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REPRESENTATIONS OF THE CANONICAL COMMUTATION RELATIONS–ALGEBRA AND THE OPERATORS OF STOCHASTIC CALCULUS

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Abstract. We study a family of representations of the canonical commutation relations (CCR)-algebra (an infinite number of degrees of freedom), which we call admissible. The family of admissible representations includes the Fock-vacuum representation. We show that, to every admissible representation, there is an associated Gaussian stochastic calculus, and we point out that the case of the Fock-vacuum CCR-representation in a natural way yields the operators of Malliavin calculus. And we thus get the operators of Malliavin's calculus of variation from a more algebraic approach than is common. And we obtain explicit and natural formulas, and rules, for the operators of stochastic calculus. Our approach makes use of a notion of symmetric (closable) pairs of operators. The Fock-vacuum representation yields a maximal symmetric pair. This duality viewpoint has the further advantage that issues with unbounded operators and dense domains can be resolved much easier than what is possible with alternative tools. With the use of CCR representation theory, we also obtain, as a byproduct, a number of new results in multi-variable operator theory which we feel are of independent interest.

Contents

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1. Introduction

Both the study of quantum fields, and of quantum statistical mechanics, entails families of representations of the canonical commutation relations (CCRs). In the case of an infinite number of degrees of freedom, it is known that we have existence of many inequivalent representations of the CCRs. Among the representations, some describe such things as a nonrelativistic infinite free Bose gas of uniform density. But the representations of the CCRs play an equally important role in the kind of infinite-dimensional analysis currently used in a calculus of variation approach to Gaussian fields, Itō integrals, including the Malliavin calculus. In the literature, the infinite-dimensional stochastic operators of derivatives and stochastic integrals are usually taken as the starting point, and the representations of the CCRs are an afterthought. Here we turn the tables. As a consequence of this, we are able to obtain a number of explicit results in an associated multi-variable spectral theory. Some of the issues involved are subtle because the operators in the representations under consideration are unbounded (by necessity), and, as a result, one must deal with delicate issues of domains of families of operators and their extensions.

The representations we study result from the Gelfand-Naimark-Segal construction (GNS) applied to certain states on the CCR-algebra. Our conclusions and main results regarding this family of CCR representations (details below, especially sects 4 and 5) hold in the general setting of Gaussian fields. But for the benefit of readers, we have also included an illustration dealing with the simplest case, that of the standard Brownian/Wiener process. Many arguments in the special case carry over to general Gaussian fields *mutatis mutandis*. In the Brownian case, our initial Hilbert space will be $\mathscr{L} = L^2(0,\infty)$.

From the initial Hilbert space $\mathscr{L}$, we build the $*$-algebra $\mathrm{CCR}(\mathscr{L})$ as in Section 2.2. We will show that the Fock state on $\mathrm{CCR}(\mathscr{L})$ corresponds to the Wiener measure $\mathbb{P}$. Moreover the corresponding representation π of $\mathrm{CCR}(\mathscr{L})$ will be acting on the Hilbert space $L^2(\Omega,\mathbb{P})$ in such a way that for every k in $\mathscr{L}$, the operator $\pi(a(k))$ is the Malliavin derivative in the direction of k. We caution that the representations of the $*$-algebra $\mathrm{CCR}(\mathscr{L})$ are by unbounded operators, but the operators in the range of the representations will be defined on a single common dense domain.

Example: There are two ways to think of systems of generators for the CCR-algebra over a fixed infinite-dimensional Hilbert space ("CCR" is short for canonical commutation relations.):

(i) an infinite-dimensional Lie algebra, or
(ii) an associative $*$-algebra.

With this in mind, (ii) will simply be the universal enveloping algebra of (i); see [Dix77]. While there is also an infinite-dimensional "Lie" group corresponding to (i), so far, we have not found it as useful as the Lie algebra itself.

All this, and related ideas, supply us with tools for an infinite-dimensional stochastic calculus. It fits in with what is called Malliavin calculus, but our present approach is different, and more natural from our point of view; and as corollaries, we obtain new and explicit results in multi-variable spectral theory which we feel are of independent interest.

