New asymptotic results in principal component analysis

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Abstract: Let X be a mean zero Gaussian random vector in a separable Hilbert space $\mathbb H$ with covariance operator $\Sigma := \mathbb E(X \otimes X)$. Let $\Sigma =$ $\sum_{r\geq 1} \mu_r P_r$ be the spectral decomposition of Σ with distinct eigenvalues $\mu_1 > \mu_2 > \ldots$ and the corresponding spectral projectors P_1, P_2, \ldots Given a sample X_1, \ldots, X_n of size n of i.i.d. copies of X, the sample covariance operator is defined as $\hat{\Sigma}_n := n^{-1} \sum_{j=1}^n X_j \otimes X_j$. The main goal of principal component analysis is to estimate spectral projectors P_1, P_2, \ldots by their empirical counterparts $\hat{P}_1, \hat{P}_2, \ldots$ properly defined in terms of spectral decomposition of the sample covariance operator $\hat{\Sigma}_n$. The aim of this paper is to study asymptotic distributions of important statistics related to this problem, in particular, of statistic $\|\hat{P}_r - P_r\|_2^2$, where $\|\cdot\|_2^2$ is the squared Hilbert–Schmidt norm. This is done in a "high-complexity" asymptotic framework in which the so called effective rank $\mathbf{r}(\Sigma) := \frac{\text{tr}(\Sigma)}{\|\Sigma\|_{\infty}}$ (tr(·) being the trace and $\|\cdot\|_{\infty}$ being the operator norm) of the true covariance Σ is becoming large simultaneously with the sample size n, but $\mathbf{r}(\Sigma) = o(n)$ as $n \to \infty$. In this setting, we prove that, in the case of one-dimensional spectral projector P_r , the properly centered and normalized statistic $\|\hat{P}_r - P_r\|_2^2$ with *data-dependent* centering and normalization converges in distribution to a Cauchy type limit. The proofs of this and other related results rely on perturbation analysis and Gaussian concentration.

AMS 2000 subject classifications: Primary 62H12.

Keywords and phrases: Sample covariance, Spectral projectors, Effective rank, Principal Component Analysis, Concentration inequalities, Asymptotic distribution, Perturbation theory.

1. Introduction

Let $X, X_1, \ldots, X_n, \ldots$ be i.i.d. random variables sampled from a Gaussian distribution in a separable Hilbert space H with zero mean and covariance operator $\Sigma := \mathbb{E} X \otimes X$ and let $\hat{\Sigma} = \hat{\Sigma}_n := n^{-1} \sum_{j=1}^n X_j \otimes X_j$ denote the sample covariance operator based on (X_1, \ldots, X_n) (X_1, \ldots, X_n) (X_1, \ldots, X_n) .¹ We will be interested in asymptotic properties

[∗]Supported in part by NSF Grants DMS-1509739, DMS-1207808, CCF-1523768 and CCF-1415498

[†]Supported in part by Simons Grant 315477 and NSF Career Grant, DMS-1454515

¹Given $u, v \in \mathbb{H}$, the tensor product $u \otimes v$ is a rank one linear operator defined as $(u \otimes v)x =$ $u\langle v, x \rangle, x \in \mathbb{H}.$

of several statistics related to spectral projectors of sample covariance $\hat{\Sigma}$ (empirical spectral projectors) that could be potentially useful in principal component analysis (PCA) and its infinite dimensional versions such as functional PCA (see, e.g., $[18]$) or kernel PCA in machine learning (see, e.g., $[19]$, $[4]$).

In the classical setting of a finite-dimensional space $\mathbb{H} = \mathbb{R}^p$ of a fixed dimension p , the large sample asymptotics of spectral characteristics of sample covariance were studied by Anderson [\[1](#page-36-1)] who derived the joint asymptotic distribution of the sample eigenvalues and the associated sample eigenvectors (see also Theorem 13.5.1 in [\[2\]](#page-36-2)). Later on, similar results were established in the infinite-dimensional case (see, e.g., [\[6](#page-36-3)]). Such an extension is rather straightforward provided that the "complexity of the problem" characterized by such parameters as the trace $tr(\Sigma)$ of the covariance operator Σ remains fixed when the sample size n tends to infinity.

In the high-dimensional setting, when the dimension p of the space grows simultaneously with the sample size n , the problem has been primarily studied for so called spiked covariance models introduced by Johnstone and co-authors (see, e.g., [\[8](#page-36-4)]). In this case, the covariance Σ has a special structure, namely,

$$
\Sigma = \sum_{j=1}^{m} \lambda_j^2 (\theta_j \otimes \theta_j) + \sigma^2 I_p,
$$

where $m < p, \theta_1, \ldots, \theta_m$ are orthonormal vectors ("principal components"), $\lambda_1^2 > \cdots > \lambda_m^2 > 0$, $\sigma^2 > 0$ and I_p is the $p \times p$ identity matrix. This means that the observation X can be represented as 2 2

$$
X = \sum_{j=1}^{m} \lambda_j \xi_j \theta_j + \sigma \sum_{j=1}^{p} \eta_j \theta_j,
$$

where $\xi_j, \eta_j, j \geq 1$ are i.i.d. standard normal random variables. Thus, X can be viewed as an observation of a "signal" $\sum_{j=1}^{m} \lambda_j \xi_j \theta_j$, consisting of m "spikes", in an independent Gaussian white noise. For such models, an elegant asymptotic theory has been developed based on the achievements of random matrix theory (see, e.g., the results of Paul [\[16](#page-37-2)] on asymptotics of eigenvectors of sample covariance in spiked covariance models and references therein). The most interesting results were obtained in the case when $\frac{p}{n} \to c$ for some constant $c \in (0, +\infty)$. In this case, however, the eigenvectors of the sample covariance $\hat{\Sigma}_n$ fail to be consistent estimators of the eigenvectors of the true covariance Σ (see Johnstone and Lu [\[8](#page-36-4)]) and this difficulty could not be overcome without further assumptions on the true eigenvectors such as, for instance, their sparsity. This led to the development of various approaches to "sparse PCA" (see, e.g., [\[7,](#page-36-5) [13,](#page-37-3) [15,](#page-37-4) [17,](#page-37-5) [21,](#page-37-6) [3\]](#page-36-6) and references therein).

In this paper, we follow a somewhat different path. It is well known that to ensure consistency of empirical spectral projectors as statistical estimators of

² assuming that the orthonormal vectors $\theta_1, \ldots, \theta_m$ are extended to an orthonormal basis $\theta_1, \ldots, \theta_p$ of \mathbb{R}^p

spectral projectors of the true covariance Σ one has to establish convergence of Σ to Σ in the operator norm. In what follows, $\|\cdot\|_{\infty}$ will denote the operator norm (for bounded operators in \mathbb{H}), $\|\cdot\|_2$ will denote the Hilbert–Schmidt norm and $\|\cdot\|_1$ will denote the nuclear norm. We also use the notation $\text{tr}(\Sigma)$ for the trace of Σ and set

$$
\mathbf{r}(\Sigma) := \frac{\mathrm{tr}(\Sigma)}{\|\Sigma\|_{\infty}}.
$$

The last quantity is always dominated by the rank of operator Σ and it is sometimes referred to as its *effective rank*. It was pointed out by Vershynin [\[20\]](#page-37-7) that the effective rank could be used to provide non-asymptotic upper bounds on the size of the operator norm $\|\Sigma - \Sigma\|_{\infty}$ with rather weak (logarithmic) dependence on the dimension and this approach was later used in statistical literature (see $[5, 14]$ $[5, 14]$). In our paper $[10]$, we proved that in the Gaussian case the size of the operator norm $\|\hat{\Sigma} - \Sigma\|_{\infty}$ can be completely characterized in terms of the effective rank $\mathbf{r}(\Sigma)$ of the true covariance Σ and its operator norm $\|\Sigma\|_{\infty}$ and that the resulting non-asymptotic bounds are dimension-free (see theorems [1](#page-3-0) and [2](#page-3-1) below). This shows that Σ is an operator norm consistent estimator of Σ provided that $\mathbf{r}(\Sigma) = o(n)$, which makes the effective rank $\mathbf{r}(\Sigma)$ an important complexity parameter of the covariance estimation problem. This also provides a dimension-free framework for such problems and allows one to study them in a "high-complexity" case (that is, when the effective rank $\mathbf{r}(\Sigma)$ could be large) without imposing any structural assumptions on the true covariance such as, for instance, spiked covariance models [\[8](#page-36-4)]. This approach has been developed in some detail in our recent papers [\[10\]](#page-36-8), [\[11\]](#page-36-9), [\[12](#page-36-10)]. The current paper continues this line of work by studying the asymptotic behavior of several important statistics under the assumptions that both $n \to \infty$ and $\mathbf{r}(\Sigma) \to \infty$, $\mathbf{r}(\Sigma) = o(n)$. This includes statistical estimators of bias of spectral projectors of Σ (empirical spectral projectors) as well as their squared Hilbert-Schmidt norm errors with a goal to develop "studentized versions" of these statistics that could be (in principle) used for statistical inference. Before stating our main results, we provide in the next section a review of the results of papers [\[10\]](#page-36-8), [\[11](#page-36-9)], [\[12\]](#page-36-10) that will be extensively used in what follows.

Throughout the paper, we write $A \leq B$ iff $A \leq CB$ for some absolute constant $C > 0$ $(A, B \ge 0)$. $A \gtrsim B$ is equivalent to $B \lesssim A$ and $A \asymp B$ is equivalent to $A \lesssim B$ and $A \gtrsim B$. Sometimes, the signs $\lesssim \gtrsim$ and \asymp could be provided with subscripts: for instance, $A \leq_{\gamma} B$ means that $A \leq CB$ with a constant C that could depend on γ .

2. Effective rank and concentration of empirical spectral projectors: a review of recent results

The following recent result (see, [\[10](#page-36-8)]) provides a complete characterization of the quantity $\mathbb{E} \|\hat{\Sigma} - \Sigma\|_{\infty}$ in terms of the operator norm $\|\Sigma\|_{\infty}$ and the effective rank $r(\Sigma)$ in the case of i.i.d. mean zero Gaussian observations.

Theorem 1. The following bound holds:

$$
\mathbb{E} \|\hat{\Sigma} - \Sigma\|_{\infty} \asymp \|\Sigma\|_{\infty} \left[\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \right].
$$
 (2.1)

In paper [\[10](#page-36-8)], it is also complemented by a concentration inequality for $\|\hat{\Sigma} \Sigma\|_{\infty}$ around its expectation:

Theorem 2. There exists a constant $C > 0$ such that for all $t \ge 1$ with probability at least $1 - e^{-t}$,

$$
\left| \|\hat{\Sigma} - \Sigma\|_{\infty} - \mathbb{E}\|\hat{\Sigma} - \Sigma\|_{\infty} \right| \le C \|\Sigma\|_{\infty} \left[\left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee 1 \right) \sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right]. \tag{2.2}
$$

It follows from (2.1) and (2.2) that with some constant $C > 0$ and with probability at least $1 - e^{-t}$

$$
\|\hat{\Sigma} - \Sigma\|_{\infty} \le C \|\Sigma\|_{\infty} \left[\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \sqrt{\sqrt{\frac{t}{n}}} \sqrt{\frac{t}{n}} \right],\tag{2.3}
$$

which, in turn, implies that for all $p \geq 1$

$$
\mathbb{E}^{1/p} \|\hat{\Sigma} - \Sigma\|_{\infty}^p \asymp_p \|\Sigma\|_{\infty} \left[\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \right]. \tag{2.4}
$$

These results showed that the sample covariance $\hat{\Sigma}$ is an operator norm consistent estimator of Σ even in the cases when the effective rank $r(\Sigma)$ becomes large as $n \to \infty$, but $\mathbf{r}(\Sigma) = o(n)$ and $\|\Sigma\|_{\infty}$ remains bounded. Thus, it becomes of interest to study the behavior of spectral projectors of sample covariance Σ (that are of crucial importance in PCA) in such an asymptotic framework. This program has been partially implemented in papers [\[11\]](#page-36-9), [\[12](#page-36-10)]. To state the main results of these papers (used in what follows), we will introduce some further definitions and notations.

Let $\Sigma = \sum_{r\geq 1} \mu_r P_r$ be the spectral representation of covariance operator Σ with distinct non zero eigenvalues μ_r , $r \geq 1$ (arranged in decreasing order) and the corresponding spectral projectors P_r , $r \geq 1$. Clearly, P_r are finite rank projectors with rank $(P_r) =: m_r$ being the multiplicity of the corresponding eigenvalue μ_r . Let $\sigma(\Sigma)$ be the spectrum of Σ . Denote by \bar{g}_r the distance from the eigenvalue μ_r to the rest of the spectrum $\sigma(\Sigma) \setminus {\mu_r}$ (the r-th "spectral gap"). It will be also convenient to consider the non zero eigenvalues $\sigma_i(\Sigma), j \geq 1$ of Σ arranged in nondecreasing order and *repeated with their multiplicities* (in the case when the number of non zero eigenvalues is finite, we extend this sequence by zeroes). With this notation, let $\Delta_r := \{j : \sigma_j(\Sigma) = \mu_r\}, r \ge 1$ and denote by \hat{P}_r the orthogonal projector onto the linear span of eigenspaces of $\hat{\Sigma}$ corresponding to its eigenvalues $\{\sigma_i(\hat{\Sigma}): j \in \Delta_r\}$. It easily follows from a well known inequality due to Weyl that

$$
\sup_{j\geq 1} |\sigma_j(\hat{\Sigma}) - \sigma_j(\Sigma)| \leq ||\hat{\Sigma} - \Sigma||_{\infty}.
$$

If $\|\hat{\Sigma} - \Sigma\|_{\infty} < \frac{\bar{g}_r}{2}$, this immediately implies that the eigenvalues $\{\sigma_j(\hat{\Sigma}): j \in$ Δ_r } form a "cluster" that belongs to the interval $(\mu_r - \frac{\bar{g}_r}{2}, \mu_r + \frac{\bar{g}_r}{2})$ and that is separated from the rest of the spectrum of $\hat{\Sigma}$ in the sense that $\sigma_j(\hat{\Sigma}) \notin$ $(\mu_r - \frac{\bar{g}_r}{2}, \mu_r + \frac{\bar{g}_r}{2})$ for all $j \notin \Delta_r$. In this case, \hat{P}_r becomes a natural estimator of P_r . It could be viewed as a random perturbation of P_r and the following result, closely related to basic facts of perturbation theory (see [\[9](#page-36-11)]), could be found in [\[11\]](#page-36-9) (see Lemmas 1 and 2 there).

Lemma 1. Let $E := \hat{\Sigma} - \Sigma$. The following bound holds:

$$
\|\hat{P}_r - P_r\|_{\infty} \le 4 \frac{\|E\|_{\infty}}{\bar{g}_r}.
$$
\n(2.5)

Moreover, denote

$$
C_r := \sum_{s \neq r} \frac{1}{\mu_r - \mu_s} P_s.
$$

Then

$$
\hat{P}_r - P_r = L_r(E) + S_r(E),
$$
\n(2.6)

where

$$
L_r(E) := C_r E P_r + P_r E C_r \tag{2.7}
$$

and

$$
||S_r(E)||_{\infty} \le 14 \left(\frac{||E||_{\infty}}{\bar{g}_r}\right)^2.
$$
\n(2.8)

Remark 1. In the case when 0 is an eigenvalue of Σ , it is convenient to extend the sum in the definition of operator C_r to $s = \infty$ with $\mu_\infty = 0$ (see, for instance, the proof of Lemma [5\)](#page-17-0). Note, however, that $P_{\infty} \Sigma = \Sigma P_{\infty} = 0$ and $P_{\infty}\hat{\Sigma} = \hat{\Sigma}P_{\infty} = 0$. Thus, this additional term in the definition of C_r does not have any impact on $L_r(E)$ (and on the parameters $A_r(\Sigma), B_r(\Sigma)$ introduced below).

This result essentially shows that the difference $\hat{P}_r - P_r$ can be represented as a sum of two terms, a linear term with respect to $E = \hat{\Sigma} - \Sigma$ denoted by $L_r(E)$ and the remainder term $S_r(E)$ for which bound (2.8) (quadratic with respect to $||E||_{\infty}^2$) holds. The linear term $L_r(E)$ could be further represented as a sum of i.i.d. mean zero random operators:

$$
L_r(E) = n^{-1} \sum_{j=1}^n (P_r X_j \otimes C_r X_j + C_r X_j \otimes P_r X_j),
$$

which easily implies simple concentration bounds and asymptotic normality results for this term. On the other hand, it follows from theorems [1](#page-3-0) and [2](#page-3-1) that with probability at least $1 - e^{-t}$

$$
||S_r(E)||_{\infty} \lesssim \frac{||\Sigma||_{\infty}^2}{\bar{g}_r^2} \left[\frac{\mathbf{r}(\Sigma)}{n} \bigvee \left(\frac{\mathbf{r}(\Sigma)}{n} \right)^2 \bigvee \frac{t}{n} \bigvee \left(\frac{t}{n} \right)^2 \right],
$$

implying that $||S_r(E)||_{\infty} = o_{\mathbb{P}}(1)$ under the assumption $r(\Sigma) = o(n)$ and $||S_r(E)||_{\infty} = o_{\mathbb{P}}(n^{-1/2})$ under the assumption $r(\Sigma) = o(n^{1/2})$ (in both cases, provided that $\frac{\|\Sigma\|_{\infty}}{\bar{a}_n}$ remains bounded). Bound on the remainder term $S_r(E)$ of \bar{g}_r the order $o_{\mathbb{P}}(n^{-1/2})$ makes this term negligible if the linear term $L_r(E)$ converges to zero with the rate $O_{\mathbb{P}}(n^{-1/2})$ (the standard rate of the central limit theorem). A more subtle analysis of bilinear forms $\langle S_r(E)u, v \rangle, u, v \in \mathbb{H}$ given in [\[11](#page-36-9)] showed that the bilinear forms concentrate around their expectations at a rate $o_{\mathbb{P}}(n^{-1/2})$ provided that $\mathbf{r}(\Sigma) = o(n)$ (which is much weaker than the assumption $\mathbf{r}(\Sigma) = o(n^{1/2})$ needed for the operator norm $||S_r(E)||_{\infty}$ to be of the order $o_{\mathbb{P}}(n^{-1/2})$). More precisely, the following result was proved for the operator

$$
R_r := R_r(E) := S_r(E) - \mathbb{E}S_r(E) = \hat{P}_r - P_r - \mathbb{E}(\hat{P}_r - P_r) - L_r(E)
$$

(see Theorem 3 in $[11]$):

Theorem 3. Suppose that, for some $\gamma \in (0,1)$,

$$
\mathbb{E}\|\hat{\Sigma} - \Sigma\|_{\infty} \le (1 - \gamma)\frac{\bar{g}_r}{2}.
$$
\n(2.9)

Then, there exists a constant $D_{\gamma} > 0$ such that, for all $u, v \in \mathbb{H}$ and for all $t \geq 1$, the following bound holds with probability at least $1 - e^{-t}$:

$$
|\langle R_r u, v \rangle| \le D_\gamma \frac{\|\Sigma\|_\infty^2}{\bar{g}_r^2} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right) \sqrt{\frac{t}{n}} \|u\| \|v\|.
$$
 (2.10)

Condition [\(2.9\)](#page-5-0) (along with concentration bound of Theorem [2\)](#page-3-1) essentially guarantees that $\|\hat{\Sigma} - \Sigma\|_{\infty} < \frac{\bar{g}_r}{2}$ with a high probability, which makes the empirical spectral projector \hat{P}_r a small random perturbation of the true spectral projector P_r and allows us to use the tools of perturbation theory. Theorem [3](#page-5-1) easily implies the following concentration bound for bilinear forms $\langle \hat{P}_r u, v \rangle$:

Corollary 1. Under the assumption of Theorem [3,](#page-5-1) with some constants D, D_{γ} 0, for all $u, v \in \mathbb{H}$ and for all $t \geq 1$ with probability at least $1 - e^{-t}$,

$$
\left| \left\langle \hat{P}_r - \mathbb{E} \hat{P}_r u, v \right\rangle \right| \le D \frac{\|\Sigma\|_{\infty}}{\bar{g}_r} \sqrt{\frac{t}{n}} \|u\| \|v\| + D_{\gamma} \frac{\|\Sigma\|_{\infty}^2}{\bar{g}_r^2} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right) \sqrt{\frac{t}{n}} \|u\| \|v\|.
$$
 (2.11)

Moreover, it is easy to see that if both u and v are either in the eigenspace of Σ corresponding to the eigenvalue μ_r , or in the orthogonal complement of this eigenspace, then the first term in the right hand side of bound [\(2.11\)](#page-5-2) could be dropped and the bound reduces to its second term.

