Fractional diffusion-type equations with exponential and logarithmic differential operators

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Abstract

We deal with some extensions of the space-fractional diffusion equation, which is satisfied by the density of a stable process (see [20]): the first equation considered here is obtained by adding an exponential differential operator expressed in terms of the Riesz-Feller derivative. We prove that this produces a random additional term in the time-argument of the corresponding stable process, which is represented by the so-called Poisson process with drift. Analogously, if we add, to the space-fractional diffusion equation, a logarithmic differential operator involving the Riesz-derivative, we obtain, as a solution, the transition semigroup of a stable process subordinated by an independent gamma subordinator with drift. Finally, we show that a non-linear extension of the space-fractional diffusion equation is satisfied by the transition density of the process obtained by time-changing the stable process with an independent linear birth process with drift.

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1 Introduction

The diffusion equation has been generalized in the fractional sense by many authors (e.g. [35], [33], [7], [2]): in particular [23] considers the time-fractional Cauchy problems, while in [12] the order of both time and space derivatives is fractional. Later, in [20] and [21], the time and space fractional diffusion equation was studied and solved analytically, also in the asymmetric case. The probabilistic expression of the solution to the diffusion equation with time-derivative of fractional order ν is given in [26], in terms of iterated stable processes (in particular, for $\nu = 1/2^n$, $n \in \mathbb{N}$, the n-times iterated Brownian motion).

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The term 'anomalous diffusion' usually indicates a diffusive process that does not follow the behavior of classical Gaussian diffusions. In the real world, anomalous diffusions are observed, for example, in turbulent plasma transport, photon diffusion, cell migration and so on. When a fractional derivative replaces the second-order derivative in a diffusion model, we get the so called superdiffusion. On the other hand, considering fractional time derivatives produces anomalous subdiffusion, slower than a classical diffusion. The stochastic time-space fractional heat-type equation has been treated in [25]. For applications to physical and financial problems, see also [16], [24], [31].

We consider here extensions of the following space-fractional diffusion equation, i.e.

$$\partial_t u(x,t) = \mathcal{D}_x^{\alpha,\theta} u(x,t),$$

where $\mathcal{D}_x^{\alpha,\theta}$ is the Riesz-Feller derivative of order $\alpha \in (0,2]$, defined below. In particular we introduce in the above equation additional terms represented by the so-called fractional exponential (or shift) operator $\mathcal{O}_{c,x}^{\alpha,\theta}$ or the fractional logarithmic operator $\mathcal{P}_{c,x}^{\alpha}$ (see Def.1 and Def.4 below). We are thus led to study the following equations, again for $\alpha \in (0,2]$,

$$\partial_t u(x,t) = \left[a\mathcal{D}_x^{\alpha,\theta} + \lambda(I - \mathcal{O}_{-1,x}^{\alpha,\theta}) \right] u(x,t) \tag{1.1}$$

$$\partial_t u(x,t) = \left[a\mathcal{D}_x^{\alpha} + \mu \mathcal{P}_{1/\rho,x}^{\alpha} \right] u(x,t) \tag{1.2}$$

under appropriate initial and boundary conditions. We prove that the solution to equation (1.1) coincides with the transition semigroup of the subordinated process defined as $\mathcal{S}_{\alpha,\theta}(at + N(t))$, $t \geq 0$, where $\mathcal{S}_{\alpha,\theta}$ is an α-stable process and N is an independent Poisson subordinator, with parameter λ. In the second case, we prove instead that equation (1.2) is satisfied by the transition semigroup of another subordinated α-stable process defined as $\mathcal{S}_\alpha(at + \Gamma(t))$, where $\Gamma(t)$, $t \geq 0$, is an independent gamma subordinator, with scale parameter $\mu > 0$. However, in both cases, the processes obtained are, for any $\alpha \in (0,2)$, pure jump models, while, only for $\alpha = 2$, they have a jump-diffusion behavior. In particular, we notice that $\mathcal{S}_\alpha(at + \Gamma(t))$ reduces, for $\alpha = 2$ and $a = 0$, to the well-known Variance Gamma (VG) process. Jump-diffusions and VG processes are applied in finance, in particular for asset pricing (see e.g. [8]). For a general $\alpha \in (0,2)$, the process $\mathcal{S}_\alpha(at + \Gamma(t))$ can be considered as a generalization of both stable and geometric stable processes (see, for example, [18]), to which it reduces in special cases.

In the last section we prove that a non-linear analogue of (1.1) is satisfied by the transition density of a stable process time-changed by an independent linear birth process with drift.

Therefore, in all these cases, the additional operator introduced in the fractional diffusion equation entails the appearance of a random element in the time argument of the corresponding process. This additional random element is represented, in the case of the fractional exponential operator, by the Poisson or birth processes, while, for the fractional logarithmic operator, it is given by the gamma process.

We remark that both equations (1.1) and (1.2) are connected to Lévy processes, while, only in the nonlinear case, we obtain a process whose finite distributions are not infinitely divisible, even though it still enjoys the Markov property.

We now introduce the notation and the basic definitions that we will use throughout the paper.

Let $X := X(t), t \geq 0$ be a one-dimensional Lévy process in $\mathbb{R}$ and f be in the Schwartz space $S(\mathbb{R})$ of rapidly decreasing functions. Then we denote by $\widetilde{f}(\xi)$ the Fourier transform of f, i.e. $\widetilde{f}(\xi) := \mathcal{F}\{f(x);\xi\} = \int_{-\infty}^{+\infty} e^{ix\xi} f(x)dx$, and by T_t the Feller semigroup associated to X, i.e.

$$(T_t f)(x) = \mathbb{E}f(x - X(t)),$$

for $f \in C_0(\mathbb{R})$, the real Banach space of continuous functions satisfying $\lim_{x\to\pm\infty} f(x) = 0$. The symbol of T_t is given by $\widehat{T_t} = e^{-t\eta}$, i.e.

$$\mathcal{F}\{T_t f;\xi\} = e^{-t\eta(\xi)}\widetilde{f}(\xi), \quad \xi \in \mathbb{R},$$

while the Lévy (or characteristic) exponent of X will be denoted by

$$\eta_X(\xi) = \frac{1}{t}\ln\left(\mathbb{E}e^{i\xi X(t)}\right),$$

(see, for example, [3]). Let $\mathcal{A}$ be the pseudo-differential operator such that $T_t = e^{t\mathcal{A}}$, then we denote by $\widehat{\mathcal{A}}$ its symbol, i.e.

$$\mathcal{F}\{\mathcal{A}f(x);\xi\} = \widehat{\mathcal{A}}(\xi)\widetilde{f}(\xi) = -\eta(\xi)\widetilde{f}(\xi), \quad \xi \in \mathbb{R}.$$

We will consider the α-stable process $\mathcal{S}_{\alpha,\theta}(t)$, $t \geq 0$, with transition density $p_{\alpha,\theta}(x;t)$ and characteristic function

$$\Phi_{\mathcal{S}_{\alpha,\theta}(t)}(\xi) := \mathbb{E}e^{i\xi\mathcal{S}_{\alpha,\theta}(t)} = \exp\{-t|\xi|^\alpha\sigma^\alpha\omega_{\alpha,\theta}(\xi)\}, \qquad \xi \in \mathbb{R},\ \alpha \in (0,2],\ \sigma > 0, \quad (1.3)$$

where $\theta = \frac{2}{\pi}\arctan\left[-\beta\tan\pi\alpha/2\right]$ and

$$\omega_{\alpha,\theta}(\xi) := \begin{cases} 1 - i\beta sign(\xi)\tan(\pi\alpha/2), & \text{if } \alpha \neq 1 \\ 1 + 2i\beta sign(\xi)\log|\xi|/\pi, & \text{if } \alpha = 1 \end{cases}, \ \beta \in [-1,1],$$

(see [29], for general references on stable r.v.'s). By assuming $\sigma = (\cos\pi\theta/2)^{1/\alpha}$, we can write (1.3) as $\Phi_{\mathcal{S}_{\alpha,\theta}(t)}(\xi) = \exp\{-t|\xi|^\alpha e^{isign(\xi)\pi\theta/2}\}$.

