ON SOME PROPERTIES OF CALIBRATED TRIFOCAL TENSORS

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Abstract. In two-view geometry, the essential matrix describes the relative position and orientation of two calibrated images. In three views, a similar role is assigned to the calibrated trifocal tensor. It is a particular case of the (uncalibrated) trifocal tensor and thus it inherits all its properties but, due to the smaller degrees of freedom, satisfies a number of additional algebraic constraints. Some of them are described in this paper. More specifically, we define a new notion — the trifocal essential matrix. On the one hand, it is a generalization of the ordinary (bifocal) essential matrix, and, on the other hand, it is closely related to the calibrated trifocal tensor. We prove the two necessary and sufficient conditions that characterize the set of trifocal essential matrices. Based on this characterization, we propose three necessary conditions on a calibrated trifocal tensor. They have a form of 15 quartic and 99 quintic polynomial equations.

1. Introduction

In multiview geometry, the fundamental matrix and the trifocal tensor describe the relative orientation of two and three (uncalibrated) images respectively. If the cameras are pre-calibrated, i.e. we are given the calibration matrices for each view, the fundamental matrix is transformed to the so-called essential matrix. It was first introduced by Longuet-Higgins in [8]. The essential matrix has fewer degrees of freedom and additional algebraic properties, compared to the fundamental matrix. A detailed investigation of these properties is given by Demazure, Faugeras, Maybank and other researchers in [1, 2, 6, 7, 9]. We shortly recall the most important of them in the next section.

The trifocal tensor for calibrated cameras (we call this entity the calibrated trifocal tensor) was first appeared in the papers by Spetsakis and Aloimonos [13] and Weng, Huang and Ahuja [16]. Later, Hartley [4] generalized the trifocal tensor for the case of uncalibrated cameras. The properties of the (uncalibrated) trifocal tensors and their characterizations have been investigated by Hartley, Shashua, Triggs and other researchers in [11, 12, 14, 15].

As well as the essential matrix, the calibrated trifocal tensor has fewer degrees of freedom and additional algebraic properties, compared to the uncalibrated case. The investigation of these properties is the main purpose of the present paper. In particular, we show that the calibrated trifocal tensor must satisfy a number of low

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degree homogeneous polynomial equations. These equations arise from the characterization constraints on a certain complex matrix associated with the calibrated trifocal tensor.

The rest of the paper is organized as follows. In Section 2, we recall some definitions and results from multiview geometry. In Section 3, we introduce a new notion — the trifocal essential matrix. On the one hand, it is a generalization of the ordinary (bifocal) essential matrix, and, on the other hand, it is closely related to the calibrated trifocal tensor. We prove the two necessary and sufficient conditions that characterize the set of trifocal essential matrices. In Section 4, based on this characterization, we propose our three necessary conditions on a calibrated trifocal tensor. They have a form of 15 quartic and 99 quintic polynomial equations in the entries of a calibrated trifocal tensor. In Section 5, we discuss the results of the paper.

2. Preliminaries

2.1. **Notation.** We preferably use $\alpha, \beta, \ldots$ for scalars, $a, b, \ldots$ for column 3-vectors, and $A, B, \ldots$ both for matrices and column 4-vectors. For a matrix A the entries are $(A)_{ij}$, the transpose is A^{T}, the determinant is $\det A$, and the trace is $\operatorname{Tr} A$. For two 3-vectors a and b the cross product is $a \times b$. For a vector a the notation $[a]_\times$ stands for the skew-symmetric matrix such that $[a]_\times b = a \times b$ for any vector b. We use I for identical matrix.

The group of 3×3 matrices satisfying $RR^{\mathrm{T}} = I$ and $\det R = 1$ is denoted by $\mathrm{SO}(3)$ in case R is real and $\mathrm{SO}(3, \mathbb{C})$ if R is allowed to have complex entries.

2.2. **Pinhole cameras.** We briefly recall some definitions and results from multiview geometry, see [2, 3, 5, 9] for details.

A *pinhole camera* is a triple (O, Π, P), where Π is the image plane, P is a central projection of points in 3-dimensional Euclidean space onto Π, and $O \notin \Pi$ is the camera centre (centre of projection P).

Let there be given coordinate frames in 3-space and in the image plane Π. Let Q be a point in 3-space represented in homogeneous coordinates as a 4-vector, and q be its image in Π represented as a 3-vector. Projection P is then given by a 3×4 homogeneous matrix, which is called the *camera matrix* and is also denoted by P. We have

$$\omega q = PQ,$$

where ω is a scale factor. For the sake of brevity, we identify further the camera (O, Π, P) with its camera matrix P.

The *focal length* is the distance between O and Π, the orthogonal projection of O onto Π is called the *principal point*. All intrinsic parameters of a camera (such as the focal length, the principal point offsets, etc.) are combined into a single upper-triangular matrix, which is called the *calibration matrix*. A camera is called *calibrated* if its calibration matrix is known.

The calibrated camera can be represented in form

$$P = \begin{bmatrix} R & t \end{bmatrix},$$

where $R \in \mathrm{SO}(3)$ is called the *rotation matrix* and $t \in \mathbb{R}^3$ is called the *translation vector*.

