A Generalized Sagnac-Wang-Fizeau formula

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Abstract

We present a special-relativistic analysis of deformable interferometers where counter propagating beams share a common optical path. The optical path is allowed to change rather arbitrarily and need not be stationary. We show that, in the absence of dispersion the phase shift has two contributions. To leading order in v/c one contribution is given by Wang empirical formula for deformable Sagnac interferometers. The second contribution is due to the stretching of the optical path and we give an explicit formula for this stretch term valid to first order in v/c. The analysis provides a unifying framework incorporating the Sagnac, Wang and Fizeau effects in a single scheme and gives a rigorous proof of Wang empirical formula.

1 Introduction

In this work we present a general framework for treating interferometers where two counter-propagating beams share a common path which starts and ends at the same point. Three interferometers in this class are Sagnac [8], Fizeau [3] and Wang [10].

The Sagnac interferometer is shown schematically in Fig. 1. It rotates in the lab as a rigid body with an angular velocity $\mathbf{\Omega}$. The phase difference of the two beams at the detector, $\Delta\Phi$, is proportional to $\mathbf{\Omega}$. In the approximation $\gamma \cong 1$, Sagnac area law says:

$$\Delta\Phi \cong \frac{4\omega}{c^2}\,\mathbf{\Omega}\cdot\mathbf{A}\,. \tag{1}$$

Here $\mathbf{A}$ is the area enclosed by the path of two beams, ω is the frequency of the light source and c the velocity of light in vacuum. In applications, the Sagnac effect is used to measure $\mathbf{\Omega}$. It is remarkable that the Sagnac effect is independent of the refractive index n [7, 1], suggesting a universal geometric origin for the effect; Hence the area law (1) holds also when a co-moving dielectric medium is put in the path of the light beams [7, 6]. In particular it holds for interferometers made with optical fibers. It has been extensively tested and lies at the heart of the Fiber Optics Gyro and Ring Laser Gyro [7, 2].

The Sagnac effect was originally born in the context of optics. However, similar ideas apply to other wave phenomena and in particular to matter waves [6, 4, 5].

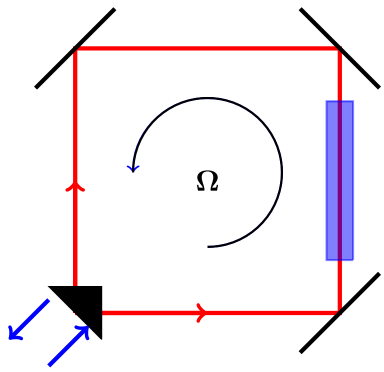

Figure 1: A Sagnac interferometer. The interferometer rotates as a rigid body in the lab with angular velocity $\boldsymbol{\Omega}$. The beam splitter is marked by the black triangle. The two counter-propagating beams are marked by red arrows; One beam propagates with and the other against the rotation. The optical path may contain co-moving dielectric medium–the blue box.

It is instructive to contrast Sagnac with Fizeau's interferometer [3], shown schematically in Fig. 2. Here the source, mirrors and detector are at rest at the lab while a section of the path goes through a flowing liquid at (essentially uniform) velocity v_0, so that one beam propagates with the flow and a counter-propagating beam against it. The phase shift $\Delta\Phi$ is given by von Laue formula which, in the absence of dispersion, reads [9]

$$\Delta\Phi = -\frac{2\omega}{c^2} v_0 L_w \,(n^2 - 1), \tag{2}$$

where L_w is the length of the optical path in water.

There are some superficial similarities between Eq. (1) and Eq. (2), but more interesting are the differences: Eq. (2) has a term that depends on the refractive index and a term that does not while Sagnac has one term independent of n. One of our aims is to put both Sagnac and Fizeau in a common framework that would give both formulas from a unified point of view and a single formula.

For a rigid rotation

$$\mathbf{v} = \boldsymbol{\Omega} \times \mathbf{r}\,, \tag{3}$$

Sagnac area law can be written as a line integral [10]

$$\frac{4\omega}{c^2} \oint \boldsymbol{\Omega} \cdot d\mathbf{A} = \frac{2\omega}{c^2} \oint \nabla \times \mathbf{v} \cdot d\mathbf{A} = \frac{2\omega}{c^2} \oint \mathbf{v} \cdot d\boldsymbol{\ell} \tag{4}$$

along the optical path. The first equality uses Eq. (3) and the last Stokes formula. For rigid body motion writing the area law as a line integral seems like a futile exercise. The advantage of the line integral and the key observation of Wang et. al. [10] is that while Sagnac area law does not even make formal sense for deformable interferometers, the