In addition to this, in [\[11\]](#page-36-9), the asymptotic normality of bilinear forms $\langle \hat{P}_r \mathbb{E} \hat{P}_r u, v \rangle, u, v \in \mathbb{H}$ was also proved in an asymptotic framework where $n \to \infty$ and $\mathbf{r}(\Sigma) = o(n)$.

Another important question studied in [\[11](#page-36-9)] concerns the structure of the bias $\mathbb{E}\hat{P}_r - P_r$ of empirical spectral projector \hat{P}_r . Namely, it was proved that the bias can be represented as the sum of two terms, the main term $P_r(\mathbb{E}\hat{P}_r - P_r)P_r$ being "aligned" with the projector P_r and the remainder T_r being of a smaller order in the operator norm (provided that $\mathbf{r}(\Sigma) = o(n)$). More specifically, the following result was proved (see Theorem 4 in [\[11\]](#page-36-9)).

Theorem 4. Suppose that, for some $\gamma \in (0,1)$, condition [\(2.9\)](#page-5-0) holds. Then, there exists a constant $D_{\gamma} > 0$ such that

$$
\mathbb{E}\hat{P}_r - P_r = P_r(\mathbb{E}\hat{P}_r - P_r)P_r + T_r
$$

with $P_rT_rP_r = 0$ and

$$
||T_r||_{\infty} \le D_{\gamma} \frac{m_r ||\Sigma||_{\infty}^2}{\bar{g}_r^2} \sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \frac{1}{\sqrt{n}}.
$$
\n(2.12)

In the case when $m_r = \text{rank}(P_r) = 1$ (so, μ_r is an eigenvalue of Σ of multiplicity 1), the structure of the bias becomes especially simple. Let $P_r = \theta_r \otimes \theta_r$, where θ_r is a unit norm eigenvector of Σ corresponding to μ_r . Then it is easy to see that

$$
\mathbb{E}\hat{P}_r - P_r = b_r P_r + T_r \tag{2.13}
$$

with $b_r = \langle (\mathbb{E}\hat{P}_r - P_r)\theta_r, \theta_r \rangle$ and T_r defined in Theorem [4.](#page-6-0) Moreover,

$$
b_r = \langle \mathbb{E}\hat{P}_r - P_r, \theta_r \otimes \theta_r \rangle = \mathbb{E}\langle \hat{\theta}_r, \theta_r \rangle^2 - 1,
$$

implying that $b_r \in [-1, 0]$. Thus, parameter b_r is an important characteristic of the bias of empirical spectral projector \hat{P}_r . It was shown in [\[11\]](#page-36-9) that, under the assumption $\mathbf{r}(\Sigma) \lesssim n$,

$$
|b_r| \lesssim \frac{\|\Sigma\|_{\infty}^2}{\bar{g}_r^2} \frac{\mathbf{r}(\Sigma)}{n}.
$$
 (2.14)

Note that this upper bound is larger than upper bound [\(2.12\)](#page-6-1) on the remainder $||T_r||_{\infty}$ by a factor $\sqrt{\mathbf{r}(\Sigma)}$.

Let now $\hat{P}_r = \hat{\theta}_r \otimes \hat{\theta}_r$ with a unit norm eigenvector $\hat{\theta}_r$ of $\hat{\Sigma}$. Since the vectors $\hat{\theta}_r, \theta_r$ are defined only up to their signs, assume without loss of generality that $\langle \hat{\theta}_r, \theta_r \rangle \geq 0$. The following result, proved in [\[11](#page-36-9)] (see Theorem 6), shows that the linear forms $\langle \hat{\theta}_r, u \rangle$ have "Bernstein type" concentration around $\sqrt{1 + b_r} \langle \theta_r, u \rangle$ with deviations of the order $O_{\mathbb{P}}(n^{-1/2})$.

Theorem 5. Suppose that condition [\(2.9\)](#page-5-0) holds for some $\gamma \in (0,1)$ and also that

$$
1 + b_r \ge \frac{\gamma}{2}.\tag{2.15}
$$

Then, there exists a constant $C_{\gamma} > 0$ such that for all $t \geq 1$ with probability at $least 1-e^{-t}$

$$
\left| \left\langle \hat{\theta}_r - \sqrt{1 + b_r} \theta_r, u \right\rangle \right| \leq C_\gamma \frac{\|\Sigma\|_\infty}{\bar{g}_r} \left(\sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right) \|u\|.
$$

Thus, if one constructs a proper estimator of the bias parameter b_r , it would be possible to improve a "naive estimator" $\langle \hat{\theta}_r, u \rangle$ of linear form $\langle \theta_r, u \rangle$ by reducing its bias. A version of such estimator based on the double sample $X_1, \ldots, X_n, \tilde{X}_1, \ldots, \tilde{X}_n$ of i.i.d. copies of X was suggested in [\[11](#page-36-9)]. If $\tilde{\Sigma} = \tilde{\Sigma}_n$ denotes the sample covariance based on $\tilde{X}_1, \ldots, \tilde{X}_n$ (the second subsample) and $\tilde{P}_r = \tilde{\theta}_r \otimes \tilde{\theta}_r$ denotes the corresponding empirical spectral projector (estimator of P_r), then the estimator \hat{b}_r of the bias parameter b_r is defined as follows:

$$
\hat{b}_r := \langle \hat{\theta}_r, \tilde{\theta}_r \rangle - 1,
$$

where the signs of $\hat{\theta}_r, \tilde{\theta}_r$ are chosen so that $\langle \hat{\theta}_r, \tilde{\theta}_r \rangle \geq 0$. Based on estimator \hat{b}_r , one can also define a bias corrected estimator $\check{\theta}_r := \frac{\hat{\theta}_r}{\sqrt{1+\hat{b}_r}}$ (which is not necessarily a unit vector) and prove the following result, showing that $\langle \check{\theta}_r, u \rangle$ is a \sqrt{n} -consistent estimator of $\langle \theta_r, u \rangle$ (at least in the case when $\mathbf{r}(\Sigma) \leq cn$ for a sufficiently small constant c :

Proposition 1. Under the assumptions and notations of Theorem [5,](#page-6-2) for some constant $C_{\gamma} > 0$ with probability at least $1 - e^{-t}$,

$$
|\hat{b}_r - b_r| \le C_\gamma \frac{\|\Sigma\|_\infty^2}{\bar{g}_r^2} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \bigvee \frac{t}{n}\right) \left(\sqrt{\frac{t}{n}} \bigvee \frac{t}{n}\right),\tag{2.16}
$$

and, for all $u \in \mathbb{H}$, with the same probability

$$
\left| \langle \check{\theta}_r - \theta_r, u \rangle \right| \le C_\gamma \frac{\|\Sigma\|_\infty}{\bar{g}_r} \left(\sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right) \|u\|.
$$
 (2.17)

In addition to this, asymptotic normality of $\langle \check{\theta}_r, u \rangle$ was also proved in [\[11\]](#page-36-9) under the assumption that $\mathbf{r}(\Sigma) = o(n)$.

Finally, we will discuss the results on normal approximation of the (squared) Hilbert–Schmidt norms $\|\hat{P}_r - P_r\|_2^2$ for an empirical spectral projector \hat{P}_r ob-tained in [\[12](#page-36-10)]. It was shown in this paper that, in the case when $\mathbf{r}(\Sigma) = o(n)$, the size of the expectation $\mathbb{E} \|\hat{P}_r - P_r\|_2^2$ could be characterized by the quantity $A_r(\Sigma) := 2\text{tr}(P_r \Sigma P_r) \text{tr}(C_r \Sigma C_r)$ (which, under mild assumption, is of the same order as $\mathbf{r}(\Sigma)$:

$$
\mathbb{E} \|\hat{P}_r - P_r\|_2^2 = (1 + o(1)) \frac{A_r(\Sigma)}{n}.
$$

A similar parameter characterizing the size of the variance $Var(\|\hat{P}_r - P_r\|_2^2)$ is defined as $B_r(\Sigma) := 2\sqrt{2} ||P_r \Sigma P_r||_2 ||C_r \Sigma C_r||_2$. Namely, the following result holds (Theorem 7 in $[12]$):

Theorem 6. Suppose condition [\(2.9\)](#page-5-0) holds for some $\gamma \in (0,1)$. Then the following bound holds with some constant $C_{\gamma} > 0$:

$$
\left| \frac{n}{B_r(\Sigma)} \text{Var}^{1/2}(\|\hat{P}_r - P_r\|_2^2) - 1 \right| \le C_\gamma m_r \frac{\|\Sigma\|_\infty^3}{\bar{g}_r^3} \frac{\mathbf{r}(\Sigma)}{B_r(\Sigma)\sqrt{n}} + \frac{m_r + 1}{n} . \tag{2.18}
$$

If $\frac{\|\Sigma\|_{\infty}}{\bar{g}_r}$ and m_r are bounded and $\frac{\mathbf{r}(\Sigma)}{B_r(\Sigma)\sqrt{n}} \to 0$, this implies that

$$
\text{Var}(\|\hat{P}_r - P_r\|_2^2) = (1 + o(1))\frac{B_r^2(\Sigma)}{n^2}.
$$

The main result of [\[12\]](#page-36-10) is the following normal approximation bounds for $\|\hat{P}_r - P_r\|_2^2$:

Theorem 7. Suppose that, for some constants $c_1, c_2 > 0$, $m_r \le c_1$ and $||\Sigma||_{\infty} \le$ $c_2\bar{q}_r$. Suppose also condition [\(2.9\)](#page-5-0) holds with some $\gamma \in (0,1)$. Then, the following bounds hold with some constant $C > 0$ depending only on γ, c_1, c_2 :

$$
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{n}{B_r(\Sigma)} \left(\|\hat{P}_r - P_r\|_2^2 - \mathbb{E} \|\hat{P}_r - P_r\|_2^2 \right) \le x \right\} - \Phi(x) \right|
$$

\n
$$
\le C \left[\frac{1}{B_r(\Sigma)} + \frac{\mathbf{r}(\Sigma)}{B_r(\Sigma)\sqrt{n}} \sqrt{\log \left(\frac{B_r(\Sigma)\sqrt{n}}{\mathbf{r}(\Sigma)} \sqrt{2} \right)} \right]
$$
(2.19)

and

$$
\sup_{x \in \mathbb{R}} \left| \mathbb{P} \left\{ \frac{\|\hat{P}_r - P_r\|_2^2 - \mathbb{E} \|\hat{P}_r - P_r\|_2^2}{\text{Var}^{1/2}(\|\hat{P}_r - P_r\|_2^2)} \le x \right\} - \Phi(x) \right|
$$

\n
$$
\le C \left[\frac{1}{B_r(\Sigma)} + \frac{\mathbf{r}(\Sigma)}{B_r(\Sigma)\sqrt{n}} \sqrt{\log \left(\frac{B_r(\Sigma)\sqrt{n}}{\mathbf{r}(\Sigma)} \sqrt{2} \right)} \right],
$$
\n(2.20)

where $\Phi(x)$ denotes the distribution function of standard normal random variable.

These bounds show that asymptotic normality of properly normalized statistic $\|\hat{P}_r - P_r\|_2^2$ holds provided that $n \to \infty$, $B_r(\Sigma) \to \infty$ and $\frac{\mathbf{r}(\Sigma)}{B_r(\Sigma)\sqrt{n}} \to 0$. In the case of p-dimensional spiked covariance models (with a fixed number of spikes), these conditions boil down to $n \to \infty$, $p \to \infty$ and $p = o(n)$.

3. Main results

We start this section with introducing a precise asymptotic framework in which $\mathbf{r}(\Sigma) \to \infty$ as $n \to \infty$. It is assumed that an observation $X = X^{(n)}$ is sampled from from a Gaussian distributions in \mathbb{H} with mean zero and covariance $\Sigma =$ $\Sigma^{(n)}$. The data consists on *n* i.i.d. copies of $X^{(n)}$: $X_1 = X_1^{(n)}, \ldots, X_n = X_n^{(n)}$ and the sample covariance $\hat{\Sigma}_n$ is based on $(X_1^{(n)},...,X_n^{(n)})$. As before, $\mu_r^{(n)}, r \ge 1$ denote distinct nonzero eigenvalues of $\Sigma^{(n)}$ arranged in decreasing order and $P_r^{(n)}, r \ge 1$ the corresponding spectral projectors. Let $\Delta_r^{(n)} := \{j : \sigma_j(\Sigma^{(n)}) =$ $\mu_r^{(n)}$ } and let $\hat{P}_r^{(n)}$ be the orthogonal projector on the linear span of eigenspaces corresponding to the eigenvalues $\{\sigma_j(\hat{\Sigma}_n), j \in \Delta_r^{(n)}\}.$

The goal is to estimate the spectral projector $P^{(n)} = P_{r_n}^{(n)}$ corresponding to the eigenvalue $\mu^{(n)} = \mu_{r_n}^{(n)}$ of $\Sigma^{(n)}$ with multiplicity $m^{(n)} = m_{r_n}^{(n)}$ and with spectral gap $\bar{g}^{(n)} = \bar{g}_{r_n}^{(n)}$. Define $C^{(n)} = C_{r_n}^{(n)} := \sum_{s \neq r_n} \frac{1}{\mu_{r_n}^{(n)}}$ $\frac{1}{\mu_{r_n}^{(n)} - \mu_s^{(n)}} P_s^{(n)}$ and let

$$
B_n := B_{r_n}(\Sigma^{(n)}) := 2\sqrt{2} ||C^{(n)}\Sigma^{(n)}C^{(n)}||_2 ||P^{(n)}\Sigma^{(n)}P^{(n)}||_2.
$$

Assumption 1. Suppose the following conditions hold:

$$
\sup_{n\geq 1} m^{(n)} < +\infty; \tag{3.1}
$$

$$
\sup_{n\geq 1} \frac{\|\Sigma^{(n)}\|_{\infty}}{\bar{g}^{(n)}} < +\infty;
$$
\n(3.2)

$$
B_n \to \infty \text{ as } n \to \infty;
$$
 (3.3)

$$
\frac{\mathbf{r}(\Sigma^{(n)})}{B_n\sqrt{n}} \to 0 \text{ as } n \to \infty.
$$
 (3.4)

Assumption [1](#page-9-0) easily implies that

$$
\mathbf{r}(\Sigma^{(n)}) \to \infty
$$
, $\mathbf{r}(\Sigma^{(n)}) = o(n)$ as $n \to \infty$.

Also, under mild additional conditions, $B_n \asymp ||\Sigma^{(n)}||_2$.

The following fact is an immediate consequence of bound [\(2.18\)](#page-7-0) and Theorem [7.](#page-8-0)

Proposition 2. Under Assumption [1,](#page-9-0)

$$
\text{Var}(\|\hat{P}^{(n)} - P^{(n)}\|_2^2) = \left(\frac{B_n}{n}\right)^2 (1 + o(1)).
$$

In addition,

$$
\frac{n\left(\|\hat{P}^{(n)} - P^{(n)}\|_2^2 - \mathbb{E}\|\hat{P}^{(n)} - P^{(n)}\|_2^2\right)}{B_n} \xrightarrow{d} Z \tag{3.5}
$$

and

$$
\frac{\left(\|\hat{P}^{(n)} - P^{(n)}\|_{2}^{2} - \mathbb{E}\|\hat{P}^{(n)} - P^{(n)}\|_{2}^{2}\right)}{\sqrt{\text{Var}(\|\hat{P}^{(n)} - P^{(n)}\|_{2}^{2})}} \xrightarrow{d} Z \text{ as } n \to \infty,
$$
 (3.6)

Z being a standard normal random variable.