2.3. **Two-view case.** Let there be given two cameras $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and $P_2 = \begin{bmatrix} A & a \end{bmatrix}$, where A is a 3×3 matrix and a is a 3-vector. Let Q be a point in 3-space, and q_k be its kth image. Then,

$$\omega_k q_k = P_k Q, \quad k = 1, 2.$$

The *incidence relation* for a pair (q_1, q_2) says

$$q_2^{\mathrm{T}} F q_1 = 0, \tag{1}$$

where matrix $F = [a]_\times A$ is called the *fundamental matrix*. It is important that relation (1) is linear in the entries of F.

It follows from the definition of matrix F that $\det F = 0$. One easily verifies that this condition is also sufficient. Thus we have

Theorem 1 ([5])**.** *A real non-zero 3×3 matrix F is a fundamental matrix if and only if*

$$\det F = 0. \tag{2}$$

The *essential matrix* E is the fundamental matrix for calibrated cameras $\hat{P}_1 = \begin{bmatrix} I & 0 \end{bmatrix}$ and $\hat{P}_2 = \begin{bmatrix} R & t \end{bmatrix}$, where $R \in \mathrm{SO}(3)$, t is a 3-vector, that is

$$E = [t]_\times R. \tag{3}$$

The matrices F and E are related by

$$F = K_2^{-\mathrm{T}} E K_1^{-1},$$

where K_k is the calibration matrix of the kth camera. It follows that the incidence relation (1) for the essential matrix becomes

$$\hat{q}_2^{\mathrm{T}} E \hat{q}_1 = 0,$$

where $\hat{q}_k = K_k^{-1} q_k$ are the so-called *normalized coordinates*. We note that $P_1 = \hat{P}_1$ and $P_2 = K_2 \hat{P}_2 \operatorname{diag}(K_1^{-1}, 1)$.

Equality (3) can be thought of as the definition of the essential matrix, i.e. it is a 3×3 non-zero skew-symmetric matrix post-multiplied by a special orthogonal matrix. Moreover, we can even consider complex essential matrices assuming that in (3) vector $t \in \mathbb{C}^3$ and matrix $R \in \mathrm{SO}(3, \mathbb{C})$.

The real fundamental matrix has 7 degrees of freedom, whereas the real essential matrix has only 5 degrees of freedom. It is translated into the following property [2, 5, 7]: two of singular values of matrix E are equal and the third is zero. The condition is also sufficient. An equivalent form of this result is given by

Theorem 2 ([1, 2])**.** *A real 3×3 matrix E is an essential matrix if and only if*

$$\det E = 0, \tag{4}$$

$$\operatorname{Tr}(EE^{\mathrm{T}})^2 - 2 \operatorname{Tr}((EE^{\mathrm{T}})^2) = 0. \tag{5}$$

We emphasize that constraints (4) and (5) characterize only *real* essential matrices. There exist non-essential complex 3×3 matrices which nevertheless satisfy both conditions (4) and (5). The most general form of such matrices will be given in the next section.

The following theorem gives another characterization constraint on the entries of essential matrix E. It is also valid in case E is complex.

Theorem 3 ([1, 2, 9])**.** *A real or complex* 3×3 *matrix* E *of rank two is an essential matrix if and only if*

$$(\mathrm{Tr}(EE^{\mathrm{T}})I - 2EE^{\mathrm{T}})E = 0_{3\times 3}. \tag{6}$$

It is interesting to note that Theorem 3 is a key for developing efficient algorithms of the essential matrix estimation from five points in two views [10].

2.4. **Three-view case.** A $(2,1)$ tensor is a valency 3 tensor with two contravariant and one covariant indices. For a $(2,1)$ tensor T we write $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$, where T_k are 3×3 matrices corresponding to the covariant index.

Let there be given three cameras $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} A & a \end{bmatrix}$ and $P_3 = \begin{bmatrix} B & b \end{bmatrix}$, where A and B are 3×3 matrices, a and b are 3-vectors. The *trifocal tensor* $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ is a $(2,1)$ tensor defined by

$$T_k = Ae_k b^{\mathrm{T}} - a e_k^{\mathrm{T}} B^{\mathrm{T}}, \tag{7}$$

where e_1, e_2, e_3 constitute the standard basis in $\mathbb{R}^3$. For a trifocal tensor T matrices T_k are called the *correlation slices.*

It is clear that $\det T_k = 0$. Let l_k and r_k be the left and right null vectors of T_k respectively. It follows from (7) that $l_k = [a]_\times A e_k$ and $r_k = [b]_\times B e_k$. Therefore the two (sextic in the entries of T_1, T_2, T_3) *epipolar constraints* hold [5, 11]:

$$\begin{aligned} \det \begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix} &= \det([a]_\times A) = 0, \\ \det \begin{bmatrix} r_1 & r_2 & r_3 \end{bmatrix} &= \det([b]_\times B) = 0. \end{aligned} \tag{8}$$

Moreover, for any scalars α, β, γ, the matrix $\alpha T_1 + \beta T_2 + \gamma T_3$ is also degenerate (its right null vector is $[b]_\times B(\alpha e_1 + \beta e_2 + \gamma e_3)$) meaning that

$$\det(\alpha T_1 + \beta T_2 + \gamma T_3) = 0. \tag{9}$$

This equality is referred to as the *extended rank constraint* [11]. It is equivalent to ten (cubic in the entries of T_1, T_2, T_3) equations each of which is a coefficient in $\alpha^i \beta^j \gamma^k$ with $i + j + k = 3$.