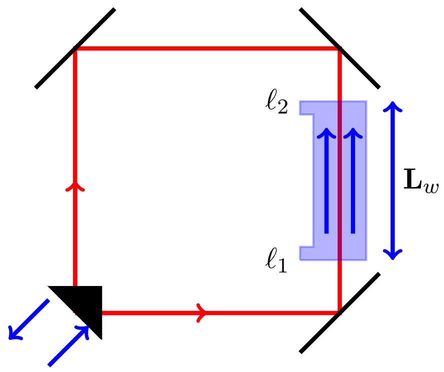

Figure 2: Schematic Fizeau interferometer. The three mirrors and beams splitter (black triangle) are at rest in the lab. The two counter-propagating beams are denoted by the red arrows. Fluid, say water, is flowing in one arm of the interferometer so one beam is moving against the flow and one with the flow.

line integral does. We therefore define

$$\Delta\Phi_{wang} = \frac{2\omega}{c^2}\oint \mathbf{v}\cdot d\boldsymbol{\ell} \tag{5}$$

In the case of non-stationary motions the integral should be understood as an integral at the time of detection in the lab.

This bring us to the third type of interferometer that we shall consider: The Wang interferometer, an example of which is shown schematically in Fig. 3. The Wang interferometer has a flexible, non-stretching fiber, moving rather arbitrarily. (The detector and light-source move with the fiber.) Wang et. al. [10] verified empirically that Eq. (5) correctly describes the interference for a variety of flexible interferometers. However, no derivation of the formula has been given there, and its domain of validity has therefore not been clarified. One of our aims is to prove Wang formula and establish its domain of validity. As we shall show Eq. (5) holds for non-stretching fibers to first order in v/c.

Here we present a unifying framework which covers the Sagnac, Fizeau and Wang effects. For fiber optics interferometers this means that we allow for flexible and stretching fibers and also allow the optical path to go through flowing fluids. The analysis is in principle elementary.

To leading order in v/c and in the absence of dispersion we find two contributions to $\Delta\Phi$:

$$\mathit{\Delta}\Phi \cong \Delta\Phi_{wang} + \Delta\Phi_{stretch}. \tag{6}$$

$\Delta\Phi_{wang}$ is Wang line integral of Eq. (5). In essence, it expresses the geometric properties of Minkowski space-time under Lorentz transformation and is independent of n. $\Delta\Phi_{stretch}$ represent the contributions of stretching. It is given by

$$\Delta\Phi_{stretch} \cong \omega\int_0^L d\ell\int_0^\ell d\tilde{\ell}\,\frac{s(\ell)-s(\tilde{\ell})}{u'(\tilde{\ell})u'(\ell)}\,, \tag{7}$$

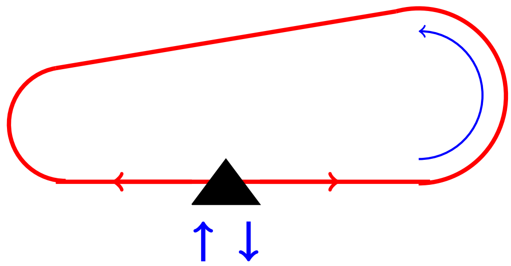

Figure 3: An example of a Wang interferometer: The optical fiber moves in the direction marked by the blue arrow. The triangle represents the co-moving beam-splitter and the two vertical arrows the light source and detector. The counter-propagating light-beams share a common path and are marked by red arrows.

where the integration variables ℓ and $\tilde{\ell}$ are length parameters along the loop, L is the total length, $u'(\ell) = c/n(\ell)$ is the velocity of light in the optical medium, and $s(\ell)$ is the local stretching rate, defined in Eq. (11) below. In the case that the motion of the optical path is non-stationary, the integral should be understood as the integral at the time of detection in the lab. It expresses the effect of change of length of given interval of the fiber between visits of the two beams. When n is constant, $\Delta\Phi_{stretch}$ is proportional to n^2.

In Wang experiments the stretching is quite small and the first term dominates. In the Fizeau experiment stretching occurs at the ends of the pipe where the velocity of the water changes. Here both terms make significant contribution and the two terms in Eq. (6) correspond to the two terms in Eq. (2) (in different order).

2 Setup of the generalized Sagnac effect

For the sake of concreteness consider a set-up of the generalized Sagnac effect involving a closed flexible and possibly stretching optical fiber moving in space. The fiber may have an arbitrary but non-dispersive refractive index $n \geq 1$.

When the fiber's radius is much smaller than the size of the loop we can think of it as a one-dimensional curve. The fiber's configuration at any given moment t is thus described, relative to the lab frame S, by the function $\mathbf{r}(\theta, t)$. Here $0 \leq \theta \leq \theta_{max}$ is a co-moving parametrization of the fiber, namely fixed θ corresponds to a fixed (material) point of the fiber. Since the fiber is a closed loop for all times t we have $\mathbf{r}(0, t) = \mathbf{r}(\theta_{max}, t)$. Note that θ is defined up to re-parametrization. Although some of the quantities defined below depend on the choice of parameter θ, the final result is of course independent of parametrization. We denote the overall fiber's length by L.