Our main goal is to develop a version of these asymptotic results for squared Hilbert–Schmidt norm error $\|\hat{P}^{(n)} - P^{(n)}\|_2^2$ of empirical spectral projector $P^{(n)}$ with a *data driven normalization* that, in principle, could lead to constructing confidence sets and statistical tests for spectral projectors of covariance operator under Assumption [1.](#page-9-0) This will be done only in the case when the target

spectral projector $P^{(n)}$ is of rank 1 (that is, $\mu^{(n)}$ is the eigenvalue of multiplicity $m^{(n)} = 1$). This problem is also related to estimation of the bias parameter $b^{(n)} = b_{r_n}^{(n)}$ of empirical spectral projector $\hat{P}^{(n)}$. This parameter and its estimator $\hat{b}^{(n)} = \hat{b}_{r_n}^{(n)}$ were introduced in Section [2.](#page-2-0) In particular, we will prove the asymptotic normality of estimator $\hat{b}^{(n)}$ with a proper normalization that depends on unknown covariances $\Sigma^{(n)}$ and derive the limit distribution of $\hat{b}^{(n)}$ with a data-driven normalization.

Let $P^{(n)} = \theta^{(n)} \otimes \theta^{(n)}$ and $\hat{P}^{(n)} = \hat{\theta}^{(n)} \otimes \hat{\theta}^{(n)}$ for unit vectors $\theta^{(n)}$, $\hat{\theta}^{(n)}$. To define the estimator $\hat{b}^{(n)}$, we need an additional independent sample $\tilde{X}_1^{(n)}, \ldots, \tilde{X}_n^{(n)}$ consisting of i.i.d. copies of $X^{(n)}$. Let $\tilde{\Sigma}_n$ denote the sample covariance based on $(\tilde{X}_1^{(n)},\ldots,\tilde{X}_n^{(n)})$ and let $\tilde{P}^{(n)} = \tilde{\theta}^{(n)} \otimes \tilde{\theta}^{(n)}$ be its empirical spectral projector corresponding to $P^{(n)}$. It will be assumed that the signs of $\hat{\theta}^{(n)}$, $\tilde{\theta}^{(n)}$ are chosen in such a way that $\langle \hat{\theta}^{(n)}, \tilde{\theta}^{(n)} \rangle \geq 0$. Define

$$
\hat{b}^{(n)} = \langle \hat{\theta}^{(n)}, \tilde{\theta}^{(n)} \rangle - 1.
$$

Theorem 8. Under Assumption [1,](#page-9-0)

$$
\frac{2n}{B_n}(\hat{b}^{(n)} - b^{(n)}) \stackrel{d}{\longrightarrow} Z \text{ as } n \to \infty,
$$

Z being a standard normal random variable.

In order to use this asymptotic normality result for statistical inference about bias parameter $b^{(n)}$, one has to find a way to estimate the normalizing factor $\frac{2n}{B_n}$ that depends on unknown covariance $\Sigma^{(n)}$. By the first claim of Proposition [2,](#page-9-1) under Assumption [1,](#page-9-0)

$$
\frac{2n}{B_n} \sim \frac{2}{\text{Var}^{1/2}(\|\hat{P}^{(n)} - P^{(n)}\|_2^2)} \text{ as } n \to \infty.
$$

Thus, equivalently, we need to estimate the variance $\text{Var}(\|\hat{P}^{(n)} - P^{(n)}\|_2^2)$. Note that

$$
\operatorname{Var}(\|\hat{P}^{(n)} - P^{(n)}\|_2^2) = \operatorname{Var}\left(\|\hat{P}^{(n)}\|_2^2 + \|P^{(n)}\|_2^2 - 2\langle \hat{P}^{(n)}, P^{(n)} \rangle\right)
$$

=
$$
\operatorname{Var}\left(2 - 2\langle \hat{P}^{(n)}, P^{(n)} \rangle\right) = 4\operatorname{Var}\left(\langle \hat{P}^{(n)}, P^{(n)} \rangle\right) = 2\mathbb{E}\left(\langle \hat{P}^{(n)}, P^{(n)} \rangle - \langle \tilde{P}^{(n)}, P^{(n)} \rangle\right)^2.
$$

To estimate the right hand side, consider the third independent sample $\bar{X}_1^{(n)}, \ldots, \bar{X}_n^{(n)}$ consisting of *n* independent copies of $X^{(n)}$ and denote by $\bar{\Sigma}_n$ the sample covariance based on $(\bar{X}_1^{(n)}, \ldots, \bar{X}_n^{(n)})$ and by $\bar{P}^{(n)} = \bar{\theta}^{(n)} \otimes \bar{\theta}^{(n)}$ its empirical spectral projector corresponding to $P^{(n)}$. Assume that the sign of $\bar{\theta}^{(n)}$ is chosen in such a way that $\langle \tilde{\theta}^{(n)}, \bar{\theta}^{(n)} \rangle \geq 0$ and define

$$
\tilde{b}^{(n)} := \langle \tilde{\theta}^{(n)}, \bar{\theta}^{(n)} \rangle - 1.
$$

We will use

$$
\langle \hat{P}^{(n)}, \tilde{P}^{(n)} \rangle = \langle \hat{\theta}^{(n)}, \tilde{\theta}^{(n)} \rangle^2 = (1 + \hat{b}^{(n)})^2
$$

as an "estimator" of $\langle \hat{P}^{(n)}, P^{(n)} \rangle$ and

$$
\langle \tilde{P}^{(n)}, \bar{P}^{(n)} \rangle = \langle \tilde{\theta}^{(n)}, \bar{\theta}^{(n)} \rangle^2 = (1 + \tilde{b}^{(n)})^2
$$

as an "estimator" of $\langle \tilde{P}^{(n)}, P^{(n)} \rangle$. To estimate Var $(||\hat{P}^{(n)} - P^{(n)}||_2^2) \sim \frac{B_n^2}{n^2}$, one can try to use the statistic $2((1 + \hat{b}^{(n)})^2 - (1 + \tilde{b}^{(n)})^2)^2$. In fact, it turns out that the sequence

$$
\frac{n}{B_n}\Big((1+\hat{b}^{(n)})^2 - (1+\tilde{b}^{(n)})^2\Big) \sim \frac{(1+\hat{b}^{(n)})^2 - (1+\tilde{b}^{(n)})^2}{\text{Var}^{1/2}(\|\hat{P}^{(n)} - P^{(n)}\|_2^2)}
$$

is asymptotically normal with mean zero and variance $\frac{3}{2}$ and

$$
\frac{\mathbb{E}\left|(1+\hat{b}^{(n)})^2 - (1+\tilde{b}^{(n)})^2\right|}{\text{Var}^{1/2}(\|\hat{P}^{(n)} - P^{(n)}\|_2^2)} \to \sqrt{\frac{3}{2}}\mathbb{E}|Z| = \sqrt{\frac{3}{\pi}} \text{ as } n \to \infty.
$$

Therefore, it might be more natural to view $\frac{\pi}{3} \left((1 + \hat{b}^{(n)})^2 - (1 + \tilde{b}^{(n)})^2 \right)^2$ as an estimator of the variance $\text{Var}(\|\hat{P}^{(n)} - P^{(n)}\|_2^2)$. In any case, we are more interested in a data driven version of Theorem [8](#page-10-0) given below.

Given $\alpha \in \mathbb{R}, \beta > 0$, let $Y_{\alpha,\beta}$ denote a random variable with density

$$
\frac{1}{2}\Big[\frac{1}{\beta}f\Big(\frac{x-\alpha}{\beta}\Big)+\frac{1}{\beta}f\Big(\frac{x+\alpha}{\beta}\Big)\Big],
$$

 $f(x) := \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$ being the standard Cauchy density. The distribution of $Y_{\alpha,\beta}$ is a mixture of two rescaled Cauchy densities with locations $\pm \alpha$ and with equal mixing probabilities. This distribution (with proper choices of parameters (α, β) occurs naturally as the distribution of the ration $\frac{\xi}{|\eta|}$ for mean zero normal random variables ξ, η . Namely, the following (probably, well known) fact holds. Its proof is rather elementary and is left to the reader.

Proposition 3. Suppose ξ, η are mean zero normal random variables with $\mathbb{E}\xi^2 = \sigma_{\xi}^2 > 0$, $\mathbb{E}\eta^2 = \sigma_{\eta}^2 > 0$ and with correlation coefficient ρ . Then

$$
\frac{\xi}{|\eta|} \stackrel{d}{=} Y_{\alpha,\beta}
$$

with $\alpha := \frac{\sigma_{\xi}}{\sigma_{\xi}}$ $\frac{\sigma_{\xi}}{\sigma_{\eta}}\rho$ and $\beta := \frac{\sigma_{\xi}}{\sigma_{\eta}}$ $\frac{\sigma_{\xi}}{\sigma_{\eta}}\sqrt{1-\rho^2}.$

We now state a data-driven version of Theorem [8.](#page-10-0)

Theorem 9. Under Assumption [1,](#page-9-0)

$$
\frac{2(\hat{b}^{(n)} - b^{(n)})}{\left| (1 + \hat{b}^{(n)})^2 - (1 + \tilde{b}^{(n)})^2 \right|} \xrightarrow{d} Y_{\alpha, \beta} \text{ as } n \to \infty,
$$

$$
\alpha := \frac{1}{2}, \ \beta := \sqrt{\frac{5}{12}}.
$$

 $where$ 2

Quite similarly, we will determine the asymptotic distribution of statistic $\|\hat{P}^{(n)} - P^{(n)}\|_2^2$ with a data-driven normalization. First note that

$$
\mathbb{E} \|\hat{P}^{(n)} - P^{(n)}\|_2^2 = \mathbb{E} \left(\|\hat{P}^{(n)}\|_2^2 + \|P^{(n)}\|_2^2 - 2\langle \hat{P}^{(n)}, P^{(n)} \rangle \right) = \mathbb{E} \left(2 - 2\langle \hat{P}^{(n)}, P^{(n)} \rangle \right)
$$

= 2 - 2\langle \mathbb{E} \hat{P}^{(n)}, P^{(n)} \rangle = 2 - 2(1 + b^{(n)})\langle P^{(n)}, P^{(n)} \rangle = -2b^{(n)} \tag{3.7}

(see Theorem [4](#page-6-0) and the comments after this theorem). In the data-driven version of [\(3.6\)](#page-9-2) we will replace $\mathbb{E} \|\hat{P}^{(n)} - P^{(n)}\|_2^2$ by its estimator $-2\hat{b}^{(n)}$ and the standard deviation $\text{Var}^{1/2}(\|\hat{P}^{(n)} - P^{(n)}\|_2^2)$ by $\left| (1 + \hat{b}^{(n)})^2 - (1 + \tilde{b}^{(n)})^2 \right|$. This yields the following result.

Theorem 10. Under Assumption [1,](#page-9-0)

$$
\frac{\|\hat{P}^{(n)} - P^{(n)}\|_2^2 + 2\hat{b}^{(n)}}{\left| (1 + \hat{b}^{(n)})^2 - (1 + \tilde{b}^{(n)})^2 \right|} \xrightarrow{d} Y_{\alpha,\beta} \text{ as } n \to \infty,
$$

$$
\frac{A}{B} := \frac{\sqrt{47}}{6}.
$$

where $\alpha := \frac{5}{6}, \ \beta :=$ 6

4. Proofs: preliminary lemmas

We start with preliminary results that will be formulated in the "non-asymptotic framework" of Section [2](#page-2-0) and the notations of that section will be used. Recall that X_1, \ldots, X_n and $\tilde{X}_1, \ldots, \tilde{X}_n$ are two samples each of size n of i.i.d. copies of X, $\hat{\Sigma}$ and $\tilde{\Sigma}$ are sample covariances based on (X_1, \ldots, X_n) and $(\tilde{X}_1, \ldots, \tilde{X}_n)$, respectively, and $E := \hat{\Sigma} - \Sigma$, $\tilde{E} := \tilde{\Sigma} - \Sigma$.

In what follows, we will use a concentration result for $\|\hat{P}_r - P_r\|_2^2 - \|L_r(E)\|_2^2$ that was obtained in [\[12\]](#page-36-10) (see Theorem 5 there) and played a crucial role in the derivation of normal approximation bound of Theorem [7.](#page-8-0)

Lemma 2. Suppose that for some $\gamma \in (0,1)$ condition [\(2.9\)](#page-5-0) holds. Then, for all $t \geq 1$, with probability at least $1 - e^{-t}$

$$
\left| \|\hat{P}_r - P_r\|_2^2 - \|L_r(E)\|_2^2 - \mathbb{E}(\|\hat{P}_r - P_r\|_2^2 - \|L_r(E)\|_2^2) \right|
$$

$$
\lesssim_{\gamma} m_r \frac{\|\Sigma\|_{\infty}^3}{\bar{g}_r^3} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t}{n} \bigvee \left(\frac{t}{n}\right)^2 \right) \sqrt{\frac{t}{n}}.
$$
 (4.1)

The first new result of this section is a useful representation for $(1 + \hat{b}_r)^2$ – $(1 + b_r)^2$ that will be crucial in our proofs.

Lemma 3. Suppose for some $\gamma \in (\frac{23}{24}, 1)$ condition [\(2.9\)](#page-5-0) holds. Then, there exists a constant $D_1 > 0$ such that the following representation holds

$$
(1 + \hat{b}_r)^2 - (1 + b_r)^2 = \langle L_r(E), L_r(\tilde{E}) \rangle - \frac{1}{2} (||L_r(E)||_2^2 - \mathbb{E}||L_r(E)||_2^2)
$$

$$
- \frac{1}{2} (||L_r(\tilde{E})||_2^2 - \mathbb{E}||L_r(\tilde{E})||_2^2) + \Upsilon_r
$$
(4.2)

V. Koltchinskii and K. Lounici/Asymptotics in principal component analysis 14

with the remainder term Υ_r that, for all $t \geq 1$, with probability at least $1 - e^{-t}$ satisfies the bound

$$
|\Upsilon_r| \le D_1 \frac{\|\Sigma\|_{\infty}^4}{\bar{g}_r^4} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t}{n} \bigvee \left(\frac{t}{n}\right)^3\right) \left(\sqrt{\frac{t}{n}} \bigvee \frac{t}{n}\right). \tag{4.3}
$$

PROOF. By the definition of \hat{b}_r , we have

$$
(1 + \hat{b}_r)^2 = \langle \hat{P}_r, \tilde{P}_r \rangle
$$

= $\langle \hat{P}_r - \mathbb{E}\hat{P}_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle + \langle \hat{P}_r - \mathbb{E}\hat{P}_r, P_r \rangle + \langle P_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle$
+ $\langle \hat{P}_r - \mathbb{E}\hat{P}_r, \mathbb{E}\tilde{P}_r - P_r \rangle + \langle \mathbb{E}\hat{P}_r - P_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle + \langle \mathbb{E}\hat{P}_r, \mathbb{E}\tilde{P}_r \rangle.$
(4.4)

In view of (2.13) , we also have

$$
\langle \mathbb{E}\hat{P}_r, \mathbb{E}\tilde{P}_r \rangle = \langle (1+b_r)P_r + T_r, (1+b_r)P_r + T_r \rangle = (1+b_r)^2 + ||T_r||_2^2,
$$

since P_r and T_r are orthogonal by definition of the latter. Thus, [\(4.4\)](#page-13-0) can be rewritten as

$$
(1+\hat{b}_r)^2 - (1+b_r)^2 = \langle \hat{P}_r - \mathbb{E}\hat{P}_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle + \langle \hat{P}_r - \mathbb{E}\hat{P}_r, P_r \rangle + \langle P_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle
$$

+ $\langle \hat{P}_r - \mathbb{E}\hat{P}_r, \mathbb{E}\tilde{P}_r - P_r \rangle + \langle \mathbb{E}\hat{P}_r - P_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle + ||T_r||_2^2.$
(4.5)

Denote

 $\hat{\varrho}_r := \big\langle \hat{P}_r - \mathbb{E} \hat{P}_r, P_r \big\rangle + \frac{1}{2}$ $\frac{1}{2} (||L_r(E)||_2^2 - \mathbb{E}||L_r(E)||_2^2), \quad \tilde{\varrho}_r := \langle \tilde{P}_r - \mathbb{E} \tilde{P}_r, P_r \rangle + \frac{1}{2}$ $\frac{1}{2} (||L_r(\tilde{E})||_2^2 - \mathbb{E}||L_r(\tilde{E})||_2^2),$

where

$$
E := \hat{\Sigma} - \Sigma, \quad \tilde{E} := \tilde{\Sigma} - \Sigma.
$$

We immediately get from [\(4.5\)](#page-13-1) that

$$
(1 + \hat{b}_r)^2 - (1 + b_r)^2 = \langle \hat{P}_r - \mathbb{E}\hat{P}_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle - \frac{1}{2} (||L_r(E)||_2^2 - \mathbb{E}||L_r(E)||_2^2) + \varrho_r
$$

$$
- \frac{1}{2} (||L_r(\tilde{E})||_2^2 - \mathbb{E}||L_r(\tilde{E})||_2^2) + \tilde{\varrho}_r + \langle \hat{P}_r - \mathbb{E}\hat{P}_r, \mathbb{E}\tilde{P}_r - P_r \rangle
$$

$$
+ \langle \mathbb{E}\hat{P}_r - P_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle + ||T_r||_2^2.
$$

Since
$$
\hat{P}_r - \mathbb{E}\hat{P}_r = L_r(E) + R_r(E), \ \tilde{P}_r - \mathbb{E}\tilde{P}_r = L_r(\tilde{E}) + R_r(\tilde{E}),
$$

\n
$$
\langle \hat{P}_r - \mathbb{E}\hat{P}_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle = \langle L_r(E) + R_r(E), L_r(\tilde{E}) + R_r(\tilde{E}) \rangle
$$
\n
$$
= \langle L_r(E), L_r(\tilde{E}) \rangle + \langle L_r(E), R_r(\tilde{E}) \rangle + \langle L_r(\tilde{E}), R_r(E) \rangle + \langle R_r(E), R_r(\tilde{E}) \rangle.
$$

Combining the last two displays, we get that representation [\(4.2\)](#page-12-0) holds with the remainder

$$
\begin{aligned} \Upsilon_r &:= \varrho_r + \tilde{\varrho}_r + \langle \hat{P}_r - \mathbb{E}\hat{P}_r, \mathbb{E}\tilde{P}_r - P_r \rangle + \langle \mathbb{E}\hat{P}_r - P_r, \tilde{P}_r - \mathbb{E}\tilde{P}_r \rangle + \|T_r\|_2^2 \\ &+ \langle L_r(E), R_r(\tilde{E}) \rangle + \langle L_r(\tilde{E}), R_r(E) \rangle + \langle R_r(E), R_r(\tilde{E}) \rangle. \end{aligned}
$$

It remains to check that Υ_r satisfies bound [\(4.3\)](#page-13-2).