There is one particular representation of the CCR version of (i) and (ii) which is especially useful for stochastic calculus. In the present paper, we call this representation the Fock vacuum-state representation. One way of realizing the representations is abstract: Begin with the Fock vacuum state (or any other state), and then pass to the corresponding GNS representation. The other way is to realize the representation with the use of a choice of a Wiener L^2-space. We prove that these two realizations are unitarily equivalent.

By stochastic calculus we mean stochastic derivatives (e.g., Malliavin derivatives), and integrals (e.g., Itō-integrals). The paper begins with the task of realizing a certain stochastic derivative operator as a closable operator acting between two Hilbert spaces.

2. Unbounded operators and the CCR-algebra

2.1. Unbounded operators between different Hilbert spaces. While the theory of unbounded operators has been focused on spectral theory where it is then natural to consider the setting of linear *endomorphisms* with dense domain in a fixed Hilbert space; many applications entail operators between distinct Hilbert spaces, say $\mathscr{H}_1$ and $\mathscr{H}_2$. Typically the facts given about the two differ greatly from one Hilbert space to the next.

Let $\mathscr{H}_i$, $i = 1, 2$, be two complex Hilbert spaces. The respective inner products will be written $\langle \cdot, \cdot \rangle_i$, with the subscript to identify the Hilbert space in question.

Definition 2.1. A linear operator T from $\mathscr{H}_1$ to $\mathscr{H}_2$ is a pair $\mathscr{D} \subset \mathscr{H}_1$, T, where $\mathscr{D}$ is a linear subspace in $\mathscr{H}_1$, and $T\varphi \in \mathscr{H}_2$ is well-defined for all $\varphi \in \mathscr{D}$.

We say that $\mathscr{D} = dom\,(T)$ is the domain of T, and

$$\mathscr{G}\,(T) = \left\{ \begin{pmatrix} \varphi \\ T\varphi \end{pmatrix} ; \varphi \in \mathscr{D} \right\} \subset \begin{pmatrix} \mathscr{H}_1 \\ \oplus \\ \mathscr{H}_2 \end{pmatrix} \tag{2.1}$$

is the graph.

If the closure $\overline{\mathscr{G}\,(T)}$ is the graph of a linear operator, we say that T is *closable.* By closure, we shall refer to closure in the norm of $\mathscr{H}_1 \oplus \mathscr{H}_2$, i.e.,

$$\left\| \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \right\|^2 = \|h_1\|_1^2 + \|h_2\|_2^2, \quad h_i \in \mathscr{H}_i. \tag{2.2}$$

If $dom\,(T)$ is dense in $\mathscr{H}_1$, we say that T is densely defined.

Definition 2.2. Let $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$ be a densely defined operator, and consider the subspace $dom\,(T^*) \subset \mathscr{H}_2$ defined as follows:

$$dom\,(T^*) = \Big\{ h_2 \in \mathscr{H}_2 \,;\, \exists C = C_{h_2} < \infty \text{ s.t. } |\langle T\varphi, h_2 \rangle_2| \leq C \,\|\varphi\|_1\,, \ \forall \varphi \in dom\,(T) \Big\} \tag{2.3}$$

Then, by Riesz' theorem, there is a unique $h_1 \in \mathscr{H}_1$ s.t.

$$\langle T\varphi, h_2 \rangle_2 = \langle \varphi, h_1 \rangle_1\,, \text{ and} \tag{2.4}$$

we set $T^* h_2 = h_1$.

Lemma 2.3. *Given a densely defined operator $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$, then T is closable if and only if $dom\,(T^*)$ is dense in $\mathscr{H}_2$.*

Proof. See [DS88]. □

Remark 2.4 (Notation and Facts).

(1) The abbreviated notation $\mathscr{H}_1 \underset{T^*}{\overset{T}{\rightleftarrows}} \mathscr{H}_2$ will be used when the domains of T and T^* are understood from the context.

(2) Let T be an operator $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$ and $\mathscr{H}_i$, $i = 1, 2$, two given Hilbert spaces. Assume $\mathscr{D} := dom\,(T)$ is dense in $\mathscr{H}_1$, and that T is *closable*. Then there is a unique *closed* operator, denoted $\overline{T}$ such that

$$\mathscr{G}\left(\overline{T}\right) = \overline{\mathscr{G}\,(T)} \tag{2.5}$$

where "—" on the RHS in (2.5) refers to norm closure in $\mathscr{H}_1 \oplus \mathscr{H}_2$, see (2.2).