We choose a parametrization where the beam splitter is located at $\theta = 0$. It is useful to imagine that the black triangle in the figures represents a pair of (putative) source and detector co-located at $\theta = 0$ with world-line $(t, \mathbf{r}(0, t))$ and rest frame S_0. The source emits monochromatic waves with frequency ω_0 (as measured in S_0).

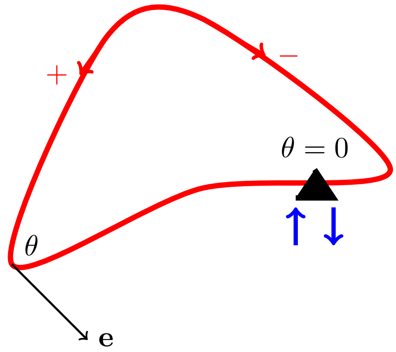

Figure 4: The parametrized curve $\mathbf{r}(\theta, t)$ is, for each fixed time t, a closed loop in space representing a fiber loop. θ is a comoving coordinate, designating fixed material points on the fiber. Shown are the tangent $\mathbf{e}$ at a point θ of the loop. The two counter-propagating beams are marked with the red arrows and the $\pm$ signs. The triangle represents the co-moving beam splitter located at $\theta = 0$ and the two blue arrows the co-moving source and detector.

Two light beams, emanating from the beam splitter, propagate in opposite directions and make one round each along the loop. The phase difference $\Delta\Phi = \Phi_+ - \Phi_-$ between the beams as they return to the beam splitter depends, among other things, on the fiber's shape and motion. Being measurable, $\Delta\Phi$ must be a Lorentz-invariant quantity.

Consider an infinitesimal fiber's segment $d\theta$, and denote its momentary rest frame by S'. The full differential of $\mathbf{r}(\theta, t)$ takes the form

$$d\mathbf{r} = (\partial_\theta \mathbf{r})\, d\theta + (\partial_t \mathbf{r})\, dt \equiv \mathbf{e}\, d\theta + \mathbf{v}\, dt \tag{8}$$

where $\mathbf{e}$ is the tangent vector and $\hat{\mathbf{e}}$ the corresponding unit vector. We denote the infinitesimal spatial displacement vector $d\boldsymbol{\ell} = \mathbf{e}\, d\theta$. The segment's velocity with respect to S (and hence the velocity of S' with respect to S) is $\mathbf{v}$ $(|\mathbf{v}| < c)$. Note that $\mathbf{e}$ depends on parametrization, but $\mathbf{v}$ and $\hat{\mathbf{e}}$ are parametrization invariant.

We denote the length of the infinitesimal segment in the lab frame by

$$d\ell = |d\boldsymbol{\ell}| = |\mathbf{e}| d\theta \tag{9}$$

The segment's proper length (namely its length in its own rest frame S') is denoted $d\ell'$. Owing to Lorentz contraction, these two quantities are related by

$$d\ell' = \left(d\ell_\perp^2 + \gamma^2 d\ell_\shortparallel^2\right)^{1/2} = \frac{\gamma}{\gamma_\perp} d\ell\,, \tag{10}$$

where $d\ell_\perp, d\ell_\shortparallel$ denote the projections of $d\boldsymbol{\ell}$ perpendicular and parallel to $\mathbf{v}$. We use here $\gamma = \left(1 - \mathbf{v}^2/c^2\right)^{-1/2}$ and $\gamma_\perp = \left(1 - {\mathbf{v}_\perp}^2/c^2\right)^{-1/2}$, where $\mathbf{v}_\perp$ is the component of $\mathbf{v}$ perpendicular to $\mathbf{e}$. By its very definition $d\ell$ (like $d\ell'$) is parametrization invariant.

If the fiber is non-stretching, $d\ell'$ of any segment is independent of time. In the case of stretching the local stretching rate — or simply the stretch — is naturally defined in the Newtonian approximation by

$$s \equiv (d\ell)^{-1}\partial_t(d\ell). \tag{11}$$

We can re-write this as

$$s = \partial_t \log d\ell = \partial_t \log |\mathbf{e}| = \hat{\mathbf{e}} \cdot \partial_\ell \mathbf{v}\,. \tag{12}$$

The last equality follows from $\partial_t \mathbf{e} = \partial_t \partial_\theta \mathbf{r} = \partial_\theta \mathbf{v}$. The partial derivative ∂_ℓ is taken at fixed t (namely $\partial_\ell = |\mathbf{e}|^{-1}\partial_\theta$).