In what follows, we frequently use bounds of Theorems [1](#page-3-0) and [2](#page-3-1) along with bound (2.3) . Under condition (2.9) , we have

$$
\|\Sigma\|_{\infty}\left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}}\bigvee\frac{\mathbf{r}(\Sigma)}{n}\right)\lesssim \frac{\bar{g}_r}{2}\leq \frac{\|\Sigma\|_{\infty}}{2}.
$$

This implies that $\frac{\mathbf{r}(\Sigma)}{n} \lesssim 1$ and $\frac{\mathbf{r}(\Sigma)}{n} \lesssim \sqrt{\frac{\mathbf{r}(\Sigma)}{n}}$. Thus, the term $\frac{\mathbf{r}(\Sigma)}{n}$ in bounds of Theorems [1](#page-3-0) and [2](#page-3-1) and [\(2.3\)](#page-3-4) could be dropped. This is done in what follows without further notice.

Our next goal is to provide a bound on the remainder term Υ_r which can be done for an arbitrary multiplicity m_r of μ_r . To this end, first note that bound (2.10) easily implies that for any symmetric operator B of finite rank m the following bound holds with probability at least $1 - e^{-t}$:

$$
\left| \langle R_r(E), B \rangle \right| \leq D_\gamma m \|B\|_\infty \frac{\|\Sigma\|_\infty^2}{\bar{g}_r^2} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t + \log(m)}{n}} \bigvee \frac{t + \log(m)}{n} \right) \sqrt{\frac{t + \log(m)}{n}}.
$$
\n
$$
(4.6)
$$

Indeed, it is enough to use the spectral representation $B = \sum_{j=1}^{m} \lambda_j (\phi_j \otimes \phi_j)$ of B with eigenvalues λ_j and orthonormal eigenvectors ϕ_j , to write

$$
\left| \langle R_r(E), B \rangle \right| \leq \sum_{j=1}^m |\lambda_j| \left| \langle R_r(E)\phi_j, \phi_j \rangle \right| \leq m \|B\|_{\infty} \max_{1 \leq j \leq m} \left| \langle R_r(E)\phi_j, \phi_j \rangle \right|,
$$

to use bound (2.10) with $t + \log(m)$ instead of t in order to control bilinear forms $\left| \langle R_r(E) \phi_j, \phi_j \rangle \right|$ and, finally, to use the union bound.

We will use bound [\(4.6\)](#page-14-0) to control the last three terms in the expression for the remainder Υ_r . To control $\langle L_r(\tilde{E}), R_r(E) \rangle$, we use [\(4.6\)](#page-14-0) conditionally on $\tilde{X}_1, \ldots, \tilde{X}_n$ with $B = L_r(\tilde{E})$ (that is of rank at most $2m_r$) to get that with probability at least $1 - e^{-t}$

$$
\left| \langle R_r(E), L_r(\tilde{E}) \rangle \right| \lesssim
$$

$$
m_r ||L_r(\tilde{E})||_{\infty} \frac{||\Sigma||_{\infty}^2}{\bar{g}_r^2} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t + \log(2m_r)}{n}} \bigvee \frac{t + \log(2m_r)}{n} \right) \sqrt{\frac{t + \log(2m_r)}{n}}.
$$

This should be combined with an upper bound on $||L_r(E)||_{\infty}$ that follows from [\(2.3\)](#page-3-4) and also holds with probability at least $1 - e^{-t}$:

$$
||L_r(\tilde{E})||_{\infty} \lesssim \frac{||\tilde{E}||_{\infty}}{\bar{g}_r} \lesssim \frac{||\Sigma||_{\infty}}{\bar{g}_r} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \bigvee \frac{t}{n}\right)
$$

(where it was also used that $||C_r||_{\infty} \le \frac{1}{\bar{g}_r}$). As a consequence, the following holds with probability at least $1 - 2e^{-t}$:

$$
\left| \langle R_r(E), L_r(\tilde{E}) \rangle \right| \lesssim_{\gamma} \n m_r \frac{\|\Sigma\|_{\infty}^3}{\bar{g}_r^3} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right) \n \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t + \log(2m_r)}{n}} \bigvee \frac{t + \log(2m_r)}{n} \right) \sqrt{\frac{t + \log(2m_r)}{n}}.
$$
\n(4.7)

Of course, a similar bound also holds for $\left| \langle L_r(E), R_r(\tilde{E}) \rangle \right|$. As to $\left| \langle R_r(E), R_r(\tilde{E}) \rangle \right|$, observe that, by [\(2.8\)](#page-4-0), [\(2.3\)](#page-3-4) and Theorem [1,](#page-3-0) we have that with probability at least $1 - e^{-t}$,

$$
||R_r(\tilde{E})||_{\infty} \le ||S_r(\tilde{E})||_{\infty} + \mathbb{E}||S_r(\tilde{E})||_{\infty} \lesssim \frac{||\tilde{E}||_{\infty}^2}{\bar{g}_r^2} + \frac{\mathbb{E}||\tilde{E}||_{\infty}^2}{\bar{g}_r^2}
$$

$$
\lesssim \frac{||\Sigma||_{\infty}^2}{\bar{g}_r^2} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \bigvee \frac{t}{n}\right)^2.
$$

Therefore, using again bound [\(2.10\)](#page-5-3) conditionally on $\tilde{X}_1, \ldots, \tilde{X}_n$ with $B =$ $R_r(\tilde{E})$ we get that with probability $1 - 2e^{-t}$

$$
\left| \langle R_r(E), R_r(\tilde{E}) \rangle \right| \lesssim_{\gamma} \n m_r \frac{\|\Sigma\|_{\infty}^4}{\bar{g}_r^4} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \right)^2 \n \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t + \log(2m_r)}{n}} \bigvee \frac{t + \log(2m_r)}{n} \right) \sqrt{\frac{t + \log(2m_r)}{n}}.
$$
\n(4.8)

To bound $||T_r||_2^2$, note that

$$
||T_r||_2^2 = \langle T_r, T_r \rangle \le ||T_r||_1 ||T_r||_{\infty},
$$

and, by the definition of T_r ,

 $||T_r||_1 \le ||\mathbb{E}\hat{P}_r - P_r||_1 + ||P_r(\mathbb{E}\hat{P}_r - P_r)|_1 \le 2m_r\mathbb{E}||\hat{P}_r - P_r||_{\infty} + m_r||\mathbb{E}\hat{P}_r - P_r||_{\infty} \le 3m_r\mathbb{E}||\hat{P}_r - P_r||_{\infty}$ Using (2.5) and Theorem [1,](#page-3-0) we get

$$
||T_r||_1 \lesssim_{\gamma} m_r \frac{||\Sigma||_{\infty}}{\bar{g}_r} \sqrt{\frac{\mathbf{r}(\Sigma)}{n}}.
$$

Therefore, by bound [\(2.12\)](#page-6-1),

$$
||T_r||_2^2 \le ||T_r||_1 ||T_r||_{\infty} \lesssim_{\gamma} m_r^2 \frac{||\Sigma||_{\infty}^3}{\bar{g}_r^3} \frac{\mathbf{r}(\Sigma)}{n} \sqrt{\frac{1}{n}}.
$$
 (4.9)

We will now control

$$
\left\langle \hat{P}_r - \mathbb{E}\hat{P}_r, \mathbb{E}\tilde{P}_r - P_r \right\rangle = \left\langle \hat{P}_r - \mathbb{E}\hat{P}_r, P_r W_r P_r \right\rangle + \left\langle \hat{P}_r - \mathbb{E}\hat{P}_r, T_r \right\rangle, \tag{4.10}
$$

where $W_r = \mathbb{E} \tilde{P}_r - P_r$. Recall that $\hat{P}_r - \mathbb{E} \hat{P}_r = L_r(E) + R_r(E)$. Since $L_r(E) =$ $P_rEC_r + C_rEP_r$ and $C_rP_r = P_rC_r = 0$, it is easy to see that

$$
\langle L_r(E), P_r W_r P_r \rangle = \langle P_r E C_r, P_r W_r P_r \rangle + \langle C_r E P_r, P_r W_r P_r \rangle = 0.
$$

Thus,

$$
\left\langle \hat{P}_r - \mathbb{E}\hat{P}_r, P_r W_r P_r \right\rangle = \left\langle R_r(E), P_r W_r P_r \right\rangle.
$$

Note that $B = P_r W_r P_r$ is an operator of rank at most m_r and, in view of [\(2.5\)](#page-4-1) and Theorem [1,](#page-3-0)

$$
||P_r W_r P_r ||_{\infty} \le ||E\tilde{P}_r - P_r ||_{\infty} \le ||\tilde{P}_r - P_r ||_{\infty}
$$

$$
\lesssim \frac{||\Sigma||_{\infty}}{\bar{g}_r} \sqrt{\frac{\mathbf{r}(\Sigma)}{n}}.
$$

Thus, bound [\(4.6\)](#page-14-0) implies that with probability at least $1 - e^{-t}$:

$$
\left| \langle \hat{P}_r - \mathbb{E} \hat{P}_r, P_r W_r P_r \rangle \right| = \left| \langle R_r(E), P_r W_r P_r \rangle \right|
$$

\n
$$
\lesssim_{\gamma} m_r \frac{\| \Sigma \|_{\infty}^3}{\bar{g}_r^3} \sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \sqrt{\sqrt{\frac{t + \log(m_r)}{n}}} \sqrt{\frac{t + \log(m_r)}{n}} \right) \sqrt{\frac{t + \log(m_r)}{n}}.
$$
\n(4.11)

On the other hand,

$$
\left| \langle \hat{P}_r - \mathbb{E} \hat{P}_r, T_r \rangle \right| \leq \|\hat{P}_r - \mathbb{E} \hat{P}_r\|_1 \|T_r\|_{\infty} \leq \left(\|\hat{P}_r - P_r\|_1 + \mathbb{E} \|\hat{P}_r - P_r\|_1 \right) \|T_r\|_{\infty}
$$

$$
\leq 2m_r \left(\|\hat{P}_r - P_r\|_{\infty} + \mathbb{E} \|\hat{P}_r - P_r\|_{\infty} \right) \|T_r\|_{\infty}.
$$

Using bounds (2.12) , (2.5) , (2.3) and Theorem [1,](#page-3-0) we get

$$
\left| \langle \hat{P}_r - \mathbb{E} \hat{P}_r, T_r \rangle \right| \lesssim_{\gamma} m_r^2 \frac{\|\Sigma\|_{\infty}^3}{\bar{g}_r^3} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right) \sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \frac{1}{\sqrt{n}}. \tag{4.12}
$$

It follows from (4.10) and bounds (4.11) , (4.12) that with probability at least $1 - 2e^{-t}$

$$
\left| \langle \hat{P}_r - \mathbb{E} \hat{P}_r, \mathbb{E} \tilde{P}_r - P_r \rangle \right|
$$

$$
\lesssim_{\gamma} m_r^2 \frac{\|\Sigma\|_{\infty}^3}{\bar{g}_r^3} \sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \left(\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \sqrt{\sqrt{\frac{t + \log(m_r)}{n}}} \sqrt{\frac{t + \log(m_r)}{n}} \right) \sqrt{\frac{t + \log(m_r)}{n}}.
$$
(4.13)

Of course, the term $\langle \tilde{P}_r - \mathbb{E} \tilde{P}_r, \mathbb{E} \hat{P}_r - P_r \rangle$ can be bounded similarly.

It remains to control ϱ_r and $\tilde{\varrho}_r$. Note that $\langle L_r(E), P_r \rangle = 0$, implying that

$$
\left\langle \hat{P}_r - \mathbb{E}\hat{P}_r, P_r \right\rangle = \left\langle L_r(E) + S_r(E) - \mathbb{E}S_r(E), P_r \right\rangle = \left\langle S_r(E) - \mathbb{E}S_r(E), P_r \right\rangle.
$$

Therefore,

$$
\varrho_r = \left\langle S_r(E), P_r \right\rangle + \frac{1}{2} ||L_r(E)||_2^2 - \mathbb{E}\left(\left\langle S_r(E), P_r \right\rangle + \frac{1}{2} ||L_r(E)||_2^2 \right).
$$

The following lemma provides a concentration inequality for the random variable $\langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2$ around its expectation (thus, implying a bound on ρ_r).

Lemma 4. Suppose that condition [\(2.9\)](#page-5-0) holds for some $\gamma \in (\frac{23}{24}, 1)$. Then, there exists a constant $L > 0$ such that for all $t \geq 1$ the following bound holds with probability at least $1 - e^{-t}$:

$$
\left| \langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2 - \mathbb{E}\left(\langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2 \right) \right|
$$

$$
\leq L m_r \frac{||\Sigma||_\infty^3}{\bar{g}_r^3} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t}{n} \bigvee \left(\frac{t}{n} \right)^2 \right) \left(\sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right).
$$
 (4.14)

Combining bounds (4.7) , (4.8) , (4.9) , (4.13) and (4.14) , it is easy to derive the following bound on Υ_r that holds with probability at least $1 - 12e^{-t}$:

$$
|\Upsilon_r| \lesssim m_r^2 \frac{\|\Sigma\|_{\infty}^4}{\bar{g}_r^4} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t + \log(2m_r)}{n} \bigvee \left(\frac{t + \log(2m_r)}{n}\right)^3\right) \left(\sqrt{\frac{t + \log(2m_r)}{n}} \bigvee \frac{t + \log(2m_r)}{n}\right)
$$

The probability bound can be written as $1 - e^{-t}$ by adjusting the constant in the inequality \lesssim . For $m_r = 1$, this yields bound [\(4.3\)](#page-13-2) completing the proof of Lemma [3.](#page-12-1) \Box

We now prove Lemma [4.](#page-17-2) To this end, we will use the following representations for operators $S_r(E)$. Given $L \subset \{1, \ldots, k+1\}$, denote $m_L := \text{card}(L)$ and

$$
J_L := \{ \vec{j} := (j_1, \dots, j_k, j_{k+1}) : j_s = r, s \in L, j_s \neq r, s \notin L \}.
$$

Denote by V_L the set of vectors $\nu = (\nu_l : l \in L^c)$ with nonnegative integer components such that $\sum_{l \in L^c} \nu_l = m_L - 1$. Finally, denote by \mathcal{L}_k the set of all $L \subset \{1, \ldots, k+1\}$ such that $L \neq \emptyset, L^c \neq \emptyset$.

Lemma 5. For all $r \geq 1$,

$$
S_r(E) = \sum_{k \ge 2} \sum_{L \in \mathcal{L}_k} (-1)^{m_L - 1} \sum_{\nu \in V_L} A_{\nu}(E), \tag{4.15}
$$

where

$$
A_{\nu}(E) := B_1 E \dots B_k E B_{k+1}
$$

with $B_l = P_r, l \in L$ and $B_l = C_r^{\nu_l+1}, l \in L^c$.

.

proof. It follows from the proof of Lemma 1 in [\[11\]](#page-36-9) that the following representation holds for $S_r(E)$:

$$
S_r(E) = -\sum_{k\geq 2} \frac{1}{2\pi i} \oint_{\gamma_r} (-1)^k [R_\Sigma(\eta)E]^k R_\Sigma(\eta) d\eta,
$$

where γ_r denotes the circle centered at μ_r of radius $\bar{g}_r/2$ with counterclockwise orientation and

$$
R_{\Sigma}(\eta) = (\Sigma - \eta I)^{-1} = \sum_{j \ge 1} \frac{1}{\mu_j - \eta} P_j
$$

denotes the resolvent of Σ .^{[3](#page-18-0)} Note also that the series in the above representation of $S_r(E)$ converges in the operator norm provided that $||E||_{\infty} < \frac{\bar{g}_r}{4}$. It follows that

$$
S_r(E) = -\sum_{k\geq 2} \frac{1}{2\pi i} \oint_{\gamma_r} (-1)^k \left[\sum_{j\geq 1} \frac{1}{\mu_j - \eta} P_j E \right]^k \sum_{j\geq 1} \frac{1}{\mu_j - \eta} P_j d\eta
$$

=
$$
\sum_{k\geq 2} \sum_{j_1, \dots, j_k, j_{k+1}\geq 1} \frac{1}{2\pi i} \oint_{\gamma_r} \frac{d\eta}{\prod_{l=1}^{k+1} (\eta - \mu_{j_l})} P_{j_1} E \dots P_{j_k} E P_{j_{k+1}}.
$$

We have

$$
\sum_{j_1,\dots,j_k,j_{k+1}\geq 1} \frac{1}{2\pi i} \oint_{\gamma_r} \frac{d\eta}{\prod_{l=1}^{k+1} (\eta - \mu_{j_l})} P_{j_1} E \dots P_{j_k} E P_{j_{k+1}}
$$

=
$$
\sum_{L \subset \{1,\dots,k+1\}} \sum_{j \in J_L} \frac{1}{2\pi i} \oint_{\gamma_r} \frac{d\eta}{(\eta - \mu_r)^{m_L} \prod_{l \in L^c} (\eta - \mu_{j_l})} P_{j_1} E \dots P_{j_k} E P_{j_{k+1}}.
$$

Using Cauchy differentiation formula, we get

$$
\frac{1}{2\pi i} \oint_{\gamma_r} \frac{d\eta}{(\eta - \mu_r)^{m_L} \prod_{l \in L^c} (\eta - \mu_{j_l})} = \frac{1}{(m_L - 1)!} \left(\prod_{l \in L^c} (\eta - \mu_{j_l})^{-1} \right)_{|\eta = \mu_r}^{(m_L - 1)}.
$$

In the cases when $L = \emptyset$ or $L^c = \emptyset$ the integral in the left hand side is equal to 0. By generalized Leibniz rule,

$$
\left(\prod_{j\in L^c}(\eta-\mu_{j_l})^{-1}\right)_{|\eta=\mu_r}^{(m_L-1)}=\sum_{\nu\in V_L}\frac{(m_L-1)!}{\prod_{l\in L^c}\nu_l!}\prod_{l\in L^c}(-1)^{\nu_l}\nu_l!(\mu_r-\mu_{j_l})^{-\nu_l-1}.
$$

Thus,

$$
\sum_{j_1,\ldots,j_k,j_{k+1}\geq 1} \frac{1}{2\pi i} \oint_{\gamma_r} \frac{d\eta}{\prod_{l=1}^{k+1} (\eta - \mu_{j_l})} P_{j_1} E \ldots P_{j_k} E P_{j_{k+1}}
$$

³In the case when 0 is an eigenvalue of Σ , the sum in the right hand side of the above formula extends to $j = \infty$ with $\mu_{\infty} = 0$. See also the remark after Lemma [1](#page-4-2)

V. Koltchinskii and K. Lounici/Asymptotics in principal component analysis 20

$$
\sum_{L \subset \mathcal{L}_k} \sum_{\vec{j} \in J_L} (-1)^{m_L - 1} \sum_{\nu \in V_L} \prod_{l \in L^c} (\mu_r - \mu_{j_l})^{-\nu_l - 1} P_{j_1} E \dots P_{j_k} E P_{j_{k+1}}.
$$

Given $\nu \in V_L$, recall that $A_{\nu}(E) = B_1 E \dots B_k E B_{k+1}$, where $B_l = P_r, l \in L$ and $B_l = C_r^{\nu_l+1}, l \in L^c$. It is easy to see that

$$
\sum_{\vec{j} \in J_L} \prod_{l \in L^c} (\mu_r - \mu_{j_l})^{-\nu_l - 1} P_{j_1} E \dots P_{j_k} E P_{j_{k+1}} = A_{\nu}(E).
$$

Therefore,

$$
\sum_{j_1,\dots,j_k,j_{k+1}\geq 1} \frac{1}{2\pi i} \oint_{\gamma_r} \frac{d\eta}{\prod_{l=1}^{k+1} (\eta - \mu_{j_l})} P_{j_1} E \dots P_{j_k} E P_{j_{k+1}} = \sum_{L \in \mathcal{L}_k} (-1)^{m_L - 1} \sum_{\nu \in V_L} A_{\nu}(E)
$$

and (4.15) follows.

and (4.15) follows.