(3) It may happen that $dom\,(T^*) = 0$. See Example 2.5 below.

Example 2.5. An operator $T : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ with dense domain s.t. $dom\,(T^*) = 0$, i.e., "extremely" non-closable.

Set $\mathscr{H}_i = L^2\,(\mu_i)$, $i = 1, 2$, where μ_1 and μ_2 are two mutually singular measures on a fixed locally compact measurable space, say X. The space $\mathscr{D} := C_c\,(X)$ is dense in both $\mathscr{H}_1$ and in $\mathscr{H}_2$ with respect to the two L^2-norms. Then, the identity mapping $T\varphi = \varphi$, $\forall \varphi \in \mathscr{D}$, becomes a Hilbert space operator $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$.

Using Definition 2.2, we see that $h_2 \in L^2\,(\mu_2)$ is in $dom\,(T^*)$ iff $\exists h_1 \in L^2\,(\mu_1)$ such that

$$\int \varphi\, h_1\, d\mu_1 = \int \varphi\, h_2\, d\mu_2, \quad \forall \varphi \in \mathscr{D}. \tag{2.6}$$

Since $\mathscr{D}$ is dense in both L^2-spaces, we get

$$\int_E h_1\, d\mu_1 = \int_E h_2\, d\mu_2, \tag{2.7}$$

where $E = supp\,(\mu_2)$.

Now suppose $h_2 \neq 0$ in $L^2\,(\mu_2)$, then there is a subset $A \subset E$ s.t. $h_2 > 0$ on A, $\mu_2\,(A) > 0$, and $\int_A h_2\, d\mu_2 > 0$. But $\int_A h_1\, d\mu_1 = \int_A h_2\, d\mu_2$, and $\int_A h_1\, d\mu_1 = 0$ since $\mu_1\,(A) = 0$. This contradiction proves that $dom\,(T^*) = 0$; and in particular T is unbounded and non-closable.

Theorem 2.6. *Let $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$ be a densely defined operator, and assume that $dom\,(T^*)$ is dense in $\mathscr{H}_2$, i.e., T is closable, then both of the operators $T^*\overline{T}$ and $\overline{T}T^*$ are densely defined, and both are selfadjoint.*

Moreover, there is a partial isometry $U : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ with initial space in $\mathscr{H}_1$ and final space in $\mathscr{H}_2$ such that

$$T = U\left(T^*\overline{T}\right)^{\frac{1}{2}} = \left(\overline{T}T^*\right)^{\frac{1}{2}} U. \tag{2.8}$$

(Eq. (2.8) is called the polar decomposition of T.)

Proof. See, e.g., [DS88]. □

2.2. **The CCR-algebra, and the Fock representations.** There are two $*$-algebras built functorially from a fixed (single) Hilbert space $\mathscr{L}$; often called the one-particle Hilbert space (in physics). The dimension $\dim \mathscr{L}$ is called *the number of degrees of freedom*. The case of interest here is when $\dim \mathscr{L} = \aleph_0$ (countably infinite). The two $*$-algebras are called the CAR, and the CCR-algebras, and they are extensively studied; see e.g., [BR81]. Of the two, only CAR($\mathscr{L}$) is a C^*-algebra.

The operators arising from representations of CCR($\mathscr{L}$) will be *unbounded*, but still having a common dense domain in the respective representation Hilbert spaces. In both cases, we have a Fock representation. For CCR($\mathscr{L}$), it is realized in the symmetric Fock space $\Gamma_{sym}(\mathscr{L})$. There are many other representations, inequivalent to the respective Fock representations.

Let $\mathscr{L}$ be as above. The CCR($\mathscr{L}$) is generated axiomatically by a system, $a(h)$, $a^*(h)$, $h \in \mathscr{L}$, subject to

$$\begin{aligned} &[a(h), a(k)] = 0, \ \forall h, k \in \mathscr{L}, \text{ and} \\ &[a(h), a^*(k)] = \langle h, k \rangle_{\mathscr{L}} \mathbb{1}. \end{aligned} \tag{2.9}$$

Notation. In (2.9), $[\cdot,\cdot]$ denotes the commutator. More specifically, if A, B are elements in a $*$-algebra, set $[A, B] := AB - BA$.