Theorem 4 ([11])**.** *A* $(2,1)$ *tensor* $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ *is a trifocal tensor if and only if it satisfies the two epipolar* (8) *and ten extended rank* (9) *constraints.*

Let q_k be the kth image of a point Q in 3-space. The *trifocal incidence relation* for a triple (q_1, q_2, q_3) says [5]

$$[q_2]_\times \Bigl(\sum_{j=1}^{3} q_{1j} T_j\Bigr) [q_3]_\times = 0_{3\times 3}. \tag{10}$$

It is important that relation (10) is linear in the entries of T.

The *calibrated trifocal tensor* $\hat{T}$ is the trifocal tensor for calibrated cameras $P_1 = \begin{bmatrix} I & 0 \end{bmatrix}$, $P_2 = \begin{bmatrix} R_2 & t_2 \end{bmatrix}$ and $P_3 = \begin{bmatrix} R_3 & t_3 \end{bmatrix}$, where $R_2, R_3 \in \mathrm{SO}(3)$, $t_2, t_3 \in \mathbb{R}^3$, i.e.

$$\hat{T}_k = R_2 e_k t_3^{\mathrm{T}} - t_2 e_k^{\mathrm{T}} R_3^{\mathrm{T}}. \tag{11}$$

The calibrated trifocal tensor is an analog of the essential matrix in three views. The tensors T and $\hat{T}$ are related by

$$T_i = K_2 \sum_{j=1}^{3} (K_1^{-\mathrm{T}})_{ij} \hat{T}_j K_3^{\mathrm{T}},$$

where K_k is the calibration matrix of the kth camera.

For any invertible 3×3 matrix M and 3-vector t, the following identity holds:

$$[M^{-1}t]_\times = \det(M^{-1}) M^{\mathrm{T}} [t]_\times M.$$

Therefore the trifocal incidence relation (10) for a calibrated trifocal tensor becomes

$$[\hat{q}_2]_\times \bigl(\sum_{j=1}^{3} \hat{q}_{1j} \hat{T}_j\bigr) [\hat{q}_3]_\times = 0_{3\times 3},$$

where $\hat{q}_k = K_k^{-1} q_k$ are the normalized coordinates.

The tensors T and $\hat{T}$ have 18 and 11 degrees of freedom respectively. It follows that matrices $\hat{T}_k$ must satisfy a number of additional algebraic constraints. Some of them are described below.

3. The Trifocal Essential Matrix and Its Characterization

A *trifocal essential matrix* is, by definition, a 3×3 matrix S which can be represented in form

$$S = s_1 t_1^{\mathrm{T}} + t_2 s_2^{\mathrm{T}}, \tag{12}$$

where $t_1, t_2, s_1, s_2 \in \mathbb{C}^3$, and vectors s_1, s_2 are such that $s_k^{\mathrm{T}} s_k = 0$, $k = 1, 2$. It is clear that matrices S, S^{T} and RSQ, where $R, Q \in \mathrm{SO}(3, \mathbb{C})$, simultaneously are (or are not) the trifocal essential matrices.

Lemma 1. *Let $a, b, c, d \in \mathbb{C}^n$. Then the (possibly) non-zero eigenvalues of matrix $M = ac^{\mathrm{T}} + bd^{\mathrm{T}}$ coincide with the eigenvalues of 2×2 matrix*

$$N = \begin{bmatrix} c^{\mathrm{T}} a & c^{\mathrm{T}} b \\ d^{\mathrm{T}} a & d^{\mathrm{T}} b \end{bmatrix}.$$

Proof. The rank of matrix M is at most 2. Let λ_1, λ_2 be the (possibly) non-zero eigenvalues of M. Then,

$$\lambda_1 + \lambda_2 = \mathrm{Tr}(M) = c^{\mathrm{T}} a + d^{\mathrm{T}} b = \mathrm{Tr}(N),$$

$$\begin{aligned} 2\lambda_1\lambda_2 = (\lambda_1 + \lambda_2)^2 - (\lambda_1^2 + \lambda_2^2) = \mathrm{Tr}(M)^2 - \mathrm{Tr}(M^2) \\ = 2(c^{\mathrm{T}} a)(d^{\mathrm{T}} b) - 2(c^{\mathrm{T}} b)(d^{\mathrm{T}} a) = 2 \det N. \end{aligned}$$

We see that λ_1, λ_2 are the eigenvalues of matrix N, as required. □

Theorem 5. *Let a 3×3 matrix S be a trifocal essential matrix. Then SS^{T} has one zero and two other equal eigenvalues.*