Remark 2. By a simple combinatorics,

$$
\operatorname{card}\left(\bigcup_{L\subset\{1,\dots,k+1\}} V_L\right) \le \sum_{m=0}^{k+1} \binom{k+1}{m}^2 = \binom{2(k+1)}{k+1} \le 2^{2(k+1)}.\tag{4.16}
$$

It is easy to check that

$$
\sum_{L \in \mathcal{L}_2} (-1)^{m_L - 1} \sum_{\nu \in V_L} A_{\nu}(E)
$$

 $= P_r EC_r EC_r + C_r EP_r EC_r + C_r EC_r EP_r - P_r EP_r EC_r^2 - P_r EC_r^2 EP_r - C_r^2 EP_r EP_r$

Using the fact that $C_rP_r = P_rC_r = 0$, this easily implies that

$$
\sum_{L \in \mathcal{L}_2} (-1)^{m_L - 1} \sum_{\nu \in V_L} \langle A_{\nu}(E), P_r \rangle = -\text{tr}(P_r E C_r^2 E P_r) = -\|P_r E C_r\|_2^2 = -\frac{1}{2} \|L_r(E)\|_2^2.
$$

Thus, we get

$$
\langle S_r(E), P_r \rangle + \frac{1}{2} \| L_r(E) \|_2^2 = \sum_{k \ge 3} \sum_{L \in \mathcal{L}_k} (-1)^{m_L - 1} \sum_{\nu \in V_L} \langle A_\nu(E), P_r \rangle.
$$

The next step is to study the concentration of the random variable $\langle S_r(E), P_r \rangle +$ $\frac{1}{2}||L_r(E)||_2^2$ around its expectation. More precisely, we study the concentration of its "truncated version"

$$
\left(\langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2\right) \varphi\left(\frac{||E||_{\infty}}{\delta}\right),\,
$$

where φ is a Lipschitz function with constant 1 on \mathbb{R}_+ , $0 \leq \varphi(s) \leq 1$, $\varphi(s) =$ $1, s \leq 1, \varphi(s) = 0, s > 2$. The value of $\delta > 0$ will be chosen below in such a way that $||E||_{\infty} \leq \delta$ with a high probability.

The main ingredient of the proof is the classical Gaussian isoperimetric inequality that easily implies the following statement.

Lemma 6. Let X_1, \ldots, X_n be i.i.d. centered Gaussian random variables in \mathbb{H} with covariance operator Σ . Let $f : \mathbb{H}^n \to \mathbb{R}$ be a function satisfying the following Lipschitz condition with some $L > 0$:

$$
\left|f(x_1,\ldots,x_n)-f(x'_1,\ldots,x'_n)\right| \leq L\left(\sum_{j=1}^n||x_j-x'_j||^2\right)^{1/2}, x_1,\ldots,x_n,x'_1,\ldots,x'_n \in \mathbb{H}.
$$

Suppose that, for a real number M,

$$
\mathbb{P}\left\{f(X_1,\ldots,X_n)\geq M\right\}\geq \frac{1}{4} \text{ and } \mathbb{P}\left\{f(X_1,\ldots,X_n)\leq M\right\}\geq \frac{1}{4}
$$

Then, there exists a numerical constant $D > 0$ such that for all $t \geq 1$

$$
\mathbb{P}\Big\{|f(X_1,\ldots,X_n)-M|\geq DL\|\Sigma\|_{\infty}^{1/2}\sqrt{t}\Big\}\leq e^{-t}.
$$

We will use Lemma [6](#page-20-0) that will be applied to the function

$$
f(X_1,\ldots,X_n) := \left(\langle S_r(E), P_r \rangle + \frac{1}{2} \|L_r(E)\|_2^2\right) \varphi\left(\frac{\|E\|_{\infty}}{\delta}\right)
$$

=
$$
\sum_{k \ge 3} \sum_{L \in \mathcal{L}_k} (-1)^{m_L - 1} \sum_{\nu \in V_L} f_{\nu,L}(X_1,\ldots,X_n),
$$

where

$$
f_{\nu,L}(X_1,\ldots,X_n) := \langle A_{\nu}(E), P_r \rangle \varphi\Big(\frac{\|E\|_{\infty}}{\delta}\Big),
$$

$$
E = \hat{\Sigma} - \Sigma, \quad \hat{\Sigma} = n^{-1} \sum_{j=1}^n X_j \otimes X_j.
$$

With a little abuse of notation, assume for now that X_1, \ldots, X_n are nonrandom vectors in H . We now have to check the Lipschitz condition for the function f .

Lemma 7. Let $\delta > 0$ and suppose that $||C_r||_{\infty} \delta \leq 1/24$. Then, there exists a numerical constant $D > 0$ such that, for all $X_1, \ldots, X_n, X'_1, \ldots, X'_n \in \mathbb{H}$,

$$
|f(X_1,\ldots,X_n)-f(X'_1,\ldots,X'_n)| \le Dm_r ||C_r||_\infty^3 \delta^2 \frac{||\Sigma||_\infty^{1/2} + \sqrt{\delta}}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2\right)^{1/2}.
$$
\n(4.17)

PROOF. Consider first each function $f_{\nu,L}$ separately. Let $L \in \mathcal{L}_k$ for some $k \geq 3$. Note that

$$
f_{\nu,L}(X_1,\ldots,X_n)=\langle B_1E\ldots B_kEB_{k+1},P_r\rangle\varphi\Big(\frac{\|E\|_{\infty}}{\delta}\Big),
$$

where $B_l = P_r, l \in L$ and $B_l = C_r^{\nu_l+1}, l \in L^c$. Therefore, we get

$$
|f_{\nu,L}(X_1,\ldots,X_n)| \leq \|B_1\|_{\infty}\ldots\|B_{k+1}\|_{\infty}\|E\|_{\infty}^k\|P_r\|_1 I(\|E\|_{\infty} \leq 2\delta) \quad (4.18)
$$

$$
\leq \|B_1\|_{\infty}\ldots\|B_{k+1}\|_{\infty}\|P_r\|_1 (2\delta)^k.
$$

For $X'_1, \ldots, X'_n \in \mathbb{H}$, denote

$$
\hat{\Sigma}' := n^{-1} \sum_{j=1}^n X'_j \otimes X'_j, \quad E' := \hat{\Sigma}' - \Sigma.
$$

Then, we get

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)|
$$

\n
$$
= \left| \langle B_1(E-E')B_2\ldots EB_kEB_{k+1}, P_r \rangle \varphi\left(\frac{\|E\|_{\infty}}{\delta}\right) \right|
$$

\n
$$
+ \langle B_1E'B_2(E-E')B_3\ldots EB_kEB_{k+1}, P_r \rangle \varphi\left(\frac{\|E\|_{\infty}}{\delta}\right) + \ldots
$$

\n
$$
+ \langle B_1E'B_2\ldots E'B_k(E-E')B_{k+1}, P_r \rangle \varphi\left(\frac{\|E\|_{\infty}}{\delta}\right)
$$

\n
$$
+ \langle B_1E'B_2\ldots E'B_kE'B_{k+1}, P_r \rangle \left(\varphi\left(\frac{\|E\|_{\infty}}{\delta}\right) - \varphi\left(\frac{\|E'\|_{\infty}}{\delta}\right)\right) \right|
$$

\n
$$
\leq k\|B_1\|_{\infty} \ldots \|B_{k+1}\|_{\infty} \|P_r\|_1 (\|E\|_{\infty} \vee \|E'\|_{\infty})^{k-1} \|E-E'\|_{\infty}
$$

\n
$$
+ \|B_1\|_{\infty} \ldots \|B_{k+1}\|_{\infty} \|P_r\|_1 \|E'\|_{\infty}^k \frac{1}{\delta} \|E-E'\|_{\infty},
$$

where we used the assumption that the Lipschitz constant of φ is 1. By symmetry, $||E'||_{\infty}$ in the right hand side can be replaced by $||E||_{\infty}$ implying that

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)|
$$
\n
$$
\leq k \|B_1\|_{\infty} \ldots \|B_{k+1}\|_{\infty} \|P_r\|_1 (\|E\|_{\infty} \vee \|E'\|_{\infty})^{k-1} \|E - E'\|_{\infty}
$$
\n
$$
+ \|B_1\|_{\infty} \ldots \|B_{k+1}\|_{\infty} \|P_r\|_1 (\|E\|_{\infty} \wedge \|E'\|_{\infty})^k \frac{1}{\delta} \|E - E'\|_{\infty}.
$$
\n(4.19)

If both $||E||_{\infty} \leq 2\delta$ and $||E'||_{\infty} \leq 2\delta$, this implies the bound

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)| \leq (4.20)
$$

$$
||B_1||_{\infty} \ldots ||B_{k+1}||_{\infty} ||P_r||_1 (k+2) (2\delta)^{k-1} ||E - E'||_{\infty}.
$$

If $||E||_{\infty} \le 2\delta$, but $||E'||_{\infty} > 2\delta$, then $f_{\nu,L}(X'_1, \ldots, X'_n) = 0$ and, by [\(4.18\)](#page-20-1),

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)| = |f_{\nu,L}(X_1,\ldots,X_n)|
$$

$$
\leq ||B_1||_{\infty} \ldots ||B_{k+1}||_{\infty} ||P_r||_1 (2\delta)^k.
$$

If, in addition, $||E-E'||_{\infty} > \delta$, then bound [\(4.20\)](#page-21-0) still holds. On the other hand, if $||E - E'||_{\infty}$ ≤ δ , then $||E'||_{\infty}$ ≤ 3 δ and we get a slightly worse bound than [\(4.20\)](#page-21-0):

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)| \leq (4.21)
$$

$$
||B_1||_{\infty} \ldots ||B_{k+1}||_{\infty} ||P_r||_1(k+2)(3\delta)^{k-1} ||E - E'||_{\infty}.
$$

The case when $||E||_{\infty} > 2\delta$ and $||E'||_{\infty} \leq 2\delta$ can be handled similarly and the case when both $||E||_{\infty} > 2\delta$ and $||E'||_{\infty} > 2\delta$ is trivial since function $f_{\nu,L}$ becomes 0. In each of these cases, bound [\(4.21\)](#page-21-1) holds.

The following bound (see Lemma 5 in [\[11](#page-36-9)]) provides a control of $||E - E'||_{\infty}$:

$$
||E - E'||_{\infty} \le \frac{4||\Sigma||_{\infty}^{1/2} + 4\sqrt{2\delta}}{\sqrt{n}} \left(\sum_{j=1}^{n} ||X_j - X'_j||^2\right)^{1/2} \sqrt{\frac{4}{n}} \sum_{j=1}^{n} ||X_j - X'_j||^2.
$$
\n(4.22)

Substituting the last bound into [\(4.21\)](#page-21-1), we get

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)| \leq (4.23)
$$

$$
\left(4\|B_1\|_{\infty}\ldots\|B_{k+1}\|_{\infty}\|P_r\|_{1}(k+2)(3\delta)^{k-1}\frac{\|\Sigma\|_{\infty}^{1/2}+\sqrt{2\delta}}{\sqrt{n}}\left(\sum_{j=1}^n\|X_j-X'_j\|^2\right)^{1/2}\right)\bigvee
$$

$$
\left(4\|B_1\|_{\infty}\ldots\|B_{k+1}\|_{\infty}\|P_r\|_{1}(k+2)(3\delta)^{k-1}\frac{1}{n}\sum_{j=1}^n\|X_j-X'_j\|^2\right).
$$
 (4.23)

In view of [\(4.18\)](#page-20-1), the left hand side is also bounded from above by

$$
2||B_1||_{\infty} \ldots ||B_{k+1}||_{\infty} ||P_r||_1 (2\delta)^k,
$$

which allows one to get from [\(4.23\)](#page-22-0) that

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)| \leq (4.24)
$$

$$
\left(4\|B_1\|_{\infty}\ldots\|B_{k+1}\|_{\infty}\|P_r\|_{1}(k+2)(3\delta)^{k-1}\frac{\|\Sigma\|_{\infty}^{1/2}+\sqrt{2\delta}}{\sqrt{n}}\left(\sum_{j=1}^n\|X_j-X'_j\|^2\right)^{1/2}\right)\bigvee
$$

$$
\left(4\|B_1\|_{\infty}\ldots\|B_{k+1}\|_{\infty}\|P_r\|_{1}(3\delta)^{k-1}\left((k+2)\frac{1}{n}\sum_{j=1}^n\|X_j-X'_j\|^2\bigwedge\delta\right)\right).
$$
 (4.24)

In the case when

$$
\left(\sum_{j=1}^{n} \|X_j - X'_j\|^2\right)^{1/2} \le \sqrt{\frac{\delta n}{k+2}},
$$

we have

$$
4||B_1||_{\infty} \dots ||B_{k+1}||_{\infty} ||P_r||_1 (3\delta)^{k-1} \left((k+2) \frac{1}{n} \sum_{j=1}^n ||X_j - X'_j||^2 \bigwedge \delta \right)
$$

\n
$$
\leq 4||B_1||_{\infty} \dots ||B_{k+1}||_{\infty} ||P_r||_1 (3\delta)^{k-1/2} \sqrt{k+2} \frac{1}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2 \right)^{1/2}.
$$

It is equally easy to check that the same bound holds in the opposite case. As a consequence, [\(4.24\)](#page-22-1) implies that

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)| \leq (4.25)
$$

$$
4||B_1||_{\infty} \ldots ||B_{k+1}||_{\infty} ||P_r||_1 (k+2)(3\delta)^{k-1} \frac{||\Sigma||_{\infty}^{1/2} + \sqrt{2\delta}}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2\right)^{1/2}.
$$

Note that

$$
||B_1||_{\infty} \dots ||B_{k+1}||_{\infty} = \prod_{l \in L^c} ||C_r^{\nu_l+1}||_{\infty} \le ||C_r||_{\infty}^{\sum_{l \in L^c} (\nu_l+1)} = ||C_r||_{\infty}^k, \quad (4.26)
$$

where we used the facts that

$$
\sum_{l \in L^c} \nu_l = m_L - 1, \ \ \text{card}(L^c) = k + 1 - m_L.
$$

Thus, we get from [\(4.25\)](#page-22-2)

$$
|f_{\nu,L}(X_1,\ldots,X_n) - f_{\nu,L}(X'_1,\ldots,X'_n)| \le
$$

$$
4||C_r||_{\infty}^k ||P_r||_1 (k+2)(3\delta)^{k-1} \frac{||\Sigma||_{\infty}^{1/2} + \sqrt{2\delta}}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2\right)^{1/2},
$$

which, taking also into account [\(4.16\)](#page-19-0), yields

$$
|f(X_1, \ldots, X_n) - f(X'_1, \ldots, X'_n)|
$$
\n
$$
\leq 4 \sum_{k \geq 3} \sum_{L \in \mathcal{L}_k} \sum_{\nu \in V_L} ||C_r||_{\infty}^k ||P_r||_1 (k+2)(3\delta)^{k-1} \frac{||\Sigma||_{\infty}^{1/2} + \sqrt{2\delta}}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2\right)^{1/2}
$$
\n
$$
\leq 4 \sum_{k \geq 3} 2^{2(k+1)} ||C_r||_{\infty}^k ||P_r||_1 (k+2)(3\delta)^{k-1} \frac{||\Sigma||_{\infty}^{1/2} + \sqrt{2\delta}}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2\right)^{1/2}
$$
\n
$$
\leq 4 \sum_{k \geq 3} (k+2) 2^{2(k+1)} 3^{k-1} \left(\frac{1}{24}\right)^{k-3} ||C_r||^3 ||P_r||_1 \delta^2 \frac{||\Sigma||_{\infty}^{1/2} + \sqrt{2\delta}}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2\right)^{1/2}
$$
\n
$$
\leq D ||C_r||^3 ||P_r||_1 \delta^2 \frac{||\Sigma||_{\infty}^{1/2} + \sqrt{\delta}}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2\right)^{1/2},
$$
\n(4.27)\n
$$
\leq D ||C_r||^3 ||P_r||_1 \delta^2 \frac{||\Sigma||_{\infty}^{1/2} + \sqrt{\delta}}{\sqrt{n}} \left(\sum_{j=1}^n ||X_j - X'_j||^2\right)^{1/2},
$$

where D is a numerical constant and we used the condition $||C_r||_{\infty} \delta \le 1/24$.

We return to the proof of Lemma [4.](#page-17-2)

PROOF. Note that, under condition (2.9) , the lower bound of Theorem [1](#page-3-0) implies that $\mathbf{r}(\Sigma) \lesssim n$. Let $t \geq 1$ and define

$$
\delta_n(t) := \mathbb{E} \|\hat{\Sigma} - \Sigma\|_{\infty} + C \|\Sigma\|_{\infty} \left[\sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right].
$$

If constant C in the above definition is sufficiently large and $\mathbf{r}(\Sigma) \lesssim n$, then it follows from Theorem [2](#page-3-1) that $||E||_{\infty} = ||\hat{\Sigma} - \Sigma||_{\infty} \leq \delta_n(t)$ with probability at least $1 - e^{-t}$. Note also that, under condition (2.9) ,

$$
\delta_n(t) \lesssim ||\Sigma||_{\infty} \left[\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right]
$$

(since $\mathbf{r}(\Sigma) \lesssim n$).