The *Fock States* ω_{Fock} on the CCR-algebra are specified as follows:

$$\omega_{Fock}(a(h)a^*(k)) = \langle h, k \rangle_{\mathscr{L}} \tag{2.10}$$

with the vacuum property

$$\omega_{Fock}(a^*(h)a(h)) = 0, \ \forall h \in \mathscr{L}; \tag{2.11}$$

For the corresponding Fock representations π we have:

$$[\pi(h), \pi^*(k)] = \langle h, k \rangle_{\mathscr{L}} I_{\Gamma_{sym}(\mathscr{L})}, \tag{2.12}$$

where $I_{\Gamma_{sym}(\mathscr{L})}$ on the RHS of (2.12) refers to the identity operator.

Some relevant papers regarding the CCR-algebra and its representations are [AW63, Arv76a, Arv76b, PS72a, PS72b, AW73, GJ87, JP91].

2.3. **An infinite-dimensional Lie algebra.** Let $\mathscr{L}$ be a separable Hilbert space, i.e., $\dim \mathscr{L} = \aleph_0$, and let CCR$(\mathscr{L})$ be the corresponding CCR-algebra. As above, its generators are denoted $a(k)$ and $a^*(l)$, for $k, l \in \mathscr{L}$. We shall need the following:

Proposition 2.7.

(1) *The "quadratic" elements in CCR$(\mathscr{L})$ of the form $a(k)a^*(l)$, $k, l \in \mathscr{L}$, span a Lie algebra $\mathfrak{g}(\mathscr{L})$ under the commutator bracket.*

(2) *We have*

$$\begin{aligned} & [a(h)a^*(k), a(l)a^*(m)] \\ = \ & \langle h, m \rangle_{\mathscr{L}} a(l)a^*(k) - \langle k, l \rangle_{\mathscr{L}} a(h)a^*(m), \end{aligned}$$

for all $h, k, l, m \in \mathscr{L}$.

(3) *If $\{\varepsilon_i\}_{i\in\mathbb{N}}$ is an ONB in $\mathscr{L}$, then the non-zero commutators are as follows: Set $\gamma_{i,j} := a(\varepsilon_i)a^*(\varepsilon_j)$, then, for $i \neq j$, we have*

$$[\gamma_{i,i}, \gamma_{j,i}] = \gamma_{j,i}; \tag{2.13}$$

$$[\gamma_{i,i}, \gamma_{i,j}] = -\gamma_{i,j}; \text{ and} \tag{2.14}$$

$$[\gamma_{j,i}, \gamma_{i,j}] = \gamma_{i,i} - \gamma_{j,j}. \tag{2.15}$$

All other commutators vanish; in particular, $\{\gamma_{i,i} \mid i \in \mathbb{N}\}$ spans an abelian sub-Lie algebra in $\mathfrak{g}(\mathscr{L})$.

Note further that, when $i \neq j$, then the three elements

$$\gamma_{i,i} - \gamma_{j,j}, \quad \gamma_{i,j}, \quad \text{and} \quad \gamma_{j,i} \tag{2.16}$$

span (over $\mathbb{R}$) an isomorphic copy of the Lie algebra $sl_2(\mathbb{R})$.

(4) *The Lie algebra generated by the first-order elements $a(h)$ and $a^*(k)$ for $h,k \in \mathscr{L}$, is called the Heisenberg Lie algebra $\mathfrak{h}(\mathscr{L})$. It is normalized by $\mathfrak{g}(\mathscr{L})$; indeed we have:*

$$[a(l)\,a^*(m), a(h)] = -\langle m,h\rangle_{\mathscr{L}}\, a(l), \text{ and}$$
$$[a(l)\,a^*(m), a^*(k)] = \langle l,k\rangle_{\mathscr{L}}\, a^*(m), \ \forall l,m,h,k \in \mathscr{L}.$$

Proof. The verification of each of the four assertions (1)-(4) uses only the fixed axioms for the CCR, i.e.,

$$\begin{cases} [a(k), a(l)] = 0, \\ [a^*(k), a^*(l)] = 0, \text{ and} \\ [a(k), a^*(l)] = \langle k,l\rangle_{\mathscr{L}}\, \mathbb{1}, \ k,l \in \mathscr{L}; \end{cases} \tag{2