We will use the *Riesz-Feller (RF) fractional derivative*, which is defined by means of its Fourier transform. Thus we consider the space $L^c(\mathcal{I})$ of functions for which the

Riemann improper integral on any open interval $\mathcal{I}$ absolutely converges (see [22]). For any $f \in L^c(\mathbb{R})$, the RF fractional derivative is defined as

$$\mathcal{F}\left\{\mathcal{D}_x^{\alpha,\theta} f(x);\xi\right\} = -\psi_{\alpha,\theta}(\xi)\mathcal{F}\left\{f(x);\xi\right\}, \quad \alpha \in (0,2], \quad |\theta| \le \min\{\alpha, 2-\alpha\}, \quad (1.4)$$

with symbol

$$\widehat{\mathcal{D}_x^{\alpha,\theta}}(\xi) = -\psi_{\alpha,\theta}(\xi) := -|\xi|^\alpha e^{i\, sign(\xi)\theta\pi/2}, \quad (1.5)$$

(see [20] and [15], p.359, up to the sign). Thus the expression in (1.5) coincides with the Lévy exponent of the α-stable r.v. $\mathcal{S}_{\alpha,\theta} := \mathcal{S}_{\alpha,\theta}(1)$. As a consequence, the RF derivative coincides with the generator of the stable process $\mathcal{S}_{\alpha,\theta}$, since it is proved in [20] that its density $p_{\alpha,\theta}(x;t)$ solves the following problem

$$\partial_t u(x,t) = \mathcal{D}_x^{\alpha,\theta} u(x,t), \qquad u(x,0) = f(x), \quad u(\pm\infty, t) = 0, \qquad x \in \mathbb{R},\ t \ge 0, \quad (1.6)$$

with $f \in L^c(\mathbb{R})$.

For $\theta = 0$, which corresponds to the symmetric case (i.e. for $\beta = 0$), the RF derivative reduces to the *Riesz derivative*, which, in its regularized form (valid also for $\alpha = 1$), can be written as

$$\mathcal{D}_x^{\alpha,0} u(x) = \begin{cases} \frac{\Gamma(1+\alpha)}{\pi} \sin\left(\frac{\alpha\pi}{2}\right) \int_0^{+\infty} \frac{f(x+z)-2f(x)+f(x-z)}{z^{1+\alpha}} dz, & \alpha \in (0,2) \\ \partial_x^2, \quad \alpha = 2 \end{cases}, \quad (1.7)$$

(see [13], p.341, for details). We will denote the latter simply as $\mathcal{D}_x^\alpha$.

For $\alpha \in (0,1)$ and $\theta = -\alpha$, which corresponds to the case of the stable subordinator (with asymmetry parameter $\beta = 1$), the fractional derivative $\mathcal{D}_x^{\alpha,\theta}$ coincides with the (left-sided) Riemann-Liouville derivative, i.e.

$$\mathcal{D}_x^{\alpha,-\alpha} u(x) = \begin{cases} \frac{(-1)}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^{+\infty} \frac{f(z)}{(x-z)^\alpha} dz, & \alpha \in (0,1) \\ -\partial_x, , \quad \alpha = 1 \end{cases}. \quad (1.8)$$

Indeed

$$\widehat{\mathcal{D}_x^{\alpha,-\alpha}}(\xi) = -\psi_{\alpha,-\alpha}(\xi) = -|\xi|^\alpha e^{-isign(\xi)\alpha\pi/2} = -(-i\xi)^\alpha, \quad (1.9)$$

see [13], p.333. In this case it is well-known that equation (1.6) is satisfied by the density of the α-stable subordinator.

2 Preliminary results

We now introduce the following pseudo-differential operators, defined in terms of the RF and the Riesz fractional derivatives (1.4) and (1.7), respectively.

Definition 1 ***(Fractional shift or exponential operator)*** *Let $f \in L^c(\mathbb{R})$ be a function s.t. $g_j f := \underbrace{\mathcal{D}_x^{\alpha,\theta}...\mathcal{D}_x^{\alpha,\theta}}_{j-times} f \in L^c(\mathbb{R})$, for any $j = 0, 1, ...$, then*

$$\mathcal{O}_{c,x}^{\alpha,\theta} f(x) := \sum_{n=0}^{\infty} \frac{c^n}{n!} \underbrace{\mathcal{D}_x^{\alpha,\theta}...\mathcal{D}_x^{\alpha,\theta}}_{n-times} f(x), \qquad c \in \mathbb{R}, , \quad \alpha \in (0, 2], \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}. \tag{2.1}$$

provided that the series converges uniformly.

Lemma 2 *The symbol of (2.1) is given, for $c \in \mathbb{R}$, by*

$$\widehat{\mathcal{O}_{c,x}^{\alpha,\theta}}(\xi) = e^{-c\psi_{\alpha,\theta}(\xi)}, \quad \alpha \in (0, 2], \quad |\theta| \leq \min\{\alpha, 2 - \alpha\}. \tag{2.2}$$

Proof. *By exploiting the uniform convergence, the Fourier transform of (2.1) can be evaluated as follows*

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{ix\xi} \mathcal{O}_{c,x}^{\alpha,\theta} f(x) dx &= \sum_{n=0}^{\infty} \frac{c^n}{n!} \int_{-\infty}^{+\infty} e^{ix\xi} g_n(x) dx \\ &= [by\ (1.4)] = -\psi_{\alpha,\theta}(\xi) \sum_{n=0}^{\infty} \frac{c^n}{n!} \int_{-\infty}^{+\infty} e^{ix\xi} g_{n-1}(x) dx \\ &= \sum_{n=0}^{\infty} \frac{(-c\psi_{\alpha,\theta}(\xi))^n}{n!} \widetilde{f}(\xi). \end{aligned}$$

■

Remark 3 *In the integer order case $\alpha = 1$ and for $\theta = -1$, we obtain, by (2.1) and (1.8), the shift operator*

$$\mathcal{O}_{c,x}^{1,-1} f(x) = e^{-c\partial_x} f(x) = f(x - c),$$

while, for $\alpha = 2$ and $\theta = 0$, we get

$$\mathcal{O}_{c,x}^{2,0} f(x) = e^{c\partial_x^2} f(x),$$

which is a special case of the generalized exponential operator considered in [10]. Note that the symbol of $e^{c\partial_x^2}$ is $e^{-c\xi^2}$.

The fractional shift operator has been introduced in [5], in the special case $\alpha \in (0, 1)$ and $\theta = -\alpha$.

We define another pseudo-differential operator in the symmetric case, i.e. for $\theta = 0$. Under this assumption the symbol (1.5) is real.

Definition 4 ***(Fractional logarithmic operator)*** *Let $f \in L^c(\mathbb{R})$ be a function s.t. $g_j f = \underbrace{\mathcal{D}_x^\alpha ... \mathcal{D}_x^\alpha}_{j-times} f \in L^c(\mathbb{R})$, for any $j = 0, ...$, then*

$$\mathcal{P}_{c,x}^\alpha f(x) := \sum_{n=1}^{\infty} \frac{c^n}{n} \underbrace{\mathcal{D}_x^\alpha ... \mathcal{D}_x^\alpha}_{n-times} f(x), \qquad c > 0, \quad \alpha \in (0, 2]. \tag{2.3}$$

provided that the series converges uniformly.

Lemma 5 *The symbol of (2.3) is given by*

$$\widehat{\mathcal{P}_{c,x}^\alpha}(\xi) = -\ln(1 + c|\xi|^\alpha), \quad c > 0, \ |\xi| < 1/c^{1/\alpha}. \tag{2.4}$$

Proof. *As before, we get*

$$\begin{aligned} \int_{-\infty}^{+\infty} e^{ix\xi} \mathcal{P}_{c,x}^\alpha f(x) dx &= \sum_{n=1}^{\infty} \frac{c^n}{n} \int_{-\infty}^{+\infty} e^{ix\xi} \underbrace{\mathcal{D}_x^\alpha ... \mathcal{D}_x^\alpha}_{n-times} f(x) dx \\ &= \sum_{n=1}^{\infty} \frac{(-c|\xi|^\alpha)^n}{n} \widetilde{f}(\xi), \end{aligned}$$

which coincides with (2.4). ■

In the integer order case $\alpha = 2$, we can apply the semigroup property of the standard derivatives, so that we have $\mathcal{P}_{c,x}^2 f(x) = -\ln\left(1 + c\frac{d^2}{dx^2}\right) f(x)$. The fractional logarithmic operator has been used in [4], in connection with the geometric stable processes.