Proof. Let S be a trifocal essential matrix, i.e. it can be represented in form (12). Matrix SS^{T} has zero eigenvalue, as $\det S = 0$. Taking into account that $s_2^{\mathrm{T}} s_2 = 0$, we get

$$SS^{\mathrm{T}} = s_1(\mu s_1^{\mathrm{T}} + \nu t_2^{\mathrm{T}}) + \nu t_2 s_1^{\mathrm{T}}, \tag{13}$$

where we have denoted $\mu = t_1^{\mathrm{T}} t_1$, $\nu = s_2^{\mathrm{T}} t_1$. By Lemma 1, the potentially non-zero eigenvalues of (13) are equal to the ones of 2×2 matrix

$$\begin{bmatrix} \nu t_2^{\mathrm{T}} s_1 & \nu(\mu s_1^{\mathrm{T}} + \nu t_2^{\mathrm{T}}) t_2 \\ 0 & \nu s_1^{\mathrm{T}} t_2 \end{bmatrix},$$

and the eigenvalues of the latter matrix are both equal to $\nu s_1^{\mathrm{T}} t_2 = (s_1^{\mathrm{T}} t_2)(s_2^{\mathrm{T}} t_1)$. Theorem 5 is proved. □

Lemma 2. *Let M be a degenerate 3×3 matrix. Then the two (possibly) non-zero eigenvalues of M coincide if and only if the entries of M are subject to*

$$\operatorname{Tr}(M)^2 - 2\operatorname{Tr}(M^2) = 0. \tag{14}$$

Proof. Let 0, λ_1, λ_2 be the eigenvalues of M. Then,

$$\operatorname{Tr}(M)^2 - 2\operatorname{Tr}(M^2) = (\lambda_1 + \lambda_2)^2 - 2(\lambda_1^2 + \lambda_2^2) = -(\lambda_1 - \lambda_2)^2.$$

It follows that $\lambda_1 = \lambda_2$ if and only if (14) holds. Lemma 2 is proved. □

Lemma 3. [9, Section 2.2] *Let $s_1, s_2 \in \mathbb{C}^3$ be a pair of non-zero vectors satisfying $s_k^{\mathrm{T}} s_k = 0$. Then there exists a matrix $R \in \mathrm{SO}(3, \mathbb{C})$ such that $Rs_1 = s_2$.*

Theorem 6. *A 3×3 matrix S is a trifocal essential matrix if and only if*

$$\det S = 0, \tag{15}$$

$$\operatorname{Tr}(SS^{\mathrm{T}})^2 - 2\operatorname{Tr}((SS^{\mathrm{T}})^2) = 0. \tag{16}$$

Proof. The"only if" part is due to Theorem 5 and Lemma 2. To prove the "if" part, let S be a 3×3 matrix satisfying (15), (16). We denote c_k the kth column of matrix S. Because matrix S is degenerate, there exists a non-zero vector a such that $Sa = 0$. There are two possibilities.

Case 1: $a^{\mathrm{T}} a \neq 0$. Scaling a and post-multiplying S by an appropriate matrix from $\mathrm{SO}(3, \mathbb{C})$, we assume without loss of generality that $a = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$. Therefore $c_3 = 0$.

Suppose first that either $c_1^{\mathrm{T}} c_1 \neq 0$ or $c_2^{\mathrm{T}} c_2 \neq 0$. Without loss of generality we assume that $c_2^{\mathrm{T}} c_2 \neq 0$. Pre-multiplying S by an appropriate rotation, we obtain

$$S = \begin{bmatrix} \lambda & \mu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The substitution of S into (16) gives

$$((\mu + \nu)^2 + \lambda^2)((\mu - \nu)^2 + \lambda^2) = 0.$$

It follows that $\lambda = i(\epsilon_1 \mu + \epsilon_2 \nu)$, where $\epsilon_k = \pm 1$. Thus,

$$S = \begin{bmatrix} i(\epsilon_1 \mu + \epsilon_2 \nu) & \mu & 0 \\ \nu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} i\epsilon_2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} \nu & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} i\epsilon_1 & 1 & 0 \end{bmatrix}.$$

Consider the case $c_1^{\mathrm{T}} c_1 = c_2^{\mathrm{T}} c_2 = 0$. Due to Lemma 3, we can pre-multiply S by an appropriate rotation to get

$$S = \begin{bmatrix} \alpha & 1 & 0 \\ \beta & i & 0 \\ \gamma & 0 & 0 \end{bmatrix},$$

where $\alpha^2 + \beta^2 + \gamma^2 = 0$. The substitution of S into (16) yields

$$4(i\alpha - \beta)^2 = 0.$$

It follows that $\beta = i\alpha$ and $\gamma = 0$. Therefore matrix S has rank one and

$$S = \begin{bmatrix} \alpha & 1 & 0 \\ i\alpha & i & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} \begin{bmatrix} \alpha & 1 & 0 \end{bmatrix} + 0s^{\mathrm{T}},$$

where s is an arbitrary 3-vector satisfying $s^{\mathrm{T}}s = 0$. Thus in either case S is a trifocal essential matrix, as required.

