}{2}}(y)\right) . \tag{A8}$$

Eq. (3.11) is written as it is since $i\,Q_l^{\frac{1}{2}}(y)$ is a real solution.

This procedure can similarly be used in Eq. (3.37), which can be written as

$$(1-t^2)\frac{d^2}{dt^2}R(t) - 3t\frac{d}{dt}R(t) - \frac{l(l+1)}{(1-t^2)}R(t) = 0 . \tag{A9}$$

Substitution of the ansatz $R(t) = P^\mu_\nu(t)A(t)$ now leads to the following equation

$$\frac{d}{dt}P^\mu_\nu(t)\left[2\frac{d}{dt}A(t)(1-t^2) - A(t)t\right] + P^\mu_\nu(t)\left[(1-t^2)\frac{d^2}{dt^2}A(t) - 3t\frac{d}{dt}A(t) - \left(\nu(\nu+1) - \frac{\mu^2}{1-t^2} + \frac{l(l+1)}{1-t^2}\right)A(t)\right] = 0 . \tag{A10}$$

As before, we assume the possibility that the coefficients of the $P^\mu_\nu(t)$ and $\frac{d}{dt}P^\mu_\nu(t)$ separately vanish. The coefficient of the latter term vanishing leads to the following simple result for A(t)

$$A(t) = (1-t^2)^{-\frac{1}{4}} . \tag{A11}$$

Substituting this equation back into Eq. (A10) leads to the following result

$$-\left[\frac{3}{4} - \nu(\nu+1)\right]t^2 - \left[\nu(\nu+1) - \frac{1}{2} + l(l+1) - \mu^2\right] = 0 , \tag{A12}$$

which is satisfied for the choice of $\nu = \frac{1}{2}$ and $\mu = l + \frac{1}{2}$. This leads to the general solution being given by

$$R_l(t) = A_l(1-t^2)^{-\frac{1}{4}} P_{\frac{1}{2}}^{l+\frac{1}{2}}(t) + B_l(1-t^2)^{-\frac{1}{4}} Q_{\frac{1}{2}}^{l+\frac{1}{2}}(t)\,, \tag{A13}$$

which is Eq. (3.38).

Appendix B: Limits of the de Sitter solutions as $y \to 1$

Let us first note that the hypergeometric functions given in Eq. (3.41) are of the form ${}_2F_1\left(a, a+1; 2a+\frac{3}{2}; y^2\right)$, where $a = \frac{l}{2}$ and $a = \frac{-l-1}{2}$ correspond to the two hypergeometric functions contained in $g_l(y)$ and $f_l(y)$ respectively. There exists a known formula for evaluating the hypergeometric functions at the point $y^2 = 1$. This formula is given by [24]

$${}_2F_1\left(a, b, c, 1\right) = \frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)} \qquad \Re\left(a+b-c\right) < 0\,;\; c \neq 0, -1, -2, ... \tag{B1}$$

This formula applies to the hypergeometric functions included in f_l and g_l, but not to their derivatives. Let us consider the functions separately to find their derivatives at $y^2 = 1$.

1. $f_l(y)$ solutions and hypergeometric functions

For the f_l solutions, we need to find the expressions explicitly in order to determine the nature of the derivatives at the point $y = 1$. For the values of $l = 0, 1, 2$ and 3, the corresponding hypergeometric functions are, respectively,

$$\begin{aligned}
&{}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; y^2\right) && (l=0)\,;\\
&{}_2F_1\left(-1, 0; -\frac{1}{2}; y^2\right) && (l=1)\,;\\
&{}_2F_1\left(-\frac{3}{2}, -\frac{1}{2}; -\frac{3}{2}; y^2\right) && (l=2)\,;\\
&{}_2F_1\left(-2, -1; -\frac{5}{2}; y^2\right) && (l=3)\,.
\end{aligned} \tag{B2}$$

Here we see that ${}_2F_1\left(a, b; c; y^2\right)$ and ${}_2F_1\left(a-1, b-1; c-2; y^2\right)$ represent two successive even (odd) solutions when $(a, b; c) = \left(\frac{-l-1}{2}, \frac{-l+1}{2}; -l+\frac{1}{2}\right)$.

Two hypergeometric functions which are contiguous are related to one another through certain differentiation formulas [24]. Let us look at the ones relevant to the f_l functions.