Remark 6 *We note that a sufficient condition for the uniform convergence of the series in (2.1) and (2.3) is that $f(x)$ is an eigenfunction of the RF fractional derivative. For example, when $\theta = -\alpha$, this is the case for $f(x) = e^{-kx}$, for $x \in \mathbb{R}$, $k > 0$. Indeed, in view of (2.2.15) in [15], p.81, we have that*

$$\mathcal{O}_{c,x}^{\alpha,-\alpha} e^{-kx} = \sum_{n=0}^{\infty} \frac{(ck^\alpha)^n}{n!} e^{-kx} = e^{-kx+ck^\alpha}.$$

Moreover the uniform convergence of the series in (2.1) and (2.3) holds for the density of the α-stable process, as the following lemma shows.

Lemma 7 *The transition density $p_{\alpha,\theta}(x;t)$ of the α-stable process $\mathcal{S}_{\alpha,\theta}(t), t \geq 0$ satisfies the following equations*

$$\mathcal{O}_{c,x}^{\alpha,\theta} u(x,t) = e^{c\partial_t} u(x,t) = u(x, t+c). \tag{2.5}$$

In the special case $\theta = 0$, we also have that

$$\mathcal{P}_{c,x}^\alpha u(x,t) = -\ln(1 - c\partial_t) u(x,t). \tag{2.6}$$

Proof. Recall that $p_{\alpha,\theta}(x;t)$ satisfies equation (1.6); thus we get that

$$\begin{aligned} \mathcal{O}^{\alpha,\theta}_{c,x} u(x,t) &= \sum_{n=0}^{\infty} \frac{c^n}{n!} \partial_t^n u(x,t) \\ &= e^{c\partial_t} u(x,t), \end{aligned} \tag{2.7}$$

which coincides with (2.5). The first step can be checked by resorting to the Fourier transform and considering (2.2):

$$\begin{aligned} \mathcal{F}\left\{\mathcal{O}^{\alpha,\theta}_{c,x} u(x,t);\xi\right\} &= e^{-c\psi_{\alpha,\theta}(\xi)} \widetilde{u}(\xi,t) = \sum_{n=0}^{\infty} \frac{c^n}{n!} \partial_t^n e^{-\psi_{\alpha,\theta}(\xi)t} \\ &= [\text{by } (1.5)] = \mathcal{F}\left\{\sum_{n=0}^{\infty} \frac{c^n}{n!} \partial_t^n u(x,t);\xi\right\}. \end{aligned}$$

Analogously we obtain (2.6) as follows:

$$\mathcal{P}^{\alpha}_{c,x} u(x,t) = \sum_{n=1}^{\infty} \frac{c^n}{n} \partial_t^n u(x,t),$$

since, by (2.4),

$$\begin{aligned} \mathcal{F}\left\{\mathcal{P}^{\alpha}_{c,x} u(x,t);\xi\right\} &= -\ln(1+c|\xi|^{\alpha}) \widetilde{u}(\xi,t) \\ &= \sum_{n=1}^{\infty} \frac{(-c|\xi|^{\alpha})^n}{n} = \sum_{n=1}^{\infty} \frac{c^n}{n} \partial_t^n e^{-|\xi|^{\alpha} t} \\ &= \mathcal{F}\left\{\sum_{n=1}^{\infty} \frac{c^n}{n} \partial_t^n u(x,t);\xi\right\}. \end{aligned}$$

■

We consider now an extension of the (space-)fractional diffusion equation (1.6), obtained by adding the fractional exponential operator. We prove that, as a consequence, the stochastic process governed by the new equation is again the stable process, but with a random time-argument represented by the Poisson process with drift, i.e. $N(t) + at$, $a, t \geq 0$. The latter has been studied in [6], where also the case of a general Lévy process subordinated by it has been analyzed.

Let $N(t)$ be a homogeneous Poisson process with parameter λ, independent from the stable process $\mathcal{S}_{\alpha,\theta}$; we define here the following subordinated process

$$\mathcal{Z}_{\alpha}(t) := \mathcal{S}_{\alpha,\theta}(at + N(t)), \qquad t \geq 0,\ a \geq 0 \tag{2.8}$$

by adding a random term in the time argument, which, in this case is represented by the Poisson process. As special case, for $\lambda \to 0$, $a = 1$, we obtain the standard stable process.

Lemma 8 ***(Fractional diffusion-type equation with exponential differential operator)*** *Let $f \in L^c(\mathbb{R})$ satisfy the condition given in Def.1. Then the following initial-value problem*

$$\begin{cases} \partial_t u(x,t) = \left[a\mathcal{D}_x^{\alpha,\theta} + \lambda(I - \mathcal{O}_{1,x}^{\alpha,\theta})\right] u(x,t) \\ u(x,0) = f(x) \end{cases}, \tag{2.9}$$

is satisfied by the transition semigroup of the process $\mathcal{Z}_\alpha$, given by

$$\mathcal{T}_t^{\mathcal{Z}} f(x) = e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \int_{\mathbb{R}} f(x-y) p_{\alpha,\theta}(y, k+at) dy. \tag{2.10}$$

Proof. By Lemma 7 it is easy to check that for the function (2.10) the fractional exponential operator $\mathcal{O}_{c,x}^{\alpha,\theta}$ is well defined. Taking the Fourier transform of (2.9), we obtain

$$\partial_t \widetilde{u}(\xi,t) = -\left[a\psi_{\alpha,\theta}(\xi) + \lambda(1 - e^{-\psi_{\alpha,\theta}(\xi)})\right] \widetilde{u}(\xi,t), \tag{2.11}$$

by (1.4) and (2.2). On the other hand, from (2.10), we have that

$$\begin{aligned} \widetilde{u}(\xi,t) &= \widetilde{f}(\xi) e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \widetilde{p}_{\alpha,\theta}(\xi, k+at) \\ &= \widetilde{f}(\xi) e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\psi_{\alpha,\theta}(\xi)(k+at)} \\ &= \widetilde{f}(\xi) \exp\{-\left[\lambda(1-e^{-\psi_{\alpha,\theta}(\xi)}) + a\psi_{\alpha,\theta}(\xi)\right] t\}, \end{aligned}$$

which, differentiated w.r.t. t, gives (2.11). ■

Remark 9 *It is easy to see that, for $a \geq 0$, the time argument in (2.8) is a subordinator and thus this time change represents a subordination. Moreover $\mathcal{Z}_\alpha$ is a Lévy process, since it is given by the composition of two independent Lévy processes. Its Lévy symbol can be obtained as follows, by considering the Laplace transform $\mathbb{E}e^{-s(N(t)+at)} = \exp\{-sat - \lambda t(1-e^{-s})\}$:*

$$\begin{aligned} \eta_{\mathcal{Z}_\alpha}(\xi) &= \frac{1}{t} \ln \left\{ \mathbb{E} \left(\mathbb{E} e^{i\xi \mathcal{S}_{\alpha,\theta}(at+N(t))} \middle| N(t) \right) \right\} = \frac{1}{t} \ln \left\{ \mathbb{E} e^{-\psi_{\alpha,\theta}(\xi)[at+N(t)]} \right\} \\ &= \frac{1}{t} \ln \left\{ e^{-a\psi_{\alpha,\theta}(\xi)t - \lambda t(1-e^{-\psi_{\alpha,\theta}(\xi)})} \right\} = -a\psi_{\alpha,\theta}(\xi) - \lambda(1-e^{-\psi_{\alpha,\theta}(\xi)}). \end{aligned}$$