Case 2: $a^{\mathrm{T}}a = 0$**.** Due to Lemma 3, we can post-multiply S by an appropriate matrix from $\mathrm{SO}(3,\mathbb{C})$ and suppose without loss of generality that $a = \begin{bmatrix}0 & 1 & i\end{bmatrix}^{\mathrm{T}}$. Therefore $c_3 = ic_2$.

By direct computation, equality (16) becomes $(c_1^{\mathrm{T}}c_1)^2 = 0$, i.e. $c_1^{\mathrm{T}}c_1 = 0$. This yields

$$S = \begin{bmatrix}\alpha & \lambda & i\lambda \\ \beta & \mu & i\mu \\ \gamma & \nu & i\nu\end{bmatrix} = \begin{bmatrix}\alpha \\ \beta \\ \gamma\end{bmatrix}\begin{bmatrix}1 & 0 & 0\end{bmatrix} + \begin{bmatrix}\lambda \\ \mu \\ \nu\end{bmatrix}\begin{bmatrix}0 & 1 & i\end{bmatrix},$$

where $\alpha^2+\beta^2+\gamma^2 = 0$, i.e. S is a trifocal essential matrix. Theorem 6 is proved. □

We notice that constraints (15), (16) coincide with constraints (4), (5) from Theorem 2. Hence, if a trifocal essential matrix is real, then it is an essential matrix.

In general, a trifocal essential matrix does not satisfy cubic constraint (6). The proof consists in exhibiting a counterexample. Let $s_1 = s_2 = \begin{bmatrix}1 & i & 0\end{bmatrix}^{\mathrm{T}}$, $t_1 = t_2 = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^{\mathrm{T}}$. Then $S = \begin{bmatrix}2 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}$ and the eigenvalues of SS^{T} are $0, 1, 1$. However,

$$(\mathrm{Tr}(SS^{\mathrm{T}})I - 2SS^{\mathrm{T}})S = -4\begin{bmatrix}1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0\end{bmatrix} \neq 0_{3\times 3}.$$

Nevertheless, there exists an analog of identity (6) for the trifocal essential matrix.

Theorem 7. *A* 3×3 *matrix* S *is a trifocal essential matrix if and only if*

$$(\mathrm{Tr}(SS^{\mathrm{T}})I - 2SS^{\mathrm{T}})^2 S = 0_{3\times 3}. \tag{17}$$

Proof. Let us prove the "only if" part. We first notice that matrix S satisfies (17) if and only if so does matrix RSQ for arbitrary $R, Q \in \mathrm{SO}(3,\mathbb{C})$. By Lemma 3, for any 3-vector s satisfying $s^{\mathrm{T}}s = 0$ there exists a matrix $R \in \mathrm{SO}(3,\mathbb{C})$ such that $Rs = \begin{bmatrix}1 & i & 0\end{bmatrix}^{\mathrm{T}}$.

Pre- and post-multiplying a trifocal essential matrix S by appropriate rotations, we assume without loss of generality that

$$S = \begin{bmatrix}1 \\ i \\ 0\end{bmatrix}\begin{bmatrix}\lambda_1 & \mu_1 & \nu_1\end{bmatrix} + \begin{bmatrix}\lambda_2 \\ \mu_2 \\ \nu_2\end{bmatrix}\begin{bmatrix}1 & i & 0\end{bmatrix} = \begin{bmatrix}\lambda_1+\lambda_2 & \mu_1+i\lambda_2 & \nu_1 \\ i\lambda_1+\mu_2 & i(\mu_1+\mu_2) & i\nu_1 \\ \nu_2 & i\nu_2 & 0\end{bmatrix}.$$

Then, by direct computation, we find

$$\begin{aligned}&(\mathrm{Tr}(SS^{\mathrm{T}})I - 2SS^{\mathrm{T}})^2 \\ &\quad = 4(\lambda_1+i\mu_1)^2\begin{bmatrix}\nu_2^2 & i\nu_2^2 & -\nu_2(\lambda_2+i\mu_2) \\ i\nu_2^2 & -\nu_2^2 & -i\nu_2(\lambda_2+i\mu_2) \\ -\nu_2(\lambda_2+i\mu_2) & -i\nu_2(\lambda_2+i\mu_2) & (\lambda_2+i\mu_2)^2\end{bmatrix}.\end{aligned}$$

It follows that (17) holds. The "only if" part is proved.

Let us prove the "if" part. Let a 3×3 matrix S satisfy (17). We first show that $\det S = 0$. Suppose, by hypothesis, that $\det S \neq 0$. Then, post-multiplying (17) by S^{-1}, we get

$$(\operatorname{Tr}(SS^{\mathrm{T}})I - 2SS^{\mathrm{T}})^2 = 0_{3\times 3}.$$

It follows that all the eigenvalues of $\operatorname{Tr}(SS^{\mathrm{T}})I - 2SS^{\mathrm{T}}$ are zeroes and

$$\operatorname{Tr}(\operatorname{Tr}(SS^{\mathrm{T}})I - 2SS^{\mathrm{T}}) = \operatorname{Tr}(SS^{\mathrm{T}}) = 0.$$

The substitution of this into (17) yields $(\det S)^5 = 0$ in contradiction to the hypothesis $\det S \neq 0$.