In the relativistic framework, the stretch is naturally defined in the same manner as above, as a scalar, but in the local rest frame, namely

$$s \equiv (d\ell')^{-1}\partial_{t'}(d\ell') = \gamma\partial_t \log(d\ell') = \gamma\partial_t \log\left(\frac{\gamma}{\gamma_\perp}|\mathbf{e}|\right)$$

which differs from the Newtonian expressions (11,12) in second order of v/c.

The phase velocity of the wave with respect to the segment's rest frame S', shall be denoted by u'. It is determined by the fiber's refractive index n, through [1]

$$u' = c/n.$$

We allow n to vary along the fiber, but we assume that it is time-independent (at any given segment), dispersion free, and polarization independent. The wave's velocity at a given segment is the same for the two directions of light propagation, namely $u'_+ = u'_- \equiv u'$.

Our main concern is the phase difference $\Delta\Phi = \Phi_+ - \Phi_-$ between waves in the two directions of propagation. This phase difference results from the difference in travel times in the two directions along the loop. We denote these travel times by $T_\pm$, and their difference by ΔT:

$$\Delta T \equiv T_+ - T_-\,.$$

With ω the lab-frame frequency of the source,

$$\Delta\Phi = \omega\,\Delta T. \tag{13}$$

In a relativistic treatment, the lab frequency ω and the rest-frame frequency ω_0 differ by time dilation: $\omega = \omega_0/\gamma_0$, where $\gamma_0 \equiv \gamma(\theta = 0)$. The relativistic expression for the phase difference is therefore

$$\Delta\Phi = \frac{\omega_0}{\gamma_0}\Delta T. \tag{14}$$

[1] For single-mode fibers whose width is comparable to λ, u' is the actual phase velocity along the fiber, and $n = c/u'$.

All that is left is to calculate the travel-time difference ΔT.

We point out that Eq. (14) is accurate to order $(v/c)^3$. This is because γ_0 is generally time dependent, and may undergo tiny changes during the short time interval ΔT. In experiments this ambiguity in the value of γ_0 is totally unimportant — even its very deviation from 1 is too small to be detected. For the sake of completeness we present the precise expression for $\Delta\Phi$, taking into account the time variation of γ_0, in the Appendix. One can write this precise expression in the form (14) but with γ_0 replaced by an appropriately averaged quantity $\bar{\gamma}_0$. The precise expression for $\Delta\Phi$ is given in Eq. (44).

3 Calculation of travel times and phase difference

In this section we first analyze the travel times of the two counter-propagating signals along an infinitesimal fiber's segment. Then we use this result to calculate the integrated time difference ΔT and phase difference $\Delta\Phi$ along the closed loop.

3.1 Contribution of infinitesimal segment

Let us consider an infinitesimal fiber's segment $d\theta$. The segment has a momentary velocity $\mathbf{v}$ with respect to S, its lab-frame length is $d\ell$, Eq. (9), and its proper length is $d\ell'$, Eq. (10). The two counter-propagating waves visit the segment $d\theta$ at two different moments $t_\pm$, and have two different (lab-frame) traversal times which we denote $dt_\pm$. We shall now explicitly calculate dt_+, the calculation of dt_- will then follow analogously. For notational simplicity, we shall now omit the "+" index in the various quantities, and recover it later at the end of this subsection, adding the "±" suffix to the relevant quantities.

It is easiest to calculate the crossing time in the rest frame S', because it is in that frame that the phase velocity takes its canonical form $u' = c/n$. We obtain right away

$$dt' = \frac{d\ell'}{u'}. \tag{15}$$

However, to compute $T_\pm$ we need to calculate the corresponding lab-frame travel time dt. This quantity is easily obtained from dt' via Lorentz transformation, as we now describe.

For concreteness let us consider a nodal point of the propagating wave, and follow its travel across the segment $d\theta$. Let A denote the event that the node enters the segment and B the event that it leaves it, as shown in Fig. 5. In the lab frame, the time difference between these two events is dt ($\equiv dt_{AB} = t_B - t_A$), and their spatial difference will be denoted $d\mathbf{r}_{AB} \equiv \mathbf{r}_B - \mathbf{r}_A$. The standard Lorentz-transformation formula for dt', applied to the spacetime interval A-B, is

$$dt' = \gamma\left(dt - \frac{\mathbf{v}\cdot d\mathbf{r}_{AB}}{c^2}\right). \tag{16}$$