The Lévy measure can be evaluated, by applying Theorem 30.1, p.197 in [30], as follows

$$\begin{aligned} \nu_{\mathcal{Z}_\alpha}(x) &= a\nu_{\mathcal{S}_{\alpha,\theta}}(x) + \lambda \int_0^{+\infty} p_{\alpha,\theta}(x,s)\delta(s-1)ds \\ &= a\left[\frac{P}{x^{1+\alpha}} 1_{(0,+\infty)}(x) + \frac{Q}{|x|^{1+\alpha}} 1_{(-\infty,0)}(x) \right] + \lambda p_{\alpha,\theta}(x,1), \end{aligned}$$

by considering that the drift coefficient of the random time argument is equal to a. For $\alpha \in (0,2)$ the diffusion coefficient is $A_{\mathcal{Z}_\alpha} = 0$ and thus the process is a pure jump process. Moreover, for $\alpha \in (0,1)$, the process has finite variation, since $\int_{|x|\leq 1} |x| \nu_{\mathcal{Z}_\alpha}(dx) < \infty$. On the other hand, since $\nu_{\mathcal{Z}_\alpha}(\mathbb{R}) = \infty$ for any α, the expected number of jumps in any finite interval is infinite, i.e. the process displays infinite activity. The drift coefficient reads

$$\begin{aligned}\gamma_{\mathcal{Z}_\alpha} &= a \int_0^{+\infty} \delta(s-1) ds \int_{|x|\leq 1} x p^s_{\alpha,\theta}(x;1) dx \\ &= a \int_{|x|\leq 1} x p_{\alpha,\theta}(x;1) dx.\end{aligned}$$

For $\alpha = 2$ and $\theta = 0$, we obtain the Lévy triplet of the process $\mathcal{Z}_2(t) = W(at+N(t)), t \geq 0$ (where $W(t), t > 0$ is a standard Brownian motion with characteristic function $e^{-\xi^2 t}$ and thus variance equal to $2t$):

$$\nu_{\mathcal{Z}_2}(x) = \lambda \int_0^{+\infty} \frac{e^{-x^2/2s}}{\sqrt{2\pi s}} \delta(s-1) ds = \lambda \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \tag{2.12}$$

by considering that the Lévy measure of the Brownian motion is zero, and

$$A_{\mathcal{Z}_2} = a, \qquad \gamma_{\mathcal{Z}_2} = 0. \tag{2.13}$$

The first two moments exist and can be obtained by deriving the characteristic function, which reads $\mathbb{E} e^{i\xi \mathcal{Z}_2(t)} = \exp\{-\xi^2 at - \lambda(1 - e^{-\xi^2})t\}$:

$$\begin{aligned}\mathbb{E}\mathcal{Z}_2(t) &= 0 \\ Var\mathcal{Z}_2(t) &= 2t\,(a+\lambda).\end{aligned}$$

Remark 10 *As a consequence of Lemma 8, we can write the generator of $\mathcal{Z}_\alpha$ as*

$$\mathcal{G}_\alpha f(x) = a\mathcal{D}_x^{\alpha,\theta} f(x) - \lambda \int_{\mathbb{R}} \left(f(x+y) - f(x)\right) p_{\alpha,\theta}(y,1) dy \tag{2.14}$$

with symbol $\widehat{\mathcal{G}_\alpha}(\xi) = -a\psi_{\alpha,\theta}(\xi) + \lambda(1 - e^{-\psi_{\alpha,\theta}(\xi)})$. Equation (2.14) can be alternatively obtained from (5.2) in [6], by considering that the generator of the stable process is the RF derivative and taking the fractional parameters in [6] equal to one. Moreover it suggests the following alternative expression of the fractional shift operator as integral transform of the standard shift operator, i.e.

$$\mathcal{O}_{c,x}^{\alpha,\theta} f(x) = \int_{\mathbb{R}} e^{cy\partial_x} p_{\alpha,\theta}(y,1) f(x) dy,$$

which, in the case $\alpha \in (0,1)$ and for $c = -1$, coincides with the definition given in [9].

Remark 11 *In the symmetric case and for $\alpha = 2$, the previous result shows that the density of the process $\mathcal{Z}_2(t) = W(at + N(t))$, $t \geq 0$ satisfies the following equation*

$$\partial_t u(x,t) = \left[a\partial_x^2 + \lambda(I - e^{-\partial_x^2}) \right] u(x,t),$$

with initial condition $u(x,0) = \delta(x)$. On the other hand, for $\alpha = 1/2$ and $\theta = -1/2$, we have another interesting special case given by $\mathcal{Z}_{1/2}(t) = \mathcal{S}_{1/2}(at + N(t))$, $t \geq 0$, where $\mathcal{S}_{1/2}(t)$ is the Lévy subordinator. The latter is known to be equal in distribution to the first passage time of a Brownian motion through the level t, i,e, $T_t := \inf\{s > 0 : W(s) \geq t\}$. Therefore we have the following equality in the sense of the finite-dimensional distributions (i.d.) with the first passage time of a Brownian motion through the trajectories of the process $N(t) + at$, i.e.

$$\mathcal{Z}_{1/2}(t) \overset{i.d.}{=} T_t^a = \inf\{s > 0 : W(s) \geq N(t) + at\}. \tag{2.15}$$

The transition density of (2.15) is thus the solution to the equation

$$\partial_t u(x,t) = \left[a\partial_x^{1/2} + \lambda(I - \mathcal{O}_{1,x}^{1/2,-1/2}) \right] u(x,t),$$

with initial condition $u(x,0) = \delta(x)$.

3 Main results

3.1 Fractional diffusion-type equation with logarithmic differential operator

We study now the extension of the fractional diffusion equation obtained by adding the logarithmic differential operator $\mathcal{P}_{c,x}^{\alpha}$ to (1.6), in the symmetric case (i.e. for $\theta = 0$). In analogy with the previous results we show that this additional term introduces a random element in the time argument of the corresponding stable process. In this case, instead of the Poisson process, we have a gamma process. We denote by $\Gamma(t), t \geq 0$ the gamma subordinator of parameters $\mu, \rho > 0$, i.e. with density

$$f_\Gamma(x,t) := \Pr\{\Gamma(t) \in dx\}/dx = \begin{cases} \frac{\rho^{\mu t}}{\Gamma(\mu t)} x^{\mu t - 1} e^{-\rho x}, & x \geq 0 \\ 0, & x < 0 \end{cases}. \tag{3.1}$$

Note that, for $\mu = 0$ the process $\Gamma(t)$ reduces to the elementary subordinator t.

Theorem 12 *Let $f \in L^c(\mathbb{R})$ satisfy the conditions given in Def.4, then the solution to the following initial-value problem*

$$\begin{cases} \partial_t u(x,t) = \left[a\mathcal{D}_x^{\alpha} + \mu\mathcal{P}_{1/\rho,x}^{\alpha} \right] u(x,t) \\ u(x,0) = f(x). \end{cases}, \quad x \in \mathbb{R},\ t > 0,\ \alpha \in (0,2], \tag{3.2}$$

coincides with the semigroup $\mathcal{T}_t^{\mathcal{X}} f(x) = \mathbb{E} f[x - \mathcal{X}_\alpha(t)]$ *of the following subordinated process*

$$\mathcal{X}_\alpha(t) = \mathcal{S}_\alpha(at + \Gamma(t)), \quad t \geq 0, \tag{3.3}$$

where $\mathcal{S}_\alpha(t), t \geq 0$, *is a symmetric stable process defined in (1.3) for* $\theta = 0$, *with density* $p_\alpha(x;t)$, *and* $\Gamma(t), t \geq 0$ *is an independent gamma subordinator with density (3.1).*

Proof. If we take the Fourier transform of the first equation in (3.2) we get, in view of (1.4) together with Lemma 5,

$$\partial_t \widetilde{u}(\xi, t) = -\left[a|\xi|^\alpha + \mu \ln(1 + |\xi|^\alpha/\rho)\right] \widetilde{u}(\xi, t).$$

Now we evaluate the characteristic function of (3.3), by considering that $\mathbb{E}e^{-s\Gamma(t)} := e^{-\Psi_\Gamma(s)t} = 1/\left(1 + s/\rho\right)^{\mu t}$:

$$\begin{aligned} \mathbb{E}e^{i\xi\mathcal{X}_\alpha(t)} &= \mathbb{E}\left[\mathbb{E}\left(e^{i\xi\mathcal{S}_\alpha(at+\Gamma(t))} \middle| \Gamma(t)\right)\right] = e^{-a|\xi|^\alpha t}\mathbb{E}e^{-|\xi|^\alpha\Gamma(t)} \\ &= \frac{e^{-a|\xi|^\alpha t}}{(1 + |\xi|^\alpha/\rho)^{\mu t}}. \end{aligned}$$