Now we prove that (17) also implies (16). Let us denote

$$\Phi(M) = (\operatorname{Tr}(MM^{\mathrm{T}})I - 2MM^{\mathrm{T}})M$$

and

$$\varphi(M) = \operatorname{Tr}(MM^{\mathrm{T}})^2 - 2\operatorname{Tr}((MM^{\mathrm{T}})^2).$$

Then it can be proved (see [2, Section 4]) that

$$\operatorname{Tr}(\Phi(M)\Phi(M)^{\mathrm{T}}) = -\operatorname{Tr}(MM^{\mathrm{T}})\varphi(M) + 12(\det M)^2 \tag{18}$$

holds for any 3×3 matrix M. We remark that

$$\Phi(S)\Phi(S)^{\mathrm{T}} = (\operatorname{Tr}(SS^{\mathrm{T}})I - 2SS^{\mathrm{T}})^2 SS^{\mathrm{T}}.$$

Therefore, if matrix S satisfies (17), then $\Phi(S)\Phi(S)^{\mathrm{T}} = 0_{3\times 3}$. Formula (18) for $M = S$ becomes

$$\operatorname{Tr}(SS^{\mathrm{T}})\varphi(S) = 0.$$

If $\varphi(S) = 0$, then we are done. Suppose that $\operatorname{Tr}(SS^{\mathrm{T}}) = 0$. Substituting this into (17) and post-multiplying by S^{T}, we get $(SS^{\mathrm{T}})^3 = 0_{3\times 3}$. It follows that all the eigenvalues of SS^{T} are zeroes and (16) evidently holds.

We see that in either case the degenerate matrix S satisfies (16). By Theorem 6, matrix S is a trifocal essential matrix. Theorem 7 is proved. □

To sum up, the above theorems imply the following statements.

- The pair of scalar constraints (15), (16) is equivalent to the single matrix constraint (17).
- The most general form of a 3×3 matrix satisfying equations (15), (16) is the trifocal essential matrix given by (12).
- If a trifocal essential matrix is real, then it is an essential matrix.
- Every essential matrix (of rank two) is a trifocal essential matrix, but the converse is not true in general.

4. Three Necessary Conditions on a Calibrated Trifocal Tensor

A new notion of trifocal essential matrix, introduced in the previous section, turns out to be closely related to calibrated trifocal tensors. The connection is established by the following lemma.

Lemma 4. *Let $\hat{T} = \begin{bmatrix}\hat{T}_1 & \hat{T}_2 & \hat{T}_3\end{bmatrix}$ be a calibrated trifocal tensor. Then a 3×3 matrix $\hat{S} = \alpha\hat{T}_1 + \beta\hat{T}_2 + \gamma\hat{T}_3$, where numbers α, β, γ are such that $\alpha^2 + \beta^2 + \gamma^2 = 0$, is a trifocal essential matrix, i.e. it can be represented in form* (12).

Proof. We notice that

$$\hat{S} = \alpha \hat{T}_1 + \beta \hat{T}_2 + \gamma \hat{T}_3 = R_2 s t_3^{\mathrm{T}} - t_2 s^{\mathrm{T}} R_3^{\mathrm{T}} = s_2 t_3^{\mathrm{T}} + (-t_2) s_3^{\mathrm{T}},$$

where $s = \begin{bmatrix}\alpha & \beta & \gamma\end{bmatrix}^{\mathrm{T}}$, and $s_k = R_k s$ are 3-vectors satisfying

$$s_k^{\mathrm{T}} s_k = s^{\mathrm{T}} R_k^{\mathrm{T}} R_k s = s^{\mathrm{T}} s = 0.$$

It follows that $\hat{S}$ is a trifocal essential matrix. Lemma 4 is proved. □

We introduce six symmetric matrices ($k = 1, 2, 3$)

$$\begin{aligned} U_k &= \hat{T}_k \hat{T}_k^{\mathrm{T}}, \\ V_k &= \hat{T}_k \hat{T}_{k+1}^{\mathrm{T}} + \hat{T}_{k+1} \hat{T}_k^{\mathrm{T}}. \end{aligned} \tag{19}$$

Here $k+1$ should be read as $k \pmod 3 + 1$, i.e. $V_3 = \hat{T}_3 \hat{T}_1^{\mathrm{T}} + \hat{T}_1 \hat{T}_3^{\mathrm{T}}$.