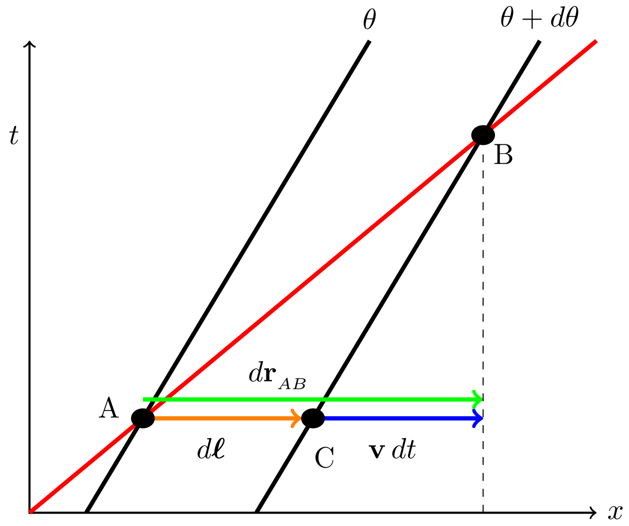

Figure 5: Space-time diagram describing the travel of a nodal point across an infinitesimal fiber's segment $d\theta$. The diagram shows the projections of all 3D vectors in one direction. The two parallel black lines are the world-lines of the two segment's edges. The red line is the world-line of the node. The lines intersect at the events A and B when the node enters and leaves the segment. Also shown are the vectors $\mathbf{e}d\theta = d\boldsymbol{\ell}$, $d\mathbf{r}_{AB}$ and $\mathbf{v}dt$ which participate in Eq. (17).

We denote by $d\boldsymbol{\ell}$ the spatial displacement vector between the two edges of the segment (at an S-frame moment of simultaneity); namely $d\boldsymbol{\ell} \equiv d\mathbf{r}_{AC} = \mathbf{e}\,d\theta$. From Fig. 5 it is clear that $d\mathbf{r}_{CB} = \mathbf{v}dt$, and hence [2]

$$d\mathbf{r}_{AB} = d\mathbf{r}_{AC} + d\mathbf{r}_{CB} = d\boldsymbol{\ell} + \mathbf{v}dt. \tag{17}$$

Extracting dt from Eq. (16) and substituting Eq. (17) we obtain

$$dt = \frac{dt'}{\gamma} + \frac{\mathbf{v}\cdot d\mathbf{r}_{AB}}{c^2} = \frac{dt'}{\gamma} + \frac{\mathbf{v}\cdot d\boldsymbol{\ell}}{c^2} + \frac{\mathbf{v}^2 dt}{c^2}. \tag{18}$$

We can now extract dt once again:

$$dt = \gamma dt' + \gamma^2 \frac{\mathbf{v}\cdot d\boldsymbol{\ell}}{c^2} = \gamma \frac{d\ell'}{u'} + \gamma^2 \frac{\mathbf{v}\cdot d\boldsymbol{\ell}}{c^2}\,.$$

So far we considered the travel time for the "+" node. When considering a "$-$" node the only change is that now $d\boldsymbol{\ell}$ is to be replaced by $-d\boldsymbol{\ell}$. We therefore obtain the exact, fully relativistic, expression for the contribution of an infinitesimal segment:

$$dt_\pm = \gamma_\pm \frac{d\ell'_\pm}{u'} \pm \gamma_\pm^2 \frac{\mathbf{v}_\pm \cdot d\boldsymbol{\ell}_\pm}{c^2}, \tag{19}$$

[2]Note that the vectors $d\mathbf{r}_{AB}$,$d\mathbf{r}_{AC}$,$d\mathbf{r}_{CB}$ need not be co-linear.

Recall that the $\pm$ nodes visit the segment at two different times $t_\pm(\theta)$. The "$\pm$" suffix in the various time-dependent quantities in Eq. (19) reflects this difference between values at t_+ and t_-. The phase velocity u' in a given segment is independent of time.

We denote by δt the contribution of the infinitesimal segment to the overall time difference ΔT:

$$\delta t \equiv dt_+ - dt_- . \tag{20}$$

The overall travel-time difference is thus

$$\Delta T = \oint \delta t \,. \tag{21}$$

3.2 The schedules $t_\pm(\theta)$

The expression (19) for $dt_\pm$ can be interpreted as a first-order ODE which determines the exact phase schedule, namely the unknowns $t_\pm(\theta)$. In the Appendix we describe in more details this ODE and its boundary condition and construct an exact formal expression for $\Delta\Phi$. However, for the rest of this paper we shall not really need this formulation. Instead, we shall now restrict the analysis to first order in the small parameter v/c (which provides an excellent approximation for all lab experiments so far).

3.3 δt to first order in v/c

To first order in v/c we may set $\gamma_\pm \cong \gamma_0 \cong 1$ and $d\ell' \cong d\ell$ in Eq. (19) which then reduces to

$$dt_\pm = \frac{d\ell_\pm}{u'} \pm \frac{\mathbf{v}_\pm \cdot d\boldsymbol{\ell}_\pm}{c^2}. \tag{22}$$

The contribution of the infinitesimal segment to the overall time difference ΔT is

$$\delta t = dt_+ - dt_- = \frac{d\ell_+ - d\ell_-}{u'} + \frac{\mathbf{v}_+ \cdot d\boldsymbol{\ell}_+ + \mathbf{v}_- \cdot d\boldsymbol{\ell}_-}{c^2} \,. \tag{23}$$