Assume that $\delta_n(t) \leq \frac{\bar{g}_r}{24}$. Since $\bar{g}_r \leq ||\Sigma||_{\infty}$, we have

$$
C\left[\sqrt{\frac{t}{n}}\bigvee\frac{t}{n}\right] \le \frac{\bar{g}_r}{24\|\Sigma\|_{\infty}} \le 1,
$$

which implies that $t \lesssim n$. Thus, in view of the upper bound of Theorem [1,](#page-3-0)

$$
\delta_n(t) \lesssim ||\Sigma||_{\infty} \left[\sqrt{\frac{\mathbf{r}(\Sigma)}{n}} \bigvee \sqrt{\frac{t}{n}} \right].
$$

For a random variable ξ , denote by $Med(\xi)$ its median. Let

$$
M := \text{Med}\left(\langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2\right).
$$

In what follows, we set $\delta := \delta_n(t)$ in the definition of function $f(X_1, \ldots, X_n)$. Suppose that $t \geq \log(4)$ (by adjusting the values of the constants the resulting bound can be easily extended to $t \geq 1$ as it is claimed in Lemma [4\)](#page-17-2). Then, we have $\mathbb{P}\{\|\hat{\Sigma} - \Sigma\|_{\infty} \ge \delta_n(t)\} \le \frac{1}{4}$, and

$$
\mathbb{P}\left\{f(X_1,\ldots,X_n)\geq M\right\} \geq \mathbb{P}\left\{f(X_1,\ldots,X_n)\geq M,\,||E||_{\infty}<\delta\right\}
$$

$$
\geq \mathbb{P}\left\{\langle S_r(E),P_r\rangle+\frac{1}{2}||L_r(E)||_2^2\geq M\right\}-\mathbb{P}\left\{\|E\|_{\infty}\geq \delta\right\}\geq \frac{1}{4}.
$$

Quite similarly, $\mathbb{P}\{f(X_1,\ldots,X_n)\leq M\}\geq \frac{1}{4}$. It follows from Lemma [6](#page-20-0) that with probability at least $1 - e^{-t}$

$$
|f(X_1, ..., X_n) - M| \le Dm_r \frac{\delta^2}{\bar{g}_r^3} ||\Sigma||_{\infty}^{1/2} \left(||\Sigma||_{\infty}^{1/2} + \sqrt{\delta} \right) \sqrt{\frac{t}{n}}
$$

$$
\le D'm_r \frac{||\Sigma||_{\infty}^3}{\bar{g}_r^3} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t}{n} \right) \sqrt{\frac{t}{n}},
$$

for some numerical constant $D' > 0$. Since on the event $\{|E\|_{\infty} \leq \delta\}$

$$
\langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2 = f(X_1, \dots, X_n),
$$

we easily obtain that with probability at least $1 - 2e^{-t}$

$$
\left| \langle S_r(E), P_r \rangle + \frac{1}{2} \| L_r(E) \|_2^2 - M \right| \le D' m_r \frac{\|\Sigma\|_{\infty}^3}{\bar{g}_r^3} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t}{n} \right) \sqrt{\frac{t}{n}}. \tag{4.28}
$$

It remains to prove a similar bound in the case when $\delta_n(t) > \frac{\bar{g}_r}{24}$. By definition of $\delta_n(t)$ and in view of assumption [\(2.9\)](#page-5-0), we get

$$
C\|\Sigma\|_{\infty}\left(\sqrt{\frac{t}{n}}\bigvee\frac{t}{n}\right) > \frac{\bar{g}_r}{24} - \mathbb{E}\|\hat{\Sigma} - \Sigma\|_{\infty} \ge \frac{\bar{g}_r}{24} - \frac{\bar{g}_r}{48} = \frac{\bar{g}_r}{48}.\tag{4.29}
$$

In view of [\(2.7\)](#page-4-3), [\(2.8\)](#page-4-0), the fact that $||P_r||_1 = m_r$ and the trace duality inequality, we obtain

$$
\left| \langle S_r(E), P_r \rangle + \frac{1}{2} \| L_r(E) \|_2^2 \right| \le \| S_r(E) \|_\infty \| P_r \|_1 + \| C_r E P_r \|_2^2
$$

$$
\le 14 \frac{\| E \|_\infty^2}{\bar{g}_r^2} \| P_r \|_1 + \| C_r \|_\infty^2 \| E \|_\infty^2 \| P_r \|_1
$$

$$
\le 15 m_r \frac{\| E \|_\infty^2}{\bar{g}_r^2}.
$$

Since $\mathbb{P}\{\Vert E\Vert_{\infty} \leq \delta\} \geq 1-e^{-t}$, we get that for all $t \geq 1$ with probability at least $1 - e^{-t}$ that

$$
\left| \langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2 \right| \leq D m_r \frac{\|\Sigma\|_{\infty}^2}{\bar{g}_r^2} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t}{n} \bigvee \left(\frac{t}{n}\right)^2 \right)
$$

for some numerical constant $D > 0$. Using this bound with $t = \log 4$, we easily get that

$$
|M| \leq \mathrm{Med}\bigg(\bigg|\langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2\bigg|\bigg) \leq Dm_r \frac{\|\Sigma\|_{\infty}^2}{\bar{g}_r^2} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{\log 4}{n} \bigvee \left(\frac{\log 4}{n}\right)^2\right).
$$

Combining the last two displays, we get that for some constant $D > 0$ and for all $t \geq 1$ with probability at least $1 - e^{-t}$

$$
\left| \langle S_r(E), P_r \rangle + \frac{1}{2} \| L_r(E) \|_2^2 - M \right| \leq D m_r \frac{\|\Sigma\|_{\infty}^2}{\bar{g}_r^2} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t}{n} \bigvee \left(\frac{t}{n} \right)^2 \right).
$$
\n(4.30)

If $\delta_n(t) > \frac{\bar{g}_r}{24}$, then [\(4.29\)](#page-24-0) holds and it follows from bound [\(4.30\)](#page-25-0) that with some constant $\overline{D} > 0$

$$
\left| \langle S_r(E), P_r \rangle + \frac{1}{2} \| L_r(E) \|_2^2 - M \right| \leq D m_r \frac{\|\Sigma\|_{\infty}^3}{\bar{g}_r^3} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{t}{n} \bigvee \left(\frac{t}{n}\right)^2 \right) \left(\sqrt{\frac{t}{n}} \bigvee \frac{t}{n} \right).
$$
\n(4.31)

Of course, in the case when $\delta_n(t) \leq \frac{\bar{g}_r}{24}$, bound [\(4.31\)](#page-25-1) also holds (it follows from bound [\(4.28\)](#page-24-1)). By integrating tail probabilities of bound [\(4.31\)](#page-25-1) that holds for all $t \geq 1$ we easily get

$$
\left| \mathbb{E} \left[\langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2 \right] - M \right| \leq \mathbb{E} \left| \left[\langle S_r(E), P_r \rangle + \frac{1}{2} ||L_r(E)||_2^2 \right] - M \right| \leq
$$

$$
\leq D m_r \frac{\|\Sigma\|_{\infty}^3}{\bar{g}_r^3} \left(\frac{\mathbf{r}(\Sigma)}{n} \bigvee \frac{1}{n} \bigvee \left(\frac{1}{n} \right)^2 \right) \sqrt{\frac{1}{n}}
$$

for some $D > 0$. Thus, we can replace the median M in bound (4.31) by the expectation which yields the bound of Lemma [4.](#page-17-2) \Box

Consider now three samples (X_1, \ldots, X_n) , $(\tilde{X}_1, \ldots, \tilde{X}_n)$ and $(\bar{X}_1, \ldots, \bar{X}_n)$ of i.i.d. copies of X with $\hat{\Sigma}, \hat{\Sigma}$ and $\bar{\Sigma}$ being the sample covariances based on the corresponding samples of size n. Let $E := \hat{\Sigma} - \Sigma, \tilde{E} := \tilde{\Sigma} - \Sigma$ and $\bar{E} := \bar{\Sigma} - \Sigma$. In view of the representation of Lemma [3,](#page-12-1) to study the asymptotic behavior of $(1+\hat{b}_r)^2 - (1+b_r)^2$ and other related statistics we will have to deal with random vectors

$$
\Xi_r := \begin{pmatrix} \sqrt{2} \langle L_r(E), L_r(\tilde{E}) \rangle \\ \sqrt{2} \langle L_r(\tilde{E}), L_r(\tilde{E}) \rangle \\ \|L_r(E)\|_2^2 - \mathbb{E} \|L_r(E)\|_2^2 \\ \|L_r(\tilde{E})\|_2^2 - \mathbb{E} \|L_r(\tilde{E})\|_2^2 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \langle P_r EC_r, P_r \tilde{E} C_r \rangle \\ 2\sqrt{2} \langle P_r \tilde{E} C_r, P_r \tilde{E} C_r \rangle \\ 2(\|P_r EC_r\|_2^2 - \mathbb{E} \|P_r EC_r\|_2^2) \\ 2(\|P_r \tilde{E} C_r\|_2^2 - \mathbb{E} \|P_r \tilde{E} C_r\|_2^2) \\ 2(\|P_r \tilde{E} C_r\|_2^2 - \mathbb{E} \|P_r \tilde{E} C_r\|_2^2) \end{pmatrix} . \tag{4.32}
$$

Let $\{\eta_{j,k}, \tilde{\eta}_{j,k}, \bar{\eta}_{j,k}, k \in \Delta_r, j \in \Delta_s, s \neq r\}$ be i.i.d. standard normal random variables. Define the random vector

$$
\Theta_r := \begin{pmatrix}\n\sqrt{2} \sum_{k \in \Delta_r} \sum_{s \neq r} \frac{\mu_s}{(\mu_s - \mu_r)^2} \sum_{j \in \Delta_s} \eta_{j,k} \tilde{\eta}_{j,k} \\
\sqrt{2} \sum_{k \in \Delta_r} \sum_{s \neq r} \frac{\mu_s}{(\mu_s - \mu_r)^2} \sum_{j \in \Delta_s} \tilde{\eta}_{j,k} \tilde{\eta}_{j,k} \\
\sum_{k \in \Delta_r} \sum_{s \neq r} \frac{\mu_s}{(\mu_s - \mu_r)^2} \sum_{j \in \Delta_s} (\eta_{j,k}^2 - 1) \\
\sum_{k \in \Delta_r} \sum_{s \neq r} \frac{\mu_s}{(\mu_s - \mu_r)^2} \sum_{j \in \Delta_s} (\tilde{\eta}_{j,k}^2 - 1) \\
\sum_{k \in \Delta_r} \sum_{s \neq r} \frac{\mu_s}{(\mu_s - \mu_r)^2} \sum_{j \in \Delta_s} (\tilde{\eta}_{j,k}^2 - 1)\n\end{pmatrix}.
$$
\n(4.33)

Lemma 8. The following representation holds:

$$
n\Xi_r = 2\mu_r \tilde{\Theta}_r + \xi,\tag{4.34}
$$

where $\tilde{\Theta}_r$ is a random vector in \mathbb{R}^5 whose distribution coincides with the distribution of Θ_r and the components ξ_j of the remainder $\xi \in \mathbb{R}^5$ satisfy the following bound: $5/2$

$$
\max_{1 \leq j \leq 5} \mathbb{E} |\xi_j| \lesssim \frac{\|\Sigma\|_{\infty}^2}{\bar{g}_r^2} \bigg(\frac{m_r^{5/2}}{\sqrt{n}} \vee \frac{m_r^3}{n} \bigg) \mathbf{r}(\Sigma).
$$

proof. Set

$$
U = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} P_r X_j \otimes C_r X_j, \quad \tilde{U} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} P_r \tilde{X}_j \otimes C_r \tilde{X}_j, \quad \bar{U} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} P_r \bar{X}_j \otimes C_r \bar{X}_j
$$

and note that

$$
n\Xi_r=\left(\begin{array}{c}2\sqrt{2}\big\langle U,\tilde{U}\big\rangle\\2\sqrt{2}\big\langle\tilde{U},\bar{U}\big\rangle\\2(\|U\|_2^2-\mathbb{E}\|U\|_2^2)\\2(\|\tilde{U}\|_2^2-\mathbb{E}\|\tilde{U}\|_2^2)\\2(\|\tilde{U}\|_2^2-\mathbb{E}\|\tilde{U}\|_2^2)\end{array}\right).
$$

$$
\Gamma_r = n^{-1} \sum_{j=1}^n P_r X_j \otimes P_r X_j, \quad \tilde{\Gamma}_r = n^{-1} \sum_{j=1}^n P_r \tilde{X}_j \otimes P_r \tilde{X}_j, \quad \bar{\Gamma}_r = n^{-1} \sum_{j=1}^n P_r \bar{X}_j \otimes P_r \bar{X}_j
$$

be the sample covariance operators based, respectively, on the "projected" samples $P_r X_j, j = 1, ..., n, P_r \tilde{X}_j, j = 1, ..., n$ and $P_r \overline{X}_j, j = 1, ..., n$ of i.i.d. centered Gaussian random variables with covariance operator $P_r \Sigma P_r = \mu_r P_r$. $\Gamma_r, \tilde{\Gamma}_r, \bar{\Gamma}_r$ can be viewed as symmetric positive semi-definite operators acting in the eigenspace of eigenvalue μ_r and they admit the following spectral decompositions:

$$
\Gamma_r = \sum_{k \in \Delta_r} \gamma_r \phi_k \otimes \phi_k, \quad \tilde{\Gamma}_r = \sum_{k \in \Delta_r} \tilde{\gamma}_r \tilde{\phi}_k \otimes \tilde{\phi}_k, \quad \bar{\Gamma}_r = \sum_{k \in \Delta_r} \bar{\gamma}_r \bar{\phi}_k \otimes \bar{\phi}_k,
$$

where $\gamma_k \geq 0$ are the eigenvalues of Γ_r with associated eigenvectors ϕ_k , $\tilde{\gamma}_r \geq 0$ are the eigenvalues of $\tilde{\Gamma}_r$ with associated eigenvectors $\tilde{\phi}_k$ and $\bar{\gamma}_r \geq 0$ are the eigenvalues of $\bar{\Gamma}_r$ with associated eigenvectors $\bar{\phi}_k$. Note also that $\{\phi_k, k \in \Delta_r\},$ $\{\tilde{\phi}_k, k \in \Delta_r\}$ and $\{\bar{\phi}_k, k \in \Delta_r\}$ are three possibly different orthonormal bases of the eigenspace of μ_r .

Let $X^{(k)}$, $\bar{X}^{(k)}$, $\bar{X}^{(k)}$, $k \in \Delta_r$ be independent copies of X (also independent of X_j , \tilde{X}_j , \bar{X}_j , $j = 1, \ldots, n$). Denote

$$
V = \sum_{k \in \Delta_r} \sqrt{\gamma_k} \phi_k \otimes C_r X^{(k)}, \quad \tilde{V} = \sum_{k \in \Delta_r} \sqrt{\tilde{\gamma}_k} \tilde{\phi}_k \otimes C_r \tilde{X}^{(k)}, \quad \bar{V} = \sum_{k \in \Delta_r} \sqrt{\tilde{\gamma}_k} \bar{\phi}_k \otimes C_r \bar{X}^{(k)}
$$

Given $\{P_r X_1, \ldots, P_r X_n, P_r \tilde{X}_1, \ldots, P_r \tilde{X}_n, P_r \bar{X}_1, \ldots, P_r \bar{X}_n\}$, the conditional distributions of (U, \tilde{U}, \bar{U}) and (V, \tilde{V}, \bar{V}) are the same. To see this note that, conditionally on $\{P_r X_1, \ldots, P_r X_n, P_r \tilde{X}_1, \ldots, P_r \tilde{X}_n, P_r \bar{X}_1, \ldots, P_r \bar{X}_n\}$, U, \tilde{U}, \bar{U} are independent centered Gaussian random operators and so are V, \tilde{V}, \bar{V} .⁴ Thus, it is enough to check that conditionally on the same random variables the covariance operators of U and V coincide (of course, the same would apply to the couples U and V, \bar{U} and V). To this end, let T denote a linear mapping from $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$ into itself such that

$$
T(u_1\otimes u_2\otimes u_3\otimes u_4)=(u_1\otimes u_3\otimes u_2\otimes u_4)
$$

(note that T is uniquely defined). By an easy computation,

$$
\mathbb{E}(U \otimes U|P_r X_j, j=1,\ldots,n) = T(\Gamma_r \otimes (C_r \Sigma C_r)) = \mathbb{E}(V \otimes V|P_r X_j, j=1,\ldots,n),
$$

which implies the claim for U and V (see also the proof of Lemma 5 in [\[12\]](#page-36-10) for more details on this argument).

Let

⁴Recall that P_rX_j and C_rX_j are independent since they are jointly Gaussian and uncorrelated (the last property follows from the fact that $P_rC_r = C_rP_r = 0$). Thus, conditionally on $\{P_r X_i\}$, U is a mean zero Gaussian random operator.