Theorem 8 (First necessary condition)**.** *Let* $\hat{T} = \begin{bmatrix}\hat{T}_1 & \hat{T}_2 & \hat{T}_3\end{bmatrix}$ *be a calibrated trifocal tensor, matrices* U_k*,* V_k *be defined in* (19)*. Then the entries of* $\hat{T}_1$*,* $\hat{T}_2$*,* $\hat{T}_3$ *are constrained by the following equations:*

$$\psi(U_3 - U_1, U_3 - U_1) - \psi(V_3, V_3) = 0, \tag{20}$$

$$\psi(U_3 - U_1, V_1) + \psi(V_2, V_3) = 0, \tag{21}$$

$$\psi(U_1 - U_2, V_1) = 0, \tag{22}$$

where $\psi(X, Y) = \operatorname{Tr}(X)\operatorname{Tr}(Y) - 2\operatorname{Tr}(XY)$*. Six more equations are obtained from* (20) – (22) *by a cyclic permutation of indices* $1 \to 2 \to 3 \to 1$*. The resulting nine equations are linearly independent.*

Proof. Let $\hat{S} = \alpha \hat{T}_1 + \beta \hat{T}_2 + \gamma \hat{T}_3$, where numbers α, β, γ are such that $\alpha^2 + \beta^2 + \gamma^2 = 0$. By Lemma 4, $\hat{S}$ is a trifocal essential matrix. By Theorem 6, the following equation holds:

$$\operatorname{Tr}(\hat{S}\hat{S}^{\mathrm{T}})^2 - 2\operatorname{Tr}((\hat{S}\hat{S}^{\mathrm{T}})^2) = 0. \tag{23}$$

The definition of matrices U_k, V_k (see (19)) permits us to write

$$\hat{S}\hat{S}^{\mathrm{T}} = \alpha^2 U_1 + \beta^2 U_2 + \gamma^2 U_3 + \alpha\beta V_1 + \beta\gamma V_2 + \gamma\alpha V_3.$$

Substituting this into (23), we find the coefficients in α^4, $\alpha^3\beta$ and $\alpha\beta^3$ taking into account that $\gamma^2 = -\alpha^2 - \beta^2$. Because α and β are arbitrary, these coefficients must vanish:

$$\alpha^4 \colon \psi(U_3 - U_1, U_3 - U_1) - \psi(V_3, V_3) = 0, \tag{24}$$

$$\alpha^3\beta \colon \psi(U_1 - U_3, V_1) - \psi(V_2, V_3) = 0, \tag{25}$$

$$\alpha\beta^3 \colon \psi(U_2 - U_3, V_1) - \psi(V_2, V_3) = 0. \tag{26}$$

Thus we get (20) = (24), (21) = −(25), and (22) = (25) − (26). It is clear that we can get six more constraints on $\hat{T}_k$ from (20) – (22) by a cyclic permutation of the indices.

Finally, the resulting nine polynomials can not be linearly dependent, since they are generated by different sets of monomials. For example, polynomials (20), (21) and (22) contain $(\hat{T}_3)_{11}^2(\hat{T}_1)_{11}^2$, $(\hat{T}_3)_{11}^2(\hat{T}_1)_{11}(\hat{T}_2)_{11}$ and $(\hat{T}_1)_{11}^3(\hat{T}_2)_{11}$ respectively, and that monomials are not contained in all the other polynomials. Theorem 8 is proved. □

Theorem 9. *Let $T = \begin{bmatrix} T_1 & T_2 & T_3 \end{bmatrix}$ be a $(2,1)$ tensor satisfying the ten extended rank constraints* (9) *and the nine constraints from Theorem 8. Then matrix $S = \alpha T_1 + \beta T_2 + \gamma T_3$ with $\alpha^2+\beta^2+\gamma^2 = 0$ is a trifocal essential matrix.*

Proof. It follows from the extended rank constraints (9) that $\det S = 0$.

Taking into account that $\gamma^2 = -\alpha^2 - \beta^2$, we conclude that the expression

$$\psi(SS^{\mathrm{T}}, SS^{\mathrm{T}}) = \mathrm{Tr}(SS^{\mathrm{T}})^2 - 2\,\mathrm{Tr}((SS^{\mathrm{T}})^2)$$

contains 9 monomials:

$$\alpha^4, \alpha^3\beta, \alpha^3\gamma, \alpha^2\beta^2, \alpha^2\beta\gamma, \alpha\beta^3, \alpha\beta^2\gamma, \beta^4, \beta^3\gamma.$$

It is directly verified that the coefficients in all of them are linear combinations of the nine polynomials from Theorem 8, i.e. $\psi(SS^{\mathrm{T}}, SS^{\mathrm{T}}) = 0$. By Theorem 6, S is a trifocal essential matrix, as required. □

Theorem 10 (Second necessary condition)**.** *Let $\hat{T} = \begin{bmatrix} \hat{T}_1 & \hat{T}_2 & \hat{T}_3 \end{bmatrix}$ be a calibrated trifocal tensor. Then the entries of $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$ are constrained by the 99 linearly independent quintic (of degree 5) polynomial equations.*

Proof. Let $\hat{S} = \alpha\hat{T}_1 + \beta\hat{T}_2 + \gamma\hat{T}_3$, where numbers α, β, γ are such that $\alpha^2+\beta^2+\gamma^2 = 0$. By Lemma 4, $\hat{S}$ is a trifocal essential matrix. By Theorem 7, the following equation holds:

$$(\mathrm{Tr}(\hat{S}\hat{S}^{\mathrm{T}})I - 2\hat{S}\hat{S}^{\mathrm{T}})^2\hat{S} = 0_{3\times3}. \tag{27}$$