The time argument in (3.3) is represented by the gamma process with drift, which is a Lévy process and also a subordinator, being strictly increasing a.s. Then the process (3.3) is itself a Lévy process and its Lévy symbol is

$$\begin{aligned} \eta_{\mathcal{X}_\alpha}(\xi) &= \frac{1}{t} \ln\left\{\mathbb{E}e^{i\xi\mathcal{X}_\alpha(t)}\right\} \\ &= -a|\xi|^\alpha - \mu \ln\left(1 + \frac{|\xi|^\alpha}{\rho}\right). \end{aligned} \tag{3.4}$$

Thus

$$\partial_t \widetilde{u}(\xi, t) = -\eta_{\mathcal{X}_\alpha}(\xi)\widetilde{u}(\xi, t)$$

so that

$$u(x, t) = \mathcal{F}^{-1}\left\{e^{-\eta_{\mathcal{X}_\alpha}(\xi)t}\widetilde{f}(\xi); x\right\} = \mathcal{T}_t^{\mathcal{X}} f(x)$$

is the solution of (3.2), where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. ■

Remark 13 *The previous results are particularly interesting in the special case* $\alpha = 2$, *since they imply that the solution to the p.d.e.*

$$\begin{aligned} \partial_t u(x, t) &= \left[a\partial_x^2 + \mu\mathcal{P}_{1/\rho,x}^2\right] u(x, t) \\ &= \left[a\partial_x^2 - \mu \ln\left(1 + \frac{\partial_x^2}{\rho}\right)\right] u(x, t), \end{aligned} \tag{3.5}$$

coincides with the transition density of the process $W(at+\Gamma(t))$, $t>0$. For $a=0$, equation (3.5) provides the generator of the variance gamma process, which can be explicitly written as

$$\mathcal{A}=-\ln\left(1+\frac{\partial_x^2}{\rho}\right),$$

by exploiting the semigroup property of the integer-order derivatives.

It is evident from (3.4) that $\mathcal{X}_\alpha$ can be considered as a generalization of both stable and geometric stable processes (see, for example, [18]), to which it reduces in the special cases $\mu=0$ and $a=0$, respectively. Again, by applying Theorem 30.1, p.197 in [30], we get, for $\alpha\in(0,2)$, the Lévy triplet:

$$\nu_{\mathcal{X}_\alpha}(\cdot)=a\,\nu_{\mathcal{S}_\alpha}(\cdot)+\mu\int_0^{+\infty}s^{-1}e^{-\rho s}p_\alpha(\cdot;s)ds, \tag{3.6}$$

$$A_{\mathcal{X}_\alpha}=0$$

and

$$\gamma_{\mathcal{X}_\alpha}=\mu\int_0^{+\infty}s^{-1}e^{-\rho s}ds\int_{|x|\le 1}xp_\alpha(x;s)dx=0,$$

since the stable process is symmetric by assumption. From (3.6) we can deduce that the asymptotic behavior of the Lévy measure at the origin, for any positive a, is polynomial, as for the stable processes, while, for the geometric stable, it is logarithmic (see [19]).

For $\alpha=2$, the Lévy measure of $\mathcal{X}_2(t)=W(at+\Gamma(t))$ is given instead by

$$\nu_{\mathcal{X}_2}(s)=\mu\int_0^{+\infty}\frac{e^{-s^2/2z}}{\sqrt{2\pi z^3}}e^{-\rho z}dz=\frac{\mu}{|s|}e^{-\sqrt{2\rho}|s|}, \tag{3.7}$$

and the diffusion and drift parameters are respectively equal to

$$A_{\mathcal{X}_2}=1,\qquad \gamma_{\mathcal{X}_2}=0. \tag{3.8}$$

We can compare (3.7) and (3.8) to the Lévy triplet of the symmetric VG process $W(\Gamma(t)),t\ge 0$ (which corresponds to the special case $a=0$): the Lévy measure is the same, but in the VG case the diffusion coefficient is equal to zero (i.e. $A=0$) and the process is a pure jump process with infinitely many jumps and finite variation, since $\int_{|x|\le 1}|x|\nu(dx)<\infty$. On the other hand here we have, in view of Theorem 21.9. in [30], an infinite variation of almost all paths of $\mathcal{X}_2$, since $A_{\mathcal{X}_2}\ne 0$, and thus it is not a pure jump process. Jump-diffusion models are extensions of pure jumps models, mixing a jump process and a diffusion process, particularly useful in option pricing (see e.g. [8]).

For $\alpha=2$, the first two moments exist and can be obtained by deriving the characteristic function, which reads $\mathbb{E}e^{-i\xi\mathcal{X}_2(t)}=e^{-a\xi^2t-\mu t\ln(1+\xi^2/\rho)}$:

$$\begin{aligned}\mathbb{E}\mathcal{X}_2(t)&=0\\ Var\mathcal{X}_2(t)&=2t\left(a+\frac{\mu}{\rho}\right).\end{aligned}$$

In the general case, for $\alpha \in (0,2)$ we can only evaluate the fractional moment of order $\gamma \in (-1, \alpha)$, by applying Theorem 3 in [32],

$$\begin{aligned}
\mathbb{E}\mathcal{X}_\alpha^\gamma(t) &= \int_t^{+\infty} \mathbb{E}\mathcal{S}_\alpha^\gamma(s) f_\Gamma(s-at,t)ds \qquad (3.9)\\
&= \frac{2^\gamma \Gamma\left(1-\frac{\gamma}{\alpha}\right)\Gamma\left(\frac{1+\gamma}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{\gamma}{2}\right)} \int_0^{+\infty} (at+s)^{\gamma/\alpha} f_\Gamma(s,t)ds \\
&= \frac{2^\gamma \Gamma\left(1-\frac{\gamma}{\alpha}\right)\Gamma\left(\frac{1+\gamma}{2}\right)}{\sqrt{\pi}\Gamma\left(1-\frac{\gamma}{2}\right)} \sum_{j=0}^{\infty} \binom{\gamma/\alpha}{j} (at)^j \int_0^{+\infty} s^{\gamma/\alpha - j} f_\Gamma(s,t)ds \\
&= \frac{\gamma 2^\gamma \Gamma\left(1-\frac{\gamma}{\alpha}\right)\Gamma\left(\frac{1+\gamma}{2}\right)\Gamma\left(\frac{\gamma}{\alpha}\right)}{\alpha\sqrt{\pi}\Gamma\left(1-\frac{\gamma}{2}\right)\Gamma(\mu t)\rho^{\gamma/\alpha}} \sum_{j=0}^{\infty} \frac{(\rho a t)^j}{j!} \frac{\Gamma\left(\frac{\gamma}{\alpha}+\mu t - j\right)}{\Gamma\left(\frac{\gamma}{\alpha}+1-j\right)} \\
&= \frac{\gamma 2^\gamma \sqrt{\pi} 2^{1-\gamma}\Gamma(\gamma)}{\alpha \sin(\pi\gamma/\alpha)\Gamma\left(1-\frac{\gamma}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)\Gamma(\mu t)\rho^{\gamma/\alpha}} {}_1\Psi_1\left(\rho a t \middle| \begin{matrix} (\gamma/\alpha+\mu t, -1) \\ (\gamma/\alpha + 1, -1) \end{matrix}\right) \\
&= \frac{2\sin(\pi\gamma/2)\Gamma(\gamma+1)}{\alpha\sqrt{\pi}\sin(\pi\gamma/\alpha)\Gamma(\mu t)\rho^{\gamma/\alpha}} {}_1\Psi_1\left(\rho a t \middle| \begin{matrix} (\gamma/\alpha+\mu t, -1) \\ (\gamma/\alpha + 1, -1) \end{matrix}\right),
\end{aligned}$$

where ${}_1\Psi_1$ denotes the generalized Wright function with $p = q = 1$ (see [15], p.56). By applying Theorem 1.5 in [15], p.58, it is easy to check that the series in (3.9) is absolutely convergent for all t.

Finally, we show that the tails' behavior of the density of $\mathcal{X}_\alpha(t)$, for any fixed t, is the same (up to a different constant) of those holding for both the stable and geometric stable random variables (see [29], p. 17, and [17], respectively).