In general, the quantities $d\ell_\pm$, $d\boldsymbol{\ell}_\pm$, $\mathbf{v}_\pm$ on the R.H.S. may be time dependent, and are different for the $\pm$ beams hence their "$\pm$" suffix. However, when $v/c \ll 1$ the various quantities in the R.H.S. change little [3] during the cycle times $T_\pm$. Since we compute $\Delta\Phi$ to first order in v/c we can safely approximate the schedule by the detection time, $t_\pm(\theta) \cong t_{det}$, in the terms proportional to $\mathbf{v}_\pm$. This gives, to first order in v/c

$$\delta t \cong \frac{d\ell_+ - d\ell_-}{u'} + 2\frac{\mathbf{v} \cdot d\boldsymbol{\ell}}{c^2} \,. \tag{24}$$

We shall refer to the two terms in the R.H.S. as the stretch and Wang's terms, which we shall respectively denote δt_{st} and δt_w. In the next two sections we address these two contributions.

[3] To systematically address this issue one may introduce the notion of adiabaticity through a one-parameter family of functions $\mathbf{r}(\theta, t) = \mathbf{r}_0(\theta, \varepsilon t)$ where ε is a small parameter.

4 Non-stretching fibers: Wang formula

In the non-stretching case $d\ell$ of a given segment is fixed and hence $d\ell_+ = d\ell_-$. The first term in the R.H.S. of Eq. (24) then cancels out, yielding

$$\delta t = 2\frac{\mathbf{v}\cdot d\boldsymbol{\ell}}{c^2} \equiv \delta t_w \,. \tag{25}$$

Accumulating the δt from the infinitesimal segments of the loop we obtain

$$\Delta T = \frac{2}{c^2}\oint \mathbf{v}\cdot d\boldsymbol{\ell}\,, \tag{26}$$

therefore we proved Wang formula, Eq. (5):

$$\Delta\Phi = \frac{2\omega}{c^2}\oint \mathbf{v}\cdot d\boldsymbol{\ell} \equiv \Delta\Phi_{wang}\,. \tag{27}$$

The integration is carried out with lab time set to the detection time t_{det}. Note that to leading order in v/c we do not need to distinguish between ω and ω_0.

5 Stretching media

In the case of stretching media $d\ell$ changes with time, and $d\ell_+ - d\ell_-$ although small, no longer vanishes. However, the smallness of v compared to c (or more precisely, compared to $u' = c/n$) implies that these quantities hardly change during the short time interval $[t_+(\theta), t_-(\theta)]$. We can then approximate

$$d\ell_+ - d\ell_- \cong (t_+ - t_-)\,\partial_t(d\ell). \tag{28}$$

From Eqs. (11) and (12)

$$\partial_t(d\ell) = s\,d\ell = d\ell\ \hat{\mathbf{e}}\cdot\partial_\ell\mathbf{v}\,. \tag{29}$$

This shows that $s\,d\ell$ is first order in $\mathbf{v}$ and hence to first order in v/c

$$\delta t_{st} = \frac{d\ell_+ - d\ell_-}{u'} \cong (t_+ - t_-)\,\frac{s}{u'}\,d\ell\,. \tag{30}$$

We still need to evaluate $t_+ - t_-$. Since s is proportional to v, we only need to evaluate $t_\pm$ at zeroth order in v/c, that is, we may pretend that the fiber is static. We thus simply use $dt_\pm = \pm d\ell/u' = \pm(|\mathbf{e}|/u')d\theta$, yielding

$$t_\pm(\theta) \cong \pm\int^\theta \frac{|\mathbf{e}(\theta')|}{u'}d\theta'\,, \tag{31}$$

with boundary conditions

$$t_+(\theta_{max}) = t_-(0) = t_{det}\,.$$

This integral may be viewed as the zeroth order of the scheduling equation (42). We therefore obtain

$$t_+(\theta) - t_-(\theta) = \int_0^\theta \frac{|\mathbf{e}(\theta')|}{u'(\theta')} d\theta' - \int_\theta^{\theta_{max}} \frac{|\mathbf{e}(\theta')|}{u'(\theta')} d\theta'$$
$$= \int_0^{\theta_{max}} \mathrm{sgn}(\theta - \theta') \frac{|\mathbf{e}(\theta')|}{u'(\theta')} d\theta' \,.$$

which we may conveniently re-express as

$$t_+(\ell) - t_-(\ell) = \int_0^L d\tilde{\ell}\, \frac{\mathrm{sgn}(\ell - \tilde{\ell})}{u'(\tilde{\ell})} \tag{32}$$

using the length parameter ℓ along the fiber, along with the corresponding integration variable $\tilde{\ell}$.