These are

$$
{}_2F_1\left(a-n,b-n;c-n;z\right)=\frac{1}{(c-n)_n}(1-z)^{c+n-a-b}z^{1+n-c}\frac{d^n}{dz^n}\left[(1-z)^{a+b-c}\,z^{c-1}\,{}_2F_1\left(a,b;c;z\right)\right]\,,
$$

$$
{}_2F_1\left(a,b;c-n;z\right)=\frac{1}{(c-n)_n}z^{1+n-c}\frac{d^n}{dz^n}\left[z^{c-1}\,{}_2F_1\left(a,b;c;z\right)\right]\,, \tag{B3}
$$

where $(k)_n=\frac{\Gamma(k+n)}{\Gamma(k)}$ is Pochhammer's symbol. Using the two relations in Eq. (B3), we can write

$$
\begin{aligned}
&{}_2F_1\left(a-n,\ b-n;c-2n;z\right)\\
&\quad=\frac{z^{1+2n-c}}{(c-n)_n(c-2n)_n}\frac{d^n}{dz^n}\left[(1-z)^{c+n-a-b}\frac{d^n}{dz^n}\left[z^{c-1}(1-z)^{a+b-c}\,{}_2F_1\left(a,b;c;z\right)\right]\right]
\end{aligned} \tag{B4}
$$

Using ${}_2F_1\left(a,b;c;z\right)={}_2F_1\left(a,a+1;2a+\frac{3}{2};z\right)$ in Eq. (B4) provides the relevant recursive relation for the f_l hypergeometric functions. To simplify the notation in what follows, let us define

$$
{}_2F_1\left(\frac{-l-1}{2},\frac{-l+1}{2};-l+\frac{1}{2};z\right)=F_l(z)\,. \tag{B5}
$$

With this definition, the solutions we seek are given by $f_l(y)=\frac{F_l(y^2)}{y^{l+1}}$. Combining Eq. (B5) with Eq. (B4), we can write

$$
F_{l+2n}(z)=\frac{z^{2n+\frac{1}{2}+l}}{\left(l+\left(n-\frac{1}{2}\right)\right)_n\left(l+\left(2n-\frac{1}{2}\right)\right)_n}\frac{d^n}{dz^n}\left[(1-z)^{n+\frac{1}{2}}\frac{d^n}{dz^n}\left[z^{-l-\frac{1}{2}}(1-z)^{-\frac{1}{2}}F_l(z)\right]\right]\,. \tag{B6}
$$

For the recursion relation, we need only consider Eq. (B6) with $n=1$,

$$
F_{l+2}(z)=\frac{z^{\frac{5}{2}+l}}{\left(l+\frac{1}{2}\right)\left(l+\frac{3}{2}\right)}\frac{d}{dz}\left[(1-z)^{\frac{3}{2}}\frac{d}{dz}\left[z^{-l-\frac{1}{2}}(1-z)^{-\frac{1}{2}}F_l(z)\right]\right]\,, \tag{B7}
$$

which upon evaluating the derivatives can be written as

$$
F_{l+2}(z)=\left(1-\frac{(2l+1)(2l+2)z}{3+8l+4l^2}\right)F_l(z)-\frac{(8l+4)z-(8l+2)z^2}{3+8l+4l^2}F_l'(z)-\frac{4(z^3-z^2)}{3+8l+4l^2}F_l''(z)\,, \tag{B8}
$$

where primes denote differentiation with respect to z. As we will see below, for even l we can extract a factor of $\sqrt{1-z}$ to write the functions $F_l(z)$ in the form $F_l(z)=\sqrt{1-z}D_l(z)$. Substituting this in Eq. (B7), we find a recursion relation for the functions $D_l(z)$,

$$
D_{l+2}(z)=\left[\left(1-\frac{2l(2l+1)z}{3+8l+4l^2}\right)D_l(z)-\frac{(8l+4)z-(8l-2)z^2}{3+8l+4l^2}D_l'(z)-\frac{4(z^3-z^2)}{3+8l+4l^2}D_l''(z)\right]\,. \tag{B9}
$$

We will now need the lowest order solutions to proceed further. The lowest order solution for even l is $f_0(z)$, which corresponds to $F_0(z) = {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; z\right)$, as shown in Eq. (B2). The hypergeometric function ${}_2F_1\left(\frac{\alpha}{2}, -\frac{\alpha}{2}; \frac{1}{2}; z\right)$ has the following known representation

$$ {}_2F_1\left(\frac{\alpha}{2}, -\frac{\alpha}{2}; \frac{1}{2}; z\right) = \cos\left(\alpha \sin^{-1}\left(\sqrt{z}\right)\right) , \tag{B10} $$

and the case where $\alpha = -1$ is the one we require. The lowest order solution for odd l is $f_1(z)$, which corresponds to $F_1(z) = {}_2F_1\left(-1, 0; -\frac{1}{2}; z\right)$. From the definition of the hypergeometric function ${}_2F_1(a, b; c; z)$

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} , \tag{B11} $$

we know that ${}_2F_1(0, b; c; z) = {}_2F_1(a, 0; c; z) = {}_2F_1(a, b; c; 0) = 1$. This tells us that the lowest order solutions are simply

$$ \begin{aligned} F_0(z) &= {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}; \frac{1}{2}; z\right) = \sqrt{1-z} , \\ F_1(z) &= {}_2F_1\left(-1, 0; -\frac{1}{2}; z\right) = 1 . \end{aligned} \tag{B12} $$