Theorem 14 *For $\alpha \in (0,2)$, we have that*

$$\begin{cases} \lim_{x\to\infty} x^\alpha P(\mathcal{X}_\alpha(t) > x) = \frac{C_{\alpha,\theta}\left(a+\frac{\mu}{\rho}\right)t}{\Gamma(1-\alpha)} \\ \lim_{x\to\infty} x^\alpha P(\mathcal{X}_\alpha(t) < -x) = \frac{C'_{\alpha,\theta}\left(a+\frac{\mu}{\rho}\right)t}{\Gamma(1-\alpha)} \end{cases}, \qquad t \geq 0, \qquad (3.10)$$

where $C_{\alpha,\theta} = \frac{1}{2}\left[1 - \frac{\tan(\pi\alpha/2)}{\tan(\pi\theta/2)}\right]$ *and* $C'_{\alpha,\theta} = \frac{1}{2}\left[1 + \frac{\tan(\pi\alpha/2)}{\tan(\pi\theta/2)}\right]$

Proof. We start by considering the process $\mathcal{S}_{\alpha,\theta}(at+\Gamma(t))$, $t \geq 0$, in the special case $\alpha \in (0,1)$, $\theta = -\alpha$, for which we can write that

$$\begin{aligned}
\int_0^{+\infty} e^{-\eta x} P(\mathcal{S}_{\alpha,-\alpha}(at+\Gamma(t)) > x)dx &= \frac{1 - \mathbb{E}e^{-\eta \mathcal{S}_{\alpha,-\alpha}(at+\Gamma(t))}}{\eta} \\
&= \frac{1 - \exp\{-a\eta^\alpha t\} - \left(1 + \frac{\eta^\alpha}{\rho}\right)^{\mu t}}{\eta} \\
&\sim \left(at + \frac{\mu t}{\rho}\right)\eta^{\alpha-1},
\end{aligned}$$

for any fixed t and for $\eta \to 0$. By applying the Tauberian theorem (see Theorem XIII-5-4, p.446, in [11]), we obtain the first equation in (3.10), with $C_{\alpha,\theta} = 1$. The case $\alpha \in (0,1)$, $|\theta| \leq \alpha$ can be obtained by equation (1.2.6) in [29], while for $\alpha \geq 1$ and $\theta = 0$ we adapt Proposition 1.3.1 in [29], p.20, to the r.v. $\mathcal{S}_\alpha(at + \Gamma(t))$, for fixed t. Let $A_{\alpha/\alpha'}$ be a (totally skewed to the right) stable r.v. of index α/α', with $\alpha' > \alpha$ and $\theta = -\alpha/\alpha', \sigma' = \left(\cos \frac{\pi\alpha}{2\alpha'}\right)^{\alpha'/\alpha}$ with Laplace transform

$$\mathbb{E}e^{-sA_{\alpha/\alpha'}} = \exp\left\{-s^{\alpha/\alpha'}\right\}$$

and let $X^\Gamma_{\alpha'}$ be a (symmetric) stable r.v. of index α', $\theta = 0$ and with

$$\mathbb{E}\left\{ e^{i\xi X^\Gamma_{\alpha'}} \middle| \Gamma \right\} = \exp\{-|\xi|^{\alpha'} [at + \Gamma(t)]^{\alpha'/\alpha}\}.$$

Thus we prove that

$$Z_\alpha := \left(A_{\alpha/\alpha'}\right)^{1/\alpha'} X_{\alpha'}$$

is a stable r.v. of index α, with parameters $\theta = 0$: indeed we can write

$$\begin{aligned}
\mathbb{E}e^{i\xi Z_\alpha} &= \mathbb{E}e^{i\xi A^{1/\alpha'}_{\alpha/\alpha'} X_{\alpha'}} = \mathbb{E}\left\{\mathbb{E}\left[e^{i\xi\left(A_{\alpha/\alpha'}\right)^{1/\alpha'} X^\Gamma_{\alpha'}} \middle| A_{\alpha/\alpha'} \right]\right\} \\
&= \mathbb{E}\left\{\mathbb{E}\left[\mathbb{E}\left(e^{i\xi\left(A_{\alpha/\alpha'}\right)^{1/\alpha'} X_{\alpha'}} \middle| \Gamma \right) \middle| A_{\alpha/\alpha'} \right]\right\} \\
&= \mathbb{E}\left\{\mathbb{E}\left[e^{-|\xi|^{\alpha'}[at+\Gamma(t)]^{\alpha'/\alpha} A_{\alpha/\alpha'}} \middle| A_{\alpha/\alpha'} \right]\right\} \\
&= \mathbb{E}e^{-|\xi|^{\alpha}[at+\Gamma(t)]},
\end{aligned}$$

which is the characteristic function of the r.v. $\mathcal{S}_\alpha(at + \Gamma(t))$, for fixed t. ■

3.2 Nonlinear fractional diffusion-type equation

We consider now a non-linear extension of the equation (1.1), defined as follows

$$\partial_t u(x,t) = \left[a\mathcal{D}^{\alpha,\theta}_x + \lambda at \left(I - \mathcal{O}^{\alpha,\theta}_{1,x}\right)\right] u(x,t) - \lambda \left(I - \mathcal{O}^{\alpha,\theta}_{1,x}\right) I^{\alpha-1}_\theta \left[xu(x,t)\right], \quad (3.11)$$

and we prove that, for $\alpha \in (1,2)$, and under the initial condition $u(x,0) = p^\theta_\alpha(x,1)$, the solution coincides with the transition density of the stable process $\mathcal{S}^\theta_\alpha$ time-changed by the so-called linear birth process with drift.

The form of the above equation is suggested by the following preliminary result. Let $B(t)$ $t \geq 0$, be a linear birth (Yule-Furry) process with one progenitor and parameter $\lambda > 0$. We recall that it is a Markov and (a.s.) non-decreasing process, with one-dimensional distribution

$$q_k(t) := \Pr\{ B(t) = k |\, B(0) = 1\} = e^{-\lambda t}(1 - e^{-\lambda t})^{k-1}, \quad k = 1, 2, ...$$

which is solution to the initial-value problem

$$\frac{d}{dt}q_k(t) = -\lambda k q_k(t) + \lambda(k-1)q_{k-1}(t), \quad q_k(0) = 1_{k=1}. \tag{3.12}$$

Lemma 15 *The density $q_a(x,t)$ of the linear birth process with positive drift, defined as $B(t)+at$, $a,t \geq 0$, satisfies the following equation:*

$$\partial_t u(x,t) = a\left[\lambda t\left(I - e^{-\partial_x}\right) - \partial_x\right]u(x,t) - \lambda\left(I - e^{-\partial_x}\right)[xu(x,t)], \qquad x \geq at+1,\ t \geq 0, \tag{3.13}$$

with initial condition $u(x,0) = \delta(x-1)$.

Proof. It is easy to check that the characteristic function of $B(t)+at$ is equal to

$$\Phi_{B(t)+at}(\xi) := \mathbb{E}e^{i\xi[B(t)+at]} = \frac{e^{-\lambda t + i\xi + i\xi at}}{1-(1-e^{-\lambda t})e^{i\xi}}, \tag{3.14}$$

so that

$$\begin{aligned}
\partial_t \Phi_{B(t)+at}(\xi) &= (-\lambda + i\xi a)\Phi_{B(t)+at}(\xi) + \lambda \frac{e^{-\lambda t + i\xi}}{1-(1-e^{-\lambda t})e^{i\xi}}\Phi_{B(t)+at}(\xi) \qquad (3.15)\\
&= i\xi a \Phi_{B(t)+at}(\xi) - \lambda \frac{1-(1-e^{-\lambda t})e^{i\xi} - e^{-\lambda t + i\xi}}{1-(1-e^{-\lambda t})e^{i\xi}}\Phi_{B(t)+at}(\xi)\\
&= i\xi a \Phi_{B(t)+at}(\xi) - \lambda(1-e^{i\xi})\frac{\Phi_{B(t)+at}(\xi)}{1-(1-e^{-\lambda t})e^{i\xi}}.
\end{aligned}$$