The quantity δt_{st} in Eq. (30) is the contribution from a given infinitesimal segment $d\ell$. Integrating along the fiber and multiplying by ω we obtain the overall stretch contribution

$$\Delta\Phi_{stretch} \cong \omega \int_0^L [t_+(\ell) - t_-(\ell)]\, \frac{s(\ell)}{u'(\ell)}\, d\ell \,. \tag{33}$$

Recall that to leading order in v/c we need not distinguish between ω and ω_0. Substituting Eq. (32) in the integrand, we bring this expression to the more explicit form

$$\Delta\Phi_{stretch} \cong \omega \int_0^L d\ell \int_0^L d\tilde{\ell}\, \frac{s(\ell)\,\mathrm{sgn}(\ell - \tilde{\ell})}{u'(\tilde{\ell}) u'(\ell)}$$
$$= \frac{\omega}{2} \int_0^L d\ell \int_0^L d\tilde{\ell}\, \frac{s(\ell) - s(\tilde{\ell})}{u'(\tilde{\ell}) u'(\ell)}\, \mathrm{sgn}(\ell - \tilde{\ell}) \tag{34}$$

which is equivalent to Eq. (7).

We make the following observations:

- $\varDelta\Phi_{stretch} = 0$ if s and u' are symmetric, i.e. $s(\ell) = s(L-\ell)$, and $u'(\ell) = u'(L-\ell)$.
- In the special case where n is constant along the fiber the R.H.S. of Eq. (32) becomes $(2\ell - L)/u'$ and Eq. (33) reduces to

  $$\varDelta\Phi_{stretch} \cong \frac{\omega n^2}{c^2} \int_0^L s(\ell)(2\ell - L)\, d\ell. \tag{35}$$

  It is proportional to the first moment of $s(l)$ relative to the middle of the fiber. One can further verify that for the validity of the last equation it is sufficient that n is constant throughout the support of s.

- If n is constant and the stretch is localized to a single point, $s(\ell) = v_0 \delta(\ell - \ell_1)$, then

  $$\varDelta\Phi_{stretch} = \frac{\omega n^2 v_0}{c^2}\, (2\ell_1 - L)\,. \tag{36}$$

  It thus gives information on the distance of the stretching point from the mid-point $L/2$.

6 The Fizeau experiment

The framework we have described is sufficiently general to accommodate Fizeau's experiment. We consider a straight pipe section in which the water flows. The laser beam enters the pipe through a glass window located at $\ell = \ell_1$, and leaves it at another glass window at $\ell = \ell_2$ with ℓ the length coordinate along the light beam. The optical path then has a section with flowing water and a complementary static section where the beam-splitter and mirrors are. The entire system is stationary.

We shall only care here about the tangential velocity component $v_{\shortparallel} = \hat{\mathbf{e}} \cdot \mathbf{v}$ of the water (the transversal velocity has no contribution at linear order). Note that $v_{\shortparallel}(\ell)$ vanishes at the two windows at $\ell = \ell_{1,2}$.

From Eq. (27) we obtain for Wang's term:

$$\Delta\Phi_{wang} = \frac{2\omega}{c^2} \int_{\ell_1}^{\ell_2} v_{\shortparallel}(\ell) d\ell. \tag{37}$$

Turning next to analyze $\Delta\Phi_{stretch}$, we first observe that the stretch occurs entirely inside the water so we can use Eq. (35). Since the laser beam is straight line inside the water, $\hat{\mathbf{e}}$ is constant there and Eq. (12) yields $s = \partial_\ell v_{\shortparallel}$. Equation (35) therefore reads

$$\Delta\Phi_{stretch} \cong \frac{\omega n^2}{c^2} \int_{\ell_1}^{\ell_2} (2\ell - L)\, \partial_\ell v_{\shortparallel}\, d\ell\,. \tag{38}$$

In a typical Fizeau experiment the pipe has a uniform cross section (as in Fig. 2), hence $v_{\shortparallel}$ has an approximately constant value v_0 along the pipe—except near the windows where it abruptly drops to zero. In such a case the entire stretch contribution comes from the near-window region, where $\partial_\ell v_{\shortparallel}$ has a δ-function shape (with amplitude v_0 at $\ell \approx \ell_1$ and $-v_0$ at $\ell \approx \ell_2$), which allows a simple evaluation of the last integral. It is simpler, however, to integrate Eq. (38) by parts:

$$\Delta\Phi_{stretch} = -\frac{2\omega n^2}{c^2} \int_{\ell_1}^{\ell_2} v_{\shortparallel}(\ell) d\ell\,. \tag{39}$$

(recalling that $v_{\shortparallel}$ always vanishes at the two windows).