Consequently, the distribution of $n \Xi_r$ coincides with the distribution of

$$
\Lambda_r:=\left(\begin{array}{c}2\sqrt{2}\big\langle V,\tilde{V}\big\rangle\\2\sqrt{2}\big\langle\tilde{V},\bar{V}\big\rangle\\2(\|V\|_2^2-\mathbb{E}\|V\|_2^2)\\2(\|\tilde{V}\|_2^2-\mathbb{E}\|\tilde{V}\|_2^2)\\2(\|\tilde{V}\|_2^2-\mathbb{E}\|\tilde{V}\|_2^2)\\ \end{array}\right).
$$

Note that

$$
\langle V, \tilde{V} \rangle = \sum_{k,l \in \Delta_r} \sqrt{\gamma_k} \sqrt{\tilde{\gamma}_l} \langle \phi_k \otimes C_r X^{(k)}, \tilde{\phi}_l \otimes C_r \tilde{X}^{(l)} \rangle = \mu_r \sum_{k,l \in \Delta_r} \langle \phi_k, \tilde{\phi}_l \rangle \langle C_r X^{(k)}, C_r \tilde{X}^{(l)} \rangle + \eta,
$$

where

$$
\eta := \sum_{k,l \in \Delta_r} (\sqrt{\gamma_k} \sqrt{\tilde{\gamma}_l} - \mu_r) \langle \phi_k, \tilde{\phi}_l \rangle \langle C_r X^{(k)}, C_r \tilde{X}^{(l)} \rangle
$$

For the remainder η , the following bound holds:

$$
|\eta| \leq \left(\sum_{k,l\in\Delta_r} (\sqrt{\gamma_k}\sqrt{\tilde{\gamma}_l} - \mu_r)^2\right)^{1/2} \left(\sum_{k\in\Delta_r} \|C_r X^{(k)}\|^2 \sum_{l\in\Delta_r} \|C_r \tilde{X}^{(l)}\|^2\right)^{1/2},
$$

which, using the independence of $\gamma_k, \tilde{\gamma}_l, C_r X^{(k)}, C_r \tilde{X}^{(l)}$ easily implies that

$$
\mathbb{E}|\eta| \leq \left(\mathbb{E} \sum_{k,l \in \Delta_r} (\sqrt{\gamma_k} \sqrt{\tilde{\gamma}_l} - \mu_r)^2 \right)^{1/2} \left(\mathbb{E} \sum_{k \in \Delta_r} \|C_r X^{(k)}\|^2 \mathbb{E} \sum_{l \in \Delta_r} \|C_r \tilde{X}^{(l)}\|^2 \right)^{1/2}
$$

$$
\leq m_r \left(\mathbb{E} \sum_{k,l \in \Delta_r} (\sqrt{\gamma_k} \sqrt{\tilde{\gamma}_l} - \mu_r)^2 \right)^{1/2} \mathbb{E} \|C_r X\|^2.
$$

Observe also that

$$
\left|\sqrt{\gamma_k}\sqrt{\tilde{\gamma}_l}-\mu_r\right|\leq \frac{\gamma_k\tilde{\gamma}_l-\mu_r^2}{\mu_r}\leq |\gamma_k-\mu_r|+|\tilde{\gamma}_l-\mu_r|+\frac{|\gamma_k-\mu_r||\tilde{\gamma}_l-\mu_r|}{\mu_r},
$$

which implies

$$
\sum_{k,l\in\Delta_r} (\sqrt{\gamma_k}\sqrt{\tilde{\gamma}_l} - \mu_r)^2 \leq 3m_r \sum_{k\in\Delta_r} (\gamma_k - \mu_r)^2 + 3m_r \sum_{l\in\Delta_r} (\tilde{\gamma}_l - \mu_r)^2 + \frac{3}{\mu_r^2} \sum_{k\in\Delta_r} (\gamma_k - \mu_r)^2 \sum_{l\in\Delta_r} (\tilde{\gamma}_l - \mu_r)^2
$$

$$
\leq 3m_r \|\Gamma_r - \mu_r P_r\|_2^2 + 3m_r \|\tilde{\Gamma}_r - \mu_r P_r\|_2^2 + \frac{3}{\mu_r^2} \|\Gamma_r - \mu_r P_r\|_2^2 \|\tilde{\Gamma}_r - \mu_r P_r\|_2^2.
$$

Hence, we get (using independence of $\Gamma_r, \tilde{\Gamma}_r$)

$$
\mathbb{E}\sum_{k,l\in\Delta_r}(\sqrt{\gamma_k}\sqrt{\tilde{\gamma}_l}-\mu_r)^2\leq 3m_r\mathbb{E}\|\Gamma_r-\mu_rP_r\|_2^2+3m_r\mathbb{E}\|\tilde{\Gamma}_r-\mu_rP_r\|_2^2+\frac{3}{\mu_r^2}\mathbb{E}\|\Gamma_r-\mu_rP_r\|_2^2\mathbb{E}\|\tilde{\Gamma}_r-\mu_rP_r\|_2^2.
$$

Since Γ_r , $\tilde{\Gamma}_r$ are sample covariances based on n i.i.d. centered Gaussian observations with the true covariance $\mu_r P_r$, we easily get

$$
\mathbb{E} \|\Gamma_r - \mu_r P_r\|_2^2 = \mathbb{E} \|\tilde{\Gamma}_r - \mu_r P_r\|_2^2 \le \frac{\mathbb{E} \|P_r X\|^4}{n} \lesssim \frac{\left(\text{tr}(\mu_r P_r)\right)^2}{n} = \frac{\mu_r^2 m_r^2}{n}.
$$

Therefore,

$$
\mathbb{E} \sum_{k,l \in \Delta_r} (\sqrt{\gamma_k} \sqrt{\tilde{\gamma}_l} - \mu_r)^2 \lesssim \frac{\mu_r^2 m_r^3}{n} + \frac{\mu_r^2 m_r^4}{n^2}.
$$

This yields the following bound on $\mathbb{E}|\eta|$:

$$
\mathbb{E}|\eta| \lesssim \left(\frac{\mu_r m_r^{5/2}}{\sqrt{n}} + \frac{\mu_r m_r^3}{n}\right) \mathbb{E}||C_r X||^2 = \left(\frac{\mu_r m_r^{5/2}}{\sqrt{n}} + \frac{\mu_r m_r^3}{n}\right) \text{tr}(C_r \Sigma C_r)
$$

$$
\lesssim \frac{||\Sigma||_{\infty}^2}{\bar{g}_r^2} \left(\frac{m_r^{5/2}}{\sqrt{n}} \vee \frac{m_r^3}{n}\right) \text{r}(\Sigma).
$$
 (4.35)

Similarly, we have

$$
||V||_2^2 = \sum_{k \in \Delta_r} \gamma_k ||\phi_k \otimes C_r X^{(k)}||_2^2 = \sum_{k \in \Delta_r} \gamma_k ||C_r X^{(k)}||^2,
$$

which implies

$$
||V||_2^2 - \mathbb{E}||V||_2^2 = \mu_r \sum_{k \in \Delta_r} \left[||C_r X^{(k)}||^2 - \mathbb{E}||C_r X^{(k)}||^2\right] + \zeta,
$$

where

$$
\zeta := \sum_{k \in \Delta_r} \Big[(\gamma_k - \mu_r) \|C_r X^{(k)}\|^2 - \mathbb{E}(\gamma_k - \mu_r) \|C_r X^{(k)}\|^2 \Big].
$$

The following bound is immediate

$$
\mathbb{E}|\zeta| \le 2\mathbb{E} \max_{k \in \Delta_r} |\gamma_k - \mu_r| \sum_{k \in \Delta_r} \mathbb{E} \|C_r X^{(k)}\|^2 \le 2m_r \mathbb{E} \|\Gamma_r - \mu_r P_r\|_{\infty} \mathbb{E} \|C_r X\|^2,
$$

where we used the independence of random variables $\gamma_k, k \in \Delta_r$ and $C_r X^{(k)}, k \in$ Δ_r . Applying the bound of Theorem [1](#page-3-0) to the sample covariance Γ_r , we easily get

$$
\mathbb{E}\|\Gamma_r - \mu_r P_r\|_{\infty} \lesssim \mu_r \bigg(\sqrt{\frac{m_r}{n}} \vee \frac{m_r}{n}\bigg).
$$

Therefore, we can conclude that

$$
\mathbb{E}|\zeta| \lesssim m_r \mu_r \left(\sqrt{\frac{m_r}{n}} \vee \frac{m_r}{n}\right) \text{tr}(C_r \Sigma C_r) \lesssim \frac{\|\Sigma\|_{\infty}^2}{\bar{g}_r^2} \left(\frac{m_r^{3/2}}{\sqrt{n}} \vee \frac{m_r^2}{n}\right) \text{r}(\Sigma). \tag{4.36}
$$

As a consequence of (4.35) , (4.36) and similar bounds for other components of vector Λ_r , we get that

$$
\begin{pmatrix} 2\sqrt{2}\langle V, \tilde{V} \rangle \\ 2\sqrt{2}\langle \tilde{V}, \bar{V} \rangle \\ 2(\|V\|_2^2 - \mathbb{E} \|V\|_2^2) \\ 2(\|\tilde{V}\|_2^2 - \mathbb{E}\|\tilde{V}\|_2^2) \\ 2(\|\tilde{V}\|_2^2 - \mathbb{E}\|\tilde{V}\|_2^2) \end{pmatrix} = 2\mu_r \tilde{\Theta}_r + \xi,
$$
 (4.37)

where

$$
\tilde{\Theta}_r = \left(\begin{array}{c}\sqrt{2}\left\langle\sum_{k\in\Delta_r}\phi_k\otimes C_rX^{(k)},\sum_{k\in\Delta_r}\tilde{\phi}_k\otimes C_r\tilde{X}^{(k)}\right\rangle\\\sqrt{2}\left\langle\sum_{k\in\Delta_r}\tilde{\phi}_k\otimes C_r\tilde{X}^{(k)},\sum_{k\in\Delta_r}\bar{\phi}_k\otimes C_r\bar{X}^{(k)}\right\rangle\\\sum_{k\in\Delta_r}\|C_rX^{(k)}\|^2 - \sum_{k\in\Delta_r}\mathbb{E}\|C_rX^{(k)}\|^2\\\sum_{k\in\Delta_r}\|C_r\tilde{X}^{(k)}\|^2 - \sum_{k\in\Delta_r}\mathbb{E}\|C_r\tilde{X}^{(k)}\|^2\\\sum_{k\in\Delta_r}\|C_r\bar{X}^{(k)}\|^2 - \sum_{k\in\Delta_r}\mathbb{E}\|C_r\bar{X}^{(k)}\|^2_2\end{array}\right)
$$

and $\xi \in \mathbb{R}^5$ is a random vector with the components satisfying the following bound: $\frac{1}{2}$

$$
\max_{1 \leq j \leq 5} \mathbb{E} |\xi_j| \lesssim \frac{\|\Sigma\|_{\infty}^2}{\bar{g}_r^2} \left(\frac{m_r^{5/2}}{\sqrt{n}} \vee \frac{m_r^3}{n} \right) \mathbf{r}(\Sigma).
$$

It remains to show that the distribution of $\tilde{\Theta}_r$ coincides with the distribution of Θ_r . To this end, note that the following representation holds:

$$
C_r X^{(k)} = \sum_{s \neq r} \frac{1}{\mu_r - \mu_s} P_s X^{(k)} = \sum_{s \neq r} \frac{\mu_s^{1/2}}{\mu_r - \mu_s} \sum_{j \in \Delta_s} \eta_{j,k} \theta_j, \ k \in \Delta_r,
$$

where, for all $s \geq 1$, θ_j , $j \in \Delta_s$ is an orthonormal basis of the eigenspace of Σ corresponding to the eigenvalue μ_s and $\{\eta_{j,k}\}\$ are i.i.d. standard normal random variables. Similarly, we have

$$
C_r \tilde{X}^{(k)} = \sum_{s \neq r} \frac{\mu_s^{1/2}}{\mu_r - \mu_s} \sum_{j \in \Delta_s} \tilde{\eta}_{j,k} \theta_j, \ k \in \Delta_r,
$$

and

$$
C_r\bar{X}^{(k)}=\sum_{s\neq r}\frac{\mu_s^{1/2}}{\mu_r-\mu_s}\sum_{j\in\Delta_s}\bar{\eta}_{j,k}\theta_j, \ k\in\Delta_r,
$$

where $\{\tilde{\eta}_{j,k}\},\{\bar{\eta}_{j,k}\}$ are i.i.d. standard normal random variables (also independent of $\{\eta_{j,k}\}\)$. Moreover, in addition $\{\eta_{j,k}\},\{\tilde{\eta}_{j,k}\},\{\bar{\eta}_{j,k}\}\)$ are independent of the samples $X_1, \ldots, X_n, \tilde{X}_1, \ldots, \tilde{X}_n, \bar{X}_1, \ldots, \bar{X}_n$. Denote

$$
\eta_j:=\sum_{k\in\Delta_r}\eta_{j,k}\theta_k,\tilde{\eta}_j:=\sum_{k\in\Delta_r}\tilde{\eta}_{j,k}\theta_k,\bar{\eta}_j:=\sum_{k\in\Delta_r}\bar{\eta}_{j,k}\theta_k.
$$

We will also need

$$
\eta_j':=\sum_{k\in\Delta_r}\eta_{j,k}\phi_k,\tilde{\eta}_j':=\sum_{k\in\Delta_r}\tilde{\eta}_{j,k}\tilde{\phi}_k,\bar{\eta}_j':=\sum_{k\in\Delta_r}\bar{\eta}_{j,k}\bar{\phi}_k.
$$

Note that conditionally on ϕ_k , $\bar{\phi}_k$, $\bar{\phi}_k$, the distributions of random vectors η'_j , $\bar{\eta}'_j$, $\bar{\eta}'_j$, $j \in$ Δ_s , $s \neq r$ is the same as the distribution of random vectors η_j , $\tilde{\eta}_j$, $\bar{\eta}_j$, $j \in \Delta_s$, $s \neq$ r (that are independent "standard normal" random vectors in the eigenspace of the eigenvalue μ_r). In addition to this,

$$
\|\eta_j\|^2 = \|\eta'_j\|^2, \|\tilde{\eta}_j\|^2 = \|\tilde{\eta}'_j\|^2, \|\bar{\eta}_j\|^2 = \|\bar{\eta}'_j\|^2.
$$

By a straightforward computation, the vector $\tilde{\Theta}_r$ can be written as follows:

$$
\tilde{\Theta}_r = \left(\begin{array}{c} \sqrt{2} \sum_{s \neq r} \frac{\mu_s}{(\mu_r - \mu_s)^2} \sum_{j \in \Delta_s} \langle \eta_j', \tilde{\eta}_j' \rangle \\ \sqrt{2} \sum_{s \neq r} \frac{\mu_s}{(\mu_r - \mu_s)^2} \sum_{j \in \Delta_s} \langle \tilde{\eta}_j', \tilde{\eta}_j' \rangle \\ \sum_{s \neq r} \frac{\mu_s}{(\mu_r - \mu_s)^2} \sum_{j \in \Delta_s} [\|\eta_j\|^2 - \mathbb{E}\|\eta_j\|^2] \\ \sum_{s \neq r} \frac{\mu_s}{(\mu_r - \mu_s)^2} \sum_{j \in \Delta_s} [\|\tilde{\eta}_j\|^2 - \mathbb{E}\|\tilde{\eta}_j\|^2] \\ \sum_{s \neq r} \frac{\mu_s}{(\mu_r - \mu_s)^2} \sum_{j \in \Delta_s} [\|\tilde{\eta}_j\|^2 - \mathbb{E}\|\tilde{\eta}_j\|^2] \end{array}\right),
$$

and it has the same distribution as

$$
\left(\begin{array}{c} \sqrt{2}\sum_{s\neq r}\frac{\mu_s}{(\mu_r-\mu_s)^2}\sum_{j\in\Delta_s}\langle\eta_j,\tilde{\eta}_j\rangle \\ \sqrt{2}\sum_{s\neq r}\frac{\mu_s}{(\mu_r-\mu_s)^2}\sum_{j\in\Delta_s}\langle\tilde{\eta}_j,\bar{\eta}_j\rangle \\ \sum_{s\neq r}\frac{\mu_s}{(\mu_r-\mu_s)^2}\sum_{j\in\Delta_s}[\|\eta_j\|^2-\mathbb{E}\|\eta_j\|^2] \\ \sum_{s\neq r}\frac{\mu_s}{(\mu_r-\mu_s)^2}\sum_{j\in\Delta_s}[\|\tilde{\eta}_j\|^2-\mathbb{E}\|\tilde{\eta}_j\|^2] \\ \sum_{s\neq r}\frac{\mu_s}{(\mu_r-\mu_s)^2}\sum_{j\in\Delta_s}[\|\tilde{\eta}_j\|^2-\mathbb{E}\|\bar{\eta}_j\|^2]\end{array}\right)=\Theta_r.
$$

This completes the proof of the lemma.

 \Box

5. Proofs: limit theorems

In this section, we turn to the proofs of theorems [8,](#page-10-0) [9](#page-11-0) and [10.](#page-12-2) Recall the asymptotic framework of Section 3 in which $X_1^{(n)}, \ldots, X_n^{(n)}, \tilde{X}_1^{(n)}, \ldots, \tilde{X}_n^{(n)}$ and $\bar{X}_1^{(n)}, \ldots, \bar{X}_n^{(n)}$ are three samples of size n each consisting of i.i.d. copies of a centered Gaussian random vector $X^{(n)}$ with covariance $\Sigma^{(n)}$. Similarly to the non-asymptotic framework, we consider the spectral decomposition $\Sigma^{(n)}$ = $\sum_{r\geq 1} \mu_r^{(n)} P_r^{(n)}$ and we are interested in the estimation of the spectral projector $P^{(n)} = P_{r_n}^{(n)}$ of $\Sigma^{(n)}$ corresponding to its eigenvalue $\mu^{(n)} = \mu_{r_n}^{(n)}$ of multiplicity $m^{(n)} = m_{r_n}^{(n)}$. We define three sample covariance operators (based on the three samples of size n :

$$
\hat{\Sigma}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} X_i^{(n)} \otimes X_i^{(n)}, \quad \tilde{\Sigma}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i^{(n)} \otimes \tilde{X}_i^{(n)}, \quad \bar{\Sigma}^{(n)} := \frac{1}{n} \sum_{i=1}^{n} \bar{X}_i^{(n)} \otimes \bar{X}_i^{(n)}
$$

and set

$$
E^{(n)} := \tilde{\Sigma}^{(n)} - \Sigma^{(n)}, \quad \tilde{E}^{(n)} := \tilde{\Sigma}^{(n)} - \Sigma^{(n)}, \quad \bar{E}^{(n)} := \bar{\Sigma}^{(n)} - \Sigma^{(n)}.
$$

Recall that $C^{(n)} = C_{r_n}^{(n)} = \sum_{s \neq r_n} \frac{1}{\mu_{r_n}^{(n)}}$ $\frac{1}{\mu_{rn}^{(n)} - \mu_s^{(n)}} P_s^{(n)}$ and

$$
B_n = B_{r_n}(\Sigma^{(n)}) = 2\sqrt{2} ||C^{(n)}\Sigma^{(n)}C^{(n)}||_2 ||P^{(n)}\Sigma^{(n)}P^{(n)}||_2.
$$

For a bounded linear operator $W : \mathbb{H} \to \mathbb{H}$, we will denote,

$$
L^{(n)}(W) = L_{r_n}^{(n)}(W) := P^{(n)}WC^{(n)} + C^{(n)}WP^{(n)}.
$$

Recall that, in theorems [8,](#page-10-0) [9](#page-11-0) and [10,](#page-12-2) it is supposed that Assumption [1](#page-9-0) is satisfied and, moreover, that $\mu^{(n)}$ is the eigenvalue of multiplicity $m^{(n)} = 1$. In this case, $\Delta_{r_n}^{(n)} = \{k_n\}$ for some $k_n \geq 1$.