We now concentrate to the last fraction and we consider the following fact

$$\begin{aligned}
\mathcal{F}\{xq_a(x,t);\xi\} &= e^{-\lambda t}\sum_{k=1}^{\infty}(1-e^{-\lambda t})^{k-1}\int_{-\infty}^{+\infty} e^{ix\xi}x\delta(x-k-at)dx \qquad (3.16)\\
&= e^{-\lambda t}\sum_{k=1}^{\infty}(1-e^{-\lambda t})^{k-1}e^{i\xi(k+at)}(k+at)\\
&= at\Phi_{B(t)+at}(\xi) + \frac{\Phi_{B(t)+at}(\xi)}{1-(1-e^{-\lambda t})e^{i\xi}},
\end{aligned}$$

so that we can rewrite (3.15) as

$$\partial_t \Phi_{B(t)+at}(\xi) = i\xi a \Phi_{B(t)+at}(\xi) - \lambda(1-e^{i\xi})\left[\mathcal{F}\{xq_a(x,t);\xi\} - at\Phi_{B(t)+at}(\xi)\right].$$

The last equation coincides with the Fourier transform of (3.13). The initial condition is satisfied, as can be checked by considering (3.14) for $t=0$. ■

Remark 16 *We note that equation (3.13), for $a \neq 0$, is an extension to the continuous domain of the equation (3.12) governing the usual birth process, to which it reduces in the limit, for $a \to 0$. In the last limiting case the shift operator $e^{-\partial_x}$ is replaced by the backward difference operator Δ, defined as $\Delta f(k) = f(k-1)$.*

The linear and non-linear birth processes have been treated in the fractional case (by considering a fractional time-derivative in (3.12)) by [27] and [28]; see also [1]. In the last reference the birth process subordinated by an independent stable subordinator (i.e. $B(\mathcal{S}_\alpha^\theta(t))$, for $\theta = -\alpha$) is considered.

We now define the following process

$$\mathcal{Y}_\alpha^\theta(t) := \mathcal{S}_\alpha^\theta(at + B(t)), \qquad t > 0, \tag{3.17}$$

where B is independent of $\mathcal{S}_\alpha^\theta$. We recall that (3.17) cannot be indicated as a 'subordinated' process, since B is not a subordinator, but we will refer to it as a randomly 'time-changed' process. For an overview on time change, see [34] and the references therein. The characteristic function of $\mathcal{Y}_\alpha^\theta(t), t \geq 0$, can be evaluated as follows

$$\begin{aligned} \Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi) \quad : &= \mathbb{E}e^{i\xi\mathcal{Y}_\alpha^\theta(t)} = \mathbb{E}\left(\mathbb{E}e^{i\xi\mathcal{S}_\alpha^\theta(at+B(t))}\Big| B(t)\right) \\ &= e^{-a\psi_{\alpha,\theta}(\xi)t}\mathbb{E}e^{-|\psi_{\alpha,\theta}(\xi)B(t)} = \frac{e^{-\lambda t - at\psi_{\alpha,\theta}(\xi) - \psi_{\alpha,\theta}(\xi)}}{1-(1-e^{-\lambda t})e^{-\psi_{\alpha,\theta}(\xi)}}, \end{aligned} \tag{3.18}$$

by recalling the Laplace transform of the Yule-Furry process, i.e.

$$\mathbb{E}e^{-sB(t)} = \frac{e^{-s-\lambda t}}{1-e^{-s}(1-e^{-\lambda t})}, \qquad \mathcal{R}e(s) > 0$$

and considering that $\mathcal{R}e(\psi_{\alpha,\theta}(\xi)) = \mathcal{R}e(|\xi|^\alpha e^{i\,sign(\xi)\theta\pi/2}) = |\xi|^\alpha\cos(\theta\pi/2) \geq 0$, for $|\theta| \leq \min\{\alpha, 2-\alpha\}$ and $\alpha \in (0, 2]$.

As it is evident from (3.18), the process $\mathcal{Y}_\alpha^\theta(t)$, $t \geq 0$ is not Lévy, even though it is still Markov.

We now restrict our analysis to the case $\alpha \in (1, 2]$, so that the following holds

$$\left|\int_{-\infty}^{+\infty} e^{i\xi x} x p_\alpha^\theta(x,t)dx\right| \leq \int_{-\infty}^{+\infty} |x|\, p_\alpha^\theta(x,t)dx < \infty \tag{3.19}$$

Lemma 17 *Let $\mathcal{F}\{xu(x,t);\xi\} := \mathbb{E}\left(\mathcal{Y}_\alpha^\theta(t)e^{i\xi\mathcal{Y}_\alpha^\theta(t)}\right)$. Then, for $\alpha \in (1,2]$ and $|\theta| \leq 2-\alpha$, the characteristic function of $\mathcal{Y}_\alpha^\theta(t), t \geq 0$, satisfies the following equation*

$$\begin{aligned} \partial_t\widetilde{u}(\xi,t) &= a[-\psi_{\alpha,\theta}(\xi) + \lambda t(1-e^{-\psi_{\alpha,\theta}(\xi)})]\widetilde{u}(\xi,t) \\ &\quad + \frac{\lambda(1-e^{-\psi_{\alpha,\theta}(\xi)})}{\alpha}|\xi|^{1-\alpha}e^{-i\,sign(\xi)(\theta-1)\pi/2}\mathcal{F}\{xu(x,t);\xi\}, \end{aligned} \tag{3.20}$$

with initial condition $\widetilde{u}(\xi,0) = e^{-\psi_{\alpha,\theta}(\xi)}$.

Proof. Let

$$u(x,t) = \mathcal{F}^{-1}\left\{\Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi); x\right\} = \sum_{k=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{k-1}p_\alpha^\theta(x, k+at) \tag{3.21}$$

be the transition density of the process $\mathcal{Y}_\alpha^\theta$, then we apply a conditioning argument and the dominated convergence theorem (considering (3.19)), to show that

$$\begin{aligned}
\mathcal{F}\{xu(x,t);\xi\} &= \mathbb{E}\left\{\mathbb{E}\left[\mathcal{S}_\alpha^\theta(B(t)+at)e^{i\xi\mathcal{S}_\alpha^\theta(B(t)+at)}\right| B(t)\right]\Big\} \qquad (3.22)\\
&= \sum_{k=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{k-1}\int_{-\infty}^{+\infty} e^{i\xi x} x p_\alpha^\theta(x,k+at)dx \\
&= \frac{1}{i}\sum_{k=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{k-1}\int_{-\infty}^{+\infty} \partial_\xi e^{i\xi x} p_\alpha^\theta(x,k+at)dx \\
&= \frac{1}{i}\sum_{k=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{k-1}\partial_\xi e^{-\psi_{\alpha,\theta}(\xi)(k+at)} \\
&= i\partial_\xi\psi_{\alpha,\theta}(\xi)e^{-\lambda t-a\psi_{\alpha,\theta}(\xi)t}\sum_{k=1}^{\infty}(1-e^{-\lambda t})^{k-1}(k+at)e^{-\psi_{\alpha,\theta}(\xi)k} \\
&= \alpha i\, sign(\xi)|\xi|^{\alpha-1}e^{i\, sign(\xi)\theta\pi/2}\left[\frac{\Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi)}{1-(1-e^{-\lambda t})e^{-\psi_{\alpha,\theta}(\xi)}} + at\Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi)\right],
\end{aligned}$$

where, in the last step, we have considered that

$$\partial_\xi\psi_{\alpha,\theta}(\xi) = \partial_\xi|\xi|^\alpha e^{i\, sign(\xi)\theta\pi/2} = \alpha\, sign(\xi)|\xi|^{\alpha-1}e^{i\, sign(\xi)\theta\pi/2}.$$

As a check, we notice that, for $\alpha \to 1^+$, $\theta = -1$, (3.22) reduces to (3.16), as it must be. By differentiating (3.18) with respect to t, we get

$$\begin{aligned}
\partial_t\Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi) &= -\left[\lambda + a\psi_{\alpha,\theta}(\xi)\right]\Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi) + \frac{\lambda e^{-\lambda t-\psi_{\alpha,\theta}(\xi)}}{1-(1-e^{-\lambda t})e^{-\psi_{\alpha,\theta}(\xi)}}\Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi) \qquad (3.23)\\
&= -a\psi_{\alpha,\theta}(\xi)\Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi) - \lambda(1-e^{-\psi_{\alpha,\theta}(\xi)})\frac{\Phi_{\mathcal{Y}_\alpha^\theta(t)}(\xi)}{1-(1-e^{-\lambda t})e^{-\psi_{\alpha,\theta}(\xi)}},
\end{aligned}$$

which, considering (3.22), coincides with (3.20). Again, as a check, (3.20) reduces to (3.13) for $\alpha \to 1^+$, $\theta = -1$. ■