Summing the two contributions (37,39) we obtain the overall Fizeau phase difference

$$\Delta\Phi = \Delta\Phi_{wang} + \Delta\Phi_{stretch} = \frac{2\omega}{c^2}(1 - n^2) \int_{\ell_1}^{\ell_2} v_{\shortparallel}(\ell) d\ell\,. \tag{40}$$

This holds for an arbitrary $v_{\shortparallel}(\ell)$. If the pipe has a uniform cross section then $v_{\shortparallel}(\ell) \approx v_0$ (away from the windows), hence the last expression reduces to

$$\Delta\Phi = \frac{2\omega v_0 L_w}{c^2} \left(1 - n^2\right), \tag{41}$$

where $L_w \equiv \ell_2 - \ell_1$ is the length of the light's orbit in water. This reproduces Laue's [9] classic result, Eq. (2).

7 Conclusion

We have described a unified framework for the phase shift in interferometers where counter-propagating beams share a common optical path and gave a rigorous proof of Wang formula. The framework encompasses the experimental setting of Sagnac, Fizeau and Wang and sheds light on all three. We have shown that, neglecting dispersion, the phase shift to leading order in v/c has, in general two contributions:

- The Wang term $\Delta\Phi_{wang}$, reflecting the geometry of Minkowski space and Lorentz transformations. It is accurate to first order in v/c, is independent of n and holds if the fiber does not stretch.
- The stretch term $\Delta\Phi_{stretch}$ which, to order v/c is given by Eq. (7). For constant n the stretch term is proportional to n^2.

The framework suggest possible new experiments involving Fizeau type interferometers where an incompressible fluid flows in tubes of varying cross sections and hence a varying velocity. Eq. (40) is a generalized Fizeau formula describing such experiments.

Appendix: Exact expression for $\Delta\Phi$

We shall present here the exact relativistic expression for the phase difference $\Delta\Phi$, which accounts for the time variation of the various quantities during the wave propagation along the fiber.

Suppose that we want to predict the value of $\Delta\Phi$ at a given detection moment $t = t_{det}$. Then we need to follow the two constant-phase curves $t_\pm(\theta)$ backward in time, along the entire closed loop, from $t = t_{det}$ to the moments of emission $t^e_\pm$ at which the two counter-propagating constant-phase curves have left the source. (Both the "source" and the "detector" are realized by the beam-splitter, located at $\theta = 0$ or equivalently θ_{max}.)

The schedule equation is a first-order ODE for the unknowns $t_\pm(\theta)$ in the range $0 \le \theta \le \theta_{max}$, which follows from Eqs. (19,10):

$$\frac{dt_\pm}{d\theta} = \pm\gamma_\pm^2 \left(\frac{|\mathbf{e}|_\pm}{(\gamma_\perp)_\pm\, u'} \pm \frac{\mathbf{v}_\pm \cdot \mathbf{e}_\pm}{c^2} \right) . \tag{42}$$

The differential equation is non-linear since the quantities on the R.H.S may be rather arbitrary functions of $t_\pm$ and θ (u' only depends on θ.)

Recalling that the "+" and "−" directions respectively correspond to increasing and decreasing θ, the boundary conditions are

$$t_+(\theta_{max}) = t_-(0) = t_{det}. \tag{43}$$

The curves $t_\pm(\theta)$ are determined, throughout the interval $0 \le \theta \le \theta_{max}$, by the first-order ODE (42) along with the boundary condition (43). The emission moments $t^e_\pm$ are then given by

$$t^e_+ = t_+(0) \ , \quad t^e_- = t_-(\theta_{max}) \, .$$

Note that the two travel times are $T_\pm = t_{det} - t^e_\pm$, and hence $\Delta T = t^e_- - t^e_+$.

Knowledge of $t^e_\pm$ will allow the precise calculation of $\Delta\Phi$. We assume that the source emits a wave with fixed rest-frame frequency ω_0. The desired quantity $\Delta\Phi$ is the difference in the phase of the emitted wave between the moments $t = t^e_+$ and $t = t^e_-$. This phase difference is ω_0 multiplied by the proper time $\Delta\tau$ of the source between these two moments. Since the source speed $\mathbf{v}_0$ is in general time-dependent, so is it Lorentz factor γ_0, and $\Delta\tau$ is thus given by an integral of $1/\gamma_0(t)$ between the two relevant moments. Therefore,

$$\Delta\Phi = \omega_0 \int_{t^e_+}^{t^e_-} \frac{dt}{\gamma_0(t)} \,. \tag{44}$$

This result may also be re-expressed, similar to Eq. (14), as

$$\Delta\Phi = \frac{\omega_0}{\bar{\gamma}_0}\Delta T, \tag{45}$$

where $1/\bar{\gamma}_0$ is defined to be the time average of $1/\gamma_0(t)$ in the range between t^e_+ and t^e_- (that is, the integral in Eq. (44) divided by ΔT). Note that a fully-precise determination of $\Delta\Phi$ requires knowledge of the two emission times $t^e_\pm$, not just their difference ΔT.

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