We notice that equality (27) is quintic in the entries of matrix $\hat{S}$. Taking into account that $\gamma^2 = -\alpha^2 - \beta^2$, every of 9 entries in the l.h.s. of (27) contains 11 monomials in variables α, β and γ. The coefficient in each of these monomials must vanish. Hence there are in total 99 quintic polynomial constraints on the entries of a calibrated trifocal tensor $\hat{T}$. These polynomials can not be linearly dependent, since they are generated by different sets of monomials. □

Remark 1. An explicit form of the quintic polynomial equations from the previous theorem is as follows:

$$(\Psi_1(U_{13}) - \Psi_1(V_3))\hat{T}_1 - \Psi_2(U_{13}, V_3)\hat{T}_3 = 0_{3\times3}, \tag{28}$$

$$\Psi_2(U_{13}, V_3)\hat{T}_1 + (\Psi_1(U_{13}) - \Psi_1(V_3))\hat{T}_3 = 0_{3\times3}, \tag{29}$$

$$\begin{aligned}(\Psi_2(U_{13}, V_2) + \Psi_2(V_1, V_3))\hat{T}_1 + \Psi_2(U_{13}, V_3)\hat{T}_2 \\ +(\Psi_2(U_{13}, V_1) - \Psi_2(V_2, V_3))\hat{T}_3 = 0_{3\times3},\end{aligned} \tag{30}$$

$$\begin{aligned}(\Psi_2(U_{13}, V_1) - \Psi_2(V_2, V_3))\hat{T}_1 + (\Psi_1(U_{13}) - \Psi_1(V_3))\hat{T}_2 \\ -(\Psi_2(U_{13}, V_2) + \Psi_2(V_1, V_3))\hat{T}_3 = 0_{3\times3},\end{aligned} \tag{31}$$

where matrices U_k, V_k are defined in (19), $U_{jk} = U_j - U_k$, and

$$\begin{aligned}\Psi(X, Y) &= (\mathrm{Tr}(X)I - 2X)(\mathrm{Tr}(Y)I - 2Y), \\ \Psi_1(X) &= \Psi(X, X), \\ \Psi_2(X, Y) &= \Psi(X, Y) + \Psi(Y, X).\end{aligned}$$

Equations (28) – (31) give $4\times9 = 36$ constraints on $\hat{T}_k$. We can get $8\times9 = 72$ more constraints from (28) – (31) by a cyclic permutation of indices $1 \to 2 \to 3 \to 1$. Thus, in total, we have 108 quintic constraints, but only 99 of them are linearly independent.

Finally, we propose the third necessary condition on a calibrated trifocal tensor. It seems not to be directly connected with the matrix $\hat{S}$. However this condition could be useful in applications, since it consists of another set of quartic polynomial equations that are satisfied by a calibrated trifocal tensor.

Theorem 11 (Third necessary condition). *Let $\hat{T} = \begin{bmatrix}\hat{T}_1 & \hat{T}_2 & \hat{T}_3\end{bmatrix}$ be a calibrated trifocal tensor. Then the entries of $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$ satisfy the following equations:*

$$\operatorname{Tr}(U_2)^2 - \operatorname{Tr}(V_3)^2 - \operatorname{Tr}(U_2^2 - V_3^2 + (U_3 - U_1)^2) = 0, \tag{32}$$

$$\operatorname{Tr}(V_2)\operatorname{Tr}(U_1 - 2U_2 - U_3) - \operatorname{Tr}(V_1)\operatorname{Tr}(V_3) + 2\operatorname{Tr}(V_2U_2) = 0, \tag{33}$$

where matrices U_k, V_k are defined in (19). *Four more equations are obtained from* (32) – (33) *by a cyclic permutation of indices $1 \to 2 \to 3 \to 1$. The resulting six equations consist of the same monomials as* (20) – (22) *and are linearly independent with them.*

Proof. Equations (32) – (33) and the four their consequences are proved by direct computation with a help of Maple.

The resulting six polynomials consist in total of 2160 monomials, which are the same as for the nine polynomials from Theorem 8. Thus we construct the 15×2160 matrix of coefficients and verify by direct computation that it has full row rank. □

5. Discussion

We have defined a new notion — the trifocal essential matrix. Algebraically it is a complex 3×3 matrix associated with a given calibrated trifocal tensor $\hat{T}$ by the contraction of $\hat{T}$ and an arbitrary 3-vector whose squared components sum to zero. In this paper, the trifocal essential matrix plays a technical role. However its geometric interpretation should help to explain why its properties are so close to the properties of ordinary (bifocal) essential matrix.

Based on the characterization of the set of trifocal essential matrices, we have given three necessary conditions on a calibrated trifocal tensor (Theorems 8, 10 and 11). They have a form of 15 quartic and 99 quintic polynomial equations. We leave the possible application of these constraints to structure-from-motion problems for further work.

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