Let $L^1_{loc}(\mathbb{R})$ be the space of the locally integrable function and let I_γ^ν denote the Feller integral with symbol ([13], p.341)

$$\widehat{I_\gamma^\nu}(\xi) = |\xi|^{-\nu}e^{-i\pi\gamma sign(\xi)/2}, \qquad |\gamma| \leq \begin{cases} \nu, & 0<\nu<1 \\ 2-\nu, & 1<\nu<2 \end{cases}, \qquad (3.24)$$

defined, for $0 < \nu < 1$, for functions in $L^1_{loc}(\mathbb{R})$ (see also [14]). Since I_γ^ν is defined only for $\nu \neq 1$, for the case $\alpha = 2$ we consider instead the Weyl integral with symbol ([13], p.333)

$$\widehat{I^1_+}(\xi) = |\xi|^{-1}e^{i\pi/2 sign(\xi)}, \qquad (3.25)$$

which can be written explicitly as

$$I_+^1[f(x)] = \int_{-\infty}^{x} f(z)dz.$$

Theorem 18 *Let* $\alpha \in (1,2)$ *and* $\theta = 2-\alpha$. *Then the density of the process* $\mathcal{Y}_\alpha^\theta(t)$, $t \geq 0$, *satisfies the following equation*

$$\partial_t u(x,t) = a\left[\mathcal{D}_x^{\alpha,\theta} + \lambda t(I - \mathcal{O}_{1,x}^{\alpha,\theta})\right] u(x,t) + \frac{\lambda}{\alpha}\left(I - \mathcal{O}_{1,x}^{\alpha,\theta}\right) I_{1-\alpha}^{\alpha-1}\left[xu(x,t)\right], \quad (3.26)$$

with initial condition $u(x,0) = p_\alpha^\theta(x,1)$, *while, for* $\alpha = 2$, $\theta = 0$, *we have instead*

$$\partial_t u(x,t) = a\left[\partial_x^2 + \lambda t(I - e^{\partial_x^2})\right] u(x,t) + \frac{\lambda}{2}(I - e^{\partial_x^2}) \int_{-\infty}^{x} zu(z,t)dz, \quad (3.27)$$

with $u(x,0) = \varphi(x)$, *where* φ *denotes the standard Gaussian density function.*

Proof. We start by noting that, for $\alpha \in (1,2]$, the function $xu(x,t)$ belongs to $L^1(\mathbb{R})$, indeed

$$\int_{-\infty}^{+\infty} xu(x,t)dx = \sum_{k=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{k-1} \int_{-\infty}^{+\infty} xp_\alpha^\theta(x,k+t)dx = 0, \quad (3.28)$$

since the location parameter of $\mathcal{S}_\alpha^\theta$ is zero by assumption. We recognize, in the last term of (3.20), the symbol given in (3.24), with $\nu = \alpha - 1 \in (0,1)$ and $\gamma = \theta - 1$. Thus we introduce the constraint $|\theta - 1| \leq \alpha - 1$, which must be considered together with the condition given in (1.4), i.e. $|\theta| \leq 2 - \alpha$, so that they are jointly satisfied only by $\theta = 2 - \alpha$. Then, considering (2.2) and (3.24), we can write the inverse Fourier transform of (3.20) as in (3.26). Equation (3.27) can be derived analogously from (3.20), by considering (3.25). The convergence of the integral in (3.27) follows from (3.28). ■

Remark 19 *It is easy to check that equation (3.26) reduces to (3.13) in the limit, for* $\alpha \to 1^+$.

Remark 20 *The solution to equation (3.27) coincides with the transition density of the process* $W(at + B(t))$, $t > 0$. *In this case, i.e. for* $\alpha = 2$, $\theta = 0$, *the first two moments exist and we have*

$$\begin{aligned} \mathbb{E}\mathcal{Y}_2(t) &= \mathbb{E}\left\{\mathbb{E}\left[W(at+B(t))|\, B(t)\right]\right\} = 0 \\ Var\mathcal{Y}_2(t) &= \sum_{k=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{k-1}\mathbb{E}\left[W(at+k)\right]^2 = 2\left(at + \frac{1}{\lambda}\right). \end{aligned}$$

In the general case, for $\alpha \in (0,2)$ we can only evaluate the fractional moment of order $\gamma \in (-1,\alpha)$, by applying the result in [29], p.18,

$$\begin{aligned}
\mathbb{E}\mathcal{Y}_\alpha^\gamma(t) &= \sum_{k=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{k-1}\mathbb{E}\mathcal{S}_\alpha^\gamma(s) \qquad (3.29)\\
&= \frac{\Gamma\left(1-\frac{\gamma}{\alpha}\right)\cos(\gamma\theta\pi/2\alpha)(1+\tan^2(\theta\pi/2)^{\gamma/2\alpha}}{\Gamma(1-\gamma)\cos(\gamma\pi/2)} \sum_{k=1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{k-1}(at+k)^{\gamma/\alpha}\\
&= \frac{\Gamma\left(1-\frac{\gamma}{\alpha}\right)\cos(\gamma\theta\pi/2\alpha)(1+\tan^2(\theta\pi/2)^{\gamma/2\alpha}}{\Gamma(1-\gamma)\cos(\gamma\pi/2)} \sum_{j=0}^{\infty} \binom{\gamma/\alpha}{j}(at)^{\gamma/\alpha-j}\mathbb{E}Z^j
\end{aligned}$$

where Z_j is geometric r.v. Z with parameter $e^{-\lambda t}$.

As far as the tails behavior of the density of $\mathcal{Y}_\alpha^\theta(t)$, for any fixed t, we prove that it has regularly varying tails, with index α. Thus it exhibits the same tails behavior of the stable process and of $\mathcal{X}_\alpha^\theta(t)$ (see Theorem 14), even though the constant, in this case, is not linear in time.

Theorem 21 *For $\alpha \in (0,2)$, we have that*

$$\begin{cases}
\lim_{x\to\infty} x^\alpha P(\mathcal{Y}_\alpha^\theta(t) > x) = \frac{C_{\alpha,\theta}\frac{1+ate^{-\lambda t}}{e^{-\lambda t}}}{\Gamma(1-\alpha)} \\
\lim_{x\to\infty} x^\alpha P(\mathcal{Y}_\alpha^\theta(t) < -x) = \frac{C'_{\alpha,\theta}\frac{1+ate^{-\lambda t}}{e^{-\lambda t}}}{\Gamma(1-\alpha)}
\end{cases}, \qquad t \geq 0, \qquad (3.30)$$

where $C_{\alpha,\theta} = \frac{1}{2}\left[1 - \frac{\tan(\pi\alpha/2)}{\tan(\pi\theta/2)}\right]$ and $C'_{\alpha,\theta} = \frac{1}{2}\left[1 + \frac{\tan(\pi\alpha/2)}{\tan(\pi\theta/2)}\right]$.

Proof. For $\alpha \in (0,1)$, $\theta = -\alpha$, we can write that

$$\begin{aligned}
\int_0^{+\infty} e^{-\eta x} P(\mathcal{S}_{\alpha,-\alpha}(at+B(t)) > x)dx &= \frac{1-\mathbb{E}e^{-\eta\mathcal{S}_{\alpha,-\alpha}(at+B(t))}}{\eta}\\
&= \frac{1-(1-e^{-\lambda t})e^{-\eta^\alpha} - e^{-\lambda t-(at+1)\eta^\alpha}}{\eta\left[1-(1-e^{-\lambda t})e^{-\eta^\alpha}\right]}\\
&\sim \frac{1-(1-e^{-\lambda t})(1-\eta^\alpha) - e^{-\lambda t}\left[1-(at+1)\eta^\alpha\right]}{\eta\left[1-(1-e^{-\lambda t})(1-\eta^\alpha)\right]}\\
&\sim \frac{1+ate^{-\lambda t}}{e^{-\lambda t}}\eta^{\alpha-1},
\end{aligned}$$

for any fixed t and for $\eta \to 0$, so that we get the first equation in (3.30), for $C_{\alpha,\theta} = 1$. The rest of the proof follows the same lines of Theorem 14. ■

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