Define the following sequences of random vectors with values in \mathbb{R}^5 :

$$
\Xi^{(n)} := \begin{pmatrix}\n\sqrt{2} \langle L^{(n)}(E^{(n)}), L^{(n)}(\tilde{E}^{(n)}) \rangle \\
\sqrt{2} \langle L^{(n)}(\tilde{E}^{(n)}), L^{(n)}(\tilde{E}^{(n)}) \rangle \\
\|L^{(n)}(E^{(n)})\|_2^2 - \mathbb{E} \|L^{(n)}(E^{(n)})\|_2^2 \\
\|L^{(n)}(\tilde{E}^{(n)})\|_2^2 - \mathbb{E} \|L^{(n)}(\tilde{E}^{(n)})\|_2^2 \\
\|L^{(n)}(\tilde{E}^{(n)})\|_2^2 - \mathbb{E} \|L^{(n)}(\tilde{E}^{(n)})\|_2^2\n\end{pmatrix}
$$

and

$$
\Theta^{(n)} := \begin{pmatrix}\n\sqrt{2} \sum_{s \neq r_n} \frac{\mu_s^{(n)}}{(\mu_{r_n}^{(n)} - \mu_s^{(n)})^2} \sum_{j \in \Delta_s} \eta_{j,k_n}^{(n)} \tilde{\eta}_{j,k_n}^{(n)} \\
\sqrt{2} \sum_{s \neq r_n} \frac{\mu_s^{(n)}}{(\mu_{r_n}^{(n)} - \mu_s^{(n)})^2} \sum_{j \in \Delta_s} \tilde{\eta}_{j,k_n}^{(n)} \tilde{\eta}_{j,k_n}^{(n)} \\
\sum_{s \neq r_n} \frac{\mu_s^{(n)}}{(\mu_{r_n}^{(n)} - \mu_s^{(n)})^2} \sum_{j \in \Delta_s} [(\eta_{j,k_n}^{(n)})^2 - 1] \\
\sum_{s \neq r_n} \frac{\mu_s^{(n)}}{(\mu_{r_n}^{(n)} - \mu_s^{(n)})^2} \sum_{j \in \Delta_s} [(\tilde{\eta}_{j,k_n}^{(n)})^2 - 1] \\
\sum_{s \neq r_n} \frac{\mu_s^{(n)}}{(\mu_{r_n}^{(n)} - \mu_s^{(n)})^2} \sum_{j \in \Delta_s} [(\bar{\eta}_{j,k_n}^{(n)})^2 - 1]\n\end{pmatrix},
$$

where $\eta_{j,k}, \tilde{\eta}_{j,k}, \bar{\eta}_{j,k}, j, k \geq 1$ are i.i.d. standard normal random variables. Denote

$$
\bar{B}_n := \left(2 \sum_{s \neq r_n} \frac{m_s^{(n)}(\mu_s^{(n)})^2}{(\mu_{r_n}^{(n)} - \mu_s^{(n)})^4}\right)^{1/2}.
$$

It is immediate to see that $B_n = 2\mu^{(n)}\bar{B}_n$ and, in view of Lemma [8,](#page-26-0)

$$
n\Xi^{(n)} = 2\mu^{(n)}\tilde{\Theta}^{(n)} + \xi^{(n)},
$$

where $\tilde{\Theta}^{(n)}$ has the same distribution as $\Theta^{(n)}$ and the remainder $\xi^{(n)} \in \mathbb{R}^5$ satisfies the bound

$$
\max_{1 \le j \le 5} \mathbb{E} |\xi_j^{(n)}| \lesssim \left(\frac{\|\Sigma^{(n)}\|_{\infty}}{\bar{g}^{(n)}} \right)^2 \frac{\mathbf{r}(\Sigma^{(n)})}{\sqrt{n}}.
$$

(where we also used the assumption that $m^{(n)} = 1$). Under Assumption [1,](#page-9-0) this implies that

$$
\frac{\xi^{(n)}}{B_n} = o_{\mathbb{P}}(1) \text{ as } n \to \infty,
$$

and we get

$$
\frac{n\Xi^{(n)}}{B_n} = \frac{\tilde{\Theta}^{(n)}}{\bar{B}_n} + o_\mathbb{P}(1). \tag{5.1}
$$

We need a simple lemma that will allow us to prove that the sequence of random variables $\frac{\tilde{\Theta}^{(n)}}{\tilde{B}_n}$ is asymptotically standard normal implying the same limit distribution for $\frac{nE^{(n)}}{B}$ $\frac{E^{(n)}}{B_n}$.

Let $\{\eta, \eta_k^{(n)}\}$ $\tilde{\eta}^{(n)}_k, \tilde{\eta}^{(n)}_k$ $\bar{\eta}^{(n)}_k, \bar{\eta}^{(n)}_k$ $\{k^{(n)}_k, k \geq 1\}$ be i.i.d. standard normal random variables and let $\lambda_k^{(n)} > 0, k \ge 1, n \ge 1$ be positive real numbers with $\sum_{k \ge 1} \lambda_k^{(n)} < \infty, n \ge 1$. Define √

$$
\vartheta_n := \left(\begin{array}{c} \sqrt{2} \sum_{k \geq 1} \lambda_k^{(n)} \eta_k^{(n)} \tilde{\eta}_k^{(n)} \\ \sqrt{2} \sum_{k \geq 1} \lambda_k^{(n)} \tilde{\eta}_k^{(n)} \bar{\eta}_k^{(n)} \\ \sum_{k \geq 1} \lambda_k^{(n)} [(\eta_k^{(n)})^2 - 1] \\ \sum_{k \geq 1} \lambda_k^{(n)} [(\tilde{\eta}_k^{(n)})^2 - 1] \\ \sum_{k \geq 1} \lambda_k^{(n)} [(\tilde{\eta}_k^{(n)})^2 - 1] \end{array} \right)
$$

and let

$$
\bar{B}_n = \left(2\sum_{k\geq 1} (\lambda_k^{(n)})^2\right)^{1/2}, \ n \geq 1.
$$

Lemma 9. If

$$
\frac{\bar{B}_n}{\sup_{k\geq 1} \lambda_k^{(n)}} \to \infty, \quad n \to \infty,
$$

then the sequence of random vectors

$$
\frac{1}{\bar{B}_n}\vartheta_n, \quad n \ge 1
$$

converges in distribution to a standard normal random vector Z_5 in \mathbb{R}^5 .

proof. The proof of this result is an easy application of Lindeberg version of the CLT. We will establish the convergence in distribution of $\langle \vartheta_n, a \rangle$ to a normal random variable $N(0, |a|^2)$ for an arbitrary $a \in \mathbb{R}^5$. For a vector $a =$ $(a_1, \ldots, a_5) \in \mathbb{R}^5$, set

$$
\vartheta_n(a,k) := a_1 \sqrt{2} \eta_k^{(n)} \tilde{\eta}_k^{(n)} + a_2 \sqrt{2} \tilde{\eta}_k^{(n)} \bar{\eta}_k^{(n)} + a_3 [(\eta_k^{(n)})^2 - 1] + a_4 [(\tilde{\eta}_k^{(n)})^2 - 1] + a_5 [(\bar{\eta}_k^{(n)})^2 - 1], k \ge 1.
$$

,

Without loss of generality, assume that $|a| = 1$. Note that r.v. $\vartheta_n(a, k), k \geq 1$ are i.i.d., $\mathbb{E} \vartheta_n(a, k) = 0$ and $\text{Var}(\vartheta_n(a, k)) = 2$. Therefore, for

$$
\zeta_n(a) := \frac{1}{\bar{B}_n} \langle \vartheta_n, a \rangle = \frac{\sum_{k \ge 1} \lambda_k^{(n)} \vartheta_n(a, k)}{\bar{B}_n}
$$

it holds that $\mathbb{E}\zeta_n(a) = 0$ and $\text{Var}(\zeta_n(a)) = 1$. In textbook versions of the central limit theorem, the result is usually stated for sums of finite triangular arrays of independent random variables. In our case, the sums are infinite. However, it is easy to reduce the problem to the finite case by truncating the series to p_n terms, where p_n is such that $\sum_{k>p_n} \lambda_k^{(n)} = o(\bar{B}_n)$. Such a reduction is rather simple and will be skipped. By the assumption of the lemma,

$$
\frac{\sup_{k\ge 1}(\lambda_k^{(n)})^2\mathbb{E}\Big[\vartheta_n^2(a,k)\Big]}{\bar{B}_n^2}=\frac{2\sup_{k\ge 1}(\lambda_k^{(n)})^2}{\bar{B}_n^2}\to 0.
$$

It remains to check that the Lindeberg condition holds. To this end, note that

$$
|\vartheta_n(a,k)| \leq \max\Bigl(\sqrt{2}|\eta^{(n)}_k||\tilde{\eta}^{(n)}_k|, \sqrt{2}|\tilde{\eta}^{(n)}_k||\bar{\eta}^{(n)}_k|, |(\eta^{(n)}_k)^2-1|, |\tilde{\eta}^{(n)}_k)^2-1|, |(\bar{\eta}^{(n)}_k)^2-1|\Bigr)
$$

and observe that the random variables involved in the maximum in the right hand side are sub-exponential. This easily implies the following bound on the tails of $\vartheta_n(a, k)$

$$
\mathbb{P}\{|\vartheta_n(a,k)|\geq t\}\leq 5e^{-ct}, t\geq 0
$$

that holds with some numerical constant $c > 0$ and for all $a \in \mathbb{R}^5$, $|a| = 1$ and all $k \geq 1$. This bound also implies that $\mathbb{E}|\vartheta_n(a,k)|^4 \leq C, a \in \mathbb{R}^5, \|a\| = 1$ for some numerical constant $C > 0$. Therefore, for all $\tau > 0$, we have

$$
\frac{\sum_{k\geq 1} (\lambda_k^{(n)})^2 \mathbb{E}\left[\vartheta_n^2(a,k)I\left(\lambda_k^{(n)}|\vartheta_n(a,k)| \geq \tau \bar{B}_n\right)\right]}{\bar{B}_n^2} \leq \frac{1}{\bar{B}_n^2} \sum_{k\geq 1} \left(\lambda_k^{(n)}\right)^2 \mathbb{E}^{1/2} |\vartheta_n(a,k)|^4 \mathbb{P}^{1/2} \left(\lambda_k^{(n)}|\vartheta_n(a,k)| \geq \tau \bar{B}_n\right)
$$

$$
\lesssim \frac{\sum_{k\geq 1} (\lambda_k^{(n)})^2}{\bar{B}_n^2} \exp\left\{-\frac{c\tau \bar{B}_n}{2 \sup_{k\geq 1} \lambda_k^{(n)}}\right\} \lesssim \exp\left\{-\frac{c\tau \bar{B}_n}{2 \sup_{k\geq 1} \lambda_k^{(n)}}\right\},
$$

which tends to 0 as $n \to \infty$ (under the condition that $\frac{\bar{B}_n}{\sup_{k \geq 1} \lambda_k^{(n)}} \to \infty$). \Box

Lemma [9](#page-33-0) will be applied to the sequence of random vectors $\Theta^{(n)}$. Under Assumption [1,](#page-9-0) the condition of the lemma holds since

$$
\frac{1}{\bar{B}_n} \sup_{s \neq r_n} \frac{\mu_s^{(n)}}{(\mu_{r_n}^{(n)} - \mu_s^{(n)})^2} = \frac{2}{B_n} \sup_{s \neq r_n} \frac{\mu^{(n)} \mu_s^{(n)}}{(\mu_{r_n}^{(n)} - \mu_s^{(n)})^2} \leq \frac{2}{B_n} \left(\frac{\|\Sigma^{(n)}\|_{\infty}}{\bar{g}^{(n)}} \right)^2 \to 0 \text{ as } n \to \infty.
$$

Thus, Lemma [9](#page-33-0) implies that $\frac{\Theta^{(n)}}{B_n} \xrightarrow{d} Z_5$ and, in view of [\(5.1\)](#page-33-1), we also have that

$$
\frac{n\Xi^{(n)}}{B_n} \xrightarrow{d} Z_5 \text{ as } n \to \infty. \tag{5.2}
$$

Under Assumption [1,](#page-9-0) Lemma [2](#page-12-3) easily implies that

$$
\frac{n}{B_n} \left(\|\hat{P}^{(n)} - P^{(n)}\|_2^2 - \mathbb{E} \|\hat{P}^{(n)} - P^{(n)}\|_2^2 \right) = \frac{n}{B_n} \langle \Xi^{(n)}, u \rangle + o_{\mathbb{P}}(1), \ u = (0, 0, 1, 0, 0). \tag{5.3}
$$

Under the same assumption, Lemma [3](#page-12-1) implies that

$$
\frac{n}{B_n}\left((1+\hat{b}^{(n)})^2 - (1+b^{(n)})^2\right) = \frac{n}{B_n}\langle \Xi^{(n)}, v \rangle + o_{\mathbb{P}}(1), \ v = \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{2}, -\frac{1}{2}, 0\right) \tag{5.4}
$$

and

$$
\frac{n}{B_n}\left((1+\tilde{b}^{(n)})^2 - (1+b^{(n)})^2\right) = \frac{n}{B_n}\langle \Xi^{(n)}, w \rangle + o_{\mathbb{P}}(1), w = \left(0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{2}, -\frac{1}{2}\right).
$$
\n(5.5)

It follows from the last two relationships that

$$
\frac{n}{B_n}\left((1+\hat{b}^{(n)})^2 - (1+\tilde{b}^{(n)})^2\right) = \frac{n}{B_n}\langle \Xi^{(n)}, v - w \rangle + o_{\mathbb{P}}(1). \tag{5.6}
$$

Proof of Theorem [8](#page-10-0). Note that

$$
\frac{n}{B_n}(\hat{b}^{(n)} - b^{(n)}) = \frac{n}{B_n} \frac{(1 + \hat{b}^{(n)})^2 - (1 + b^{(n)})^2}{2 + \hat{b}^{(n)} + b^{(n)}}.
$$
(5.7)

Under Assumption [1,](#page-9-0) Proposition [1](#page-7-1) implies that $|\hat{b}^{(n)} - b^{(n)}| = O_{\mathbb{P}}$ $\sqrt{\mathbf{r}(\Sigma^{(n)})}$ n . Recall also that $|b^{(n)}| \leq \frac{\|\Sigma^{(n)}\|_{\infty}^2}{\bar{g}_r^2}$ $\mathbf{r}(\Sigma^{(n)})$ $\frac{1}{n}$ (see bound [\(2.14\)](#page-6-4)). Thus, under As-sumption [1,](#page-9-0) we get that $b^{(n)} = o(1)$ and $\hat{b}^{(n)} = o_{\mathbb{P}}(1)$. These facts along with representations [\(5.7\)](#page-35-0), [\(5.4\)](#page-35-1) and also with [\(5.2\)](#page-34-0) imply that $\frac{2n}{B_n}(\hat{b}^{(n)} - b^{(n)})$ converges in distribution to the same limit as $\frac{n}{B_n}((1+\hat{b}^{(n)})^2 - (1+b^{(n)})^2)$, which is the distribution of the random variable $\langle Z_5, w \rangle$. Since $|w| = 1, \langle Z_5, w \rangle$ is a standard normal random variable, which completes the proof of Theorem [8.](#page-10-0)

Proof of Theorem [10](#page-12-2). Recall that

$$
\mathbb{E} \|\hat{P}^{(n)} - P^{(n)}\|_2^2 = -2b^{(n)}
$$

(see (3.7)). The following representation holds:

$$
\frac{\|\hat{P}^{(n)} - P^{(n)}\|_{2}^{2} + 2\hat{b}^{(n)}}{\|(1 + \hat{b}^{(n)})^{2} - (1 + \tilde{b}^{(n)})^{2}\|_{2}^{2}} = \frac{\|\hat{P}^{(n)} - P^{(n)}\|_{2}^{2} + 2b^{(n)}}{\|(1 + \hat{b}^{(n)})^{2} - (1 + \tilde{b}^{(n)})^{2}\|_{2}^{2}} + \frac{2(\hat{b}^{(n)} - b^{(n)})}{\|(1 + \hat{b}^{(n)})^{2} - (1 + \tilde{b}^{(n)})^{2}\|_{2}^{2}} = \frac{\frac{n}{B_{n}} \left(\|\hat{P}^{(n)} - P^{(n)}\|_{2}^{2} - \mathbb{E}\|\hat{P}^{(n)} - P^{(n)}\|_{2}^{2}\right)}{\left|\frac{n}{B_{n}} \left((1 + \hat{b}^{(n)})^{2} - (1 + \tilde{b}^{(n)})^{2} \right)\right|} + \frac{\frac{2n}{B_{n}} \left(\hat{b}^{(n)} - b^{(n)} \right)}{\left|\frac{n}{B_{n}} \left((1 + \hat{b}^{(n)})^{2} - (1 + \tilde{b}^{(n)})^{2} \right)\right|} \tag{5.8}
$$

.

In view of (5.2) , (5.3) , (5.6) and the combination of (5.7) with (5.4) , we easily conclude that the sequence of random variables

$$
\frac{\|\hat{P}^{(n)} - P^{(n)}\|_2^2 + 2\hat{b}^{(n)}}{|(1 + \hat{b}^{(n)})^2 - (1 + \tilde{b}^{(n)})^2|}
$$

converges in distribution to $\frac{\langle Z_5, u+v \rangle}{\langle Z_5, v-w \rangle}$. Using Proposition [3,](#page-11-1) it is easy to show that $\frac{\langle Z_5, u+v \rangle}{\langle Z_5, v-w \rangle} \stackrel{d}{=} Y_{\frac{5}{6}, \frac{\sqrt{47}}{6}}$. This completes the proof of Theorem [10.](#page-12-2)

Proof of Theorem [9](#page-11-0) is quite similar.

 \Box

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