We will use these to derive the expressions for the $f_l(y)$ functions given in Eq. (3.43). We begin with the even l solutions. It can be seen that all even l solutions are of the form $F_l(z) = \sqrt{1-z}D_l(z)$. This follows directly from the fact that $F_0(z) = \sqrt{1-z}D_0(z)$, where $D_0(z) = 1$, and Eq. (B9). The operator in Eq. (B9) takes a polynomial and produces another polynomial of one order higher. The only exception is $D_0(z) = 1$ which, when inserted into Eq. (B9), produces $D_2(z) = 1$. We can also calculate directly that $F_2(z) = \sqrt{1-z} = \sqrt{1-z}D_2(z)$. It follows from Eq. (B9) that $D_{2k}(z)$ is a polynomial of order $(k-1)$ in z.

For $F_l(z)$ corresponding to odd l, we can use the recursion relation of Eq. (B8) directly. For example, substitution of $F_1(z) = 1$ in Eq. (B8) leads to $F_3(z) = 1 - \frac{4}{5}z$, substituting this result back in Eq. (B8) results in $F_5(z) = 1 - \frac{4}{3}z + \frac{8}{21}z^2$, etc. Thus $F_{2k+1}(z)$ is a polynomial of order k in z.

We can now make a change of variable to $z = y^2$ in all the $F_l(z)$ solutions, and consider $f_l(y) = \frac{F_l(y^2)}{y^{l+1}}$ to find the $f_l(y)$ solutions shown in Eq. (3.43).

2. Limits of the $g_l(y)$ solutions

The limit of the g_l hypergeometric functions and their derivatives are most easily determined by making use of the following integral representation for hypergeometric functions

$$ {}_2F_1\left(a,b,c,x\right) = \frac{\Gamma\left(c\right)}{\Gamma\left(b\right)\Gamma\left(c-b\right)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a}dt \qquad (\Re(c)>\Re(b)>0)\ , \tag{B13}$$

As we have seen, the f_l hypergeometric functions have $c \leq b$ for all choices of l, with the equality holding true for the $l=0$ case. Hence, we could not use Eq. (B13) for those functions, and had to make use of the treatment described earlier. For the g_l hypergeometric functions, $a=\frac{l}{2}$, which guarantees that ${}_2F_1\left(a,a+1,2a+\frac{3}{2},y^2\right)$ always has $c>b>0$. This also holds true for the derivative of this function on account of Eq. (3.44). We can thus consider Eq. (B13) in terms of the g_l hypergeometric functions we are dealing with, in which case we have

$$ {}_2F_1\left(\frac{l}{2},\frac{l}{2}+1,l+\frac{3}{2},y^2\right) = \frac{\Gamma\left(l+\frac{3}{2}\right)}{\Gamma\left(\frac{l}{2}+1\right)\Gamma\left(\frac{l+1}{2}\right)}\int_0^1\left(\frac{t}{1-ty^2}\right)^{\frac{l}{2}}(1-t)^{\frac{l-1}{2}}dt\,, \tag{B14}$$

while the derivative of this function takes the form

$$ \partial_y\left({}_2F_1\left(\frac{l}{2},\frac{l}{2}+1,l+\frac{3}{2},y^2\right)\right) = \frac{\Gamma\left(l+\frac{3}{2}\right)ly}{\Gamma\left(\frac{l}{2}+1\right)\Gamma\left(\frac{l+1}{2}\right)}\int_0^1\left(\frac{t}{1-ty^2}\right)^{\frac{l}{2}+1}(1-t)^{\frac{l-1}{2}}dt\,, \tag{B15}$$

The explicit representation of the g_l hypergeometric functions and their derivatives, in terms of elementary functions, can now be derived using thse equations for specific choices of l. Since we are interested in the nature of the limit of these functions as $y\to 1$ for any choice of l, we can simply take this limit in the above expressions, and then evaluate the integrals. This amounts to the evaluation of standard integrals. We find that Eq. (B13) gives us the following finite result

$$ {}_2F_1\left(\frac{l}{2},\frac{l}{2}+1,l+\frac{3}{2},1\right) = \frac{2\sqrt{\pi}}{(l+1)}\frac{\Gamma\left(l+\frac{3}{2}\right)}{\left(\Gamma\left(\frac{l+1}{2}\right)\right)^2}\,, \tag{B16}$$

while Eq. (B15) diverges for all choices of l.

Similarly, the $y\to 0$ limits can also be determined very simply by using substituting $y=0$ in Eq. (B14) and (B15). One can easily find that ${}_2F_1\left(\frac{l}{2},\frac{l}{2}+1,l+\frac{3}{2},0\right)=1$, by working out the integral. The derivative of this function at $y=0$ vanishes on account of an overall factor of y, and the fact that the integral is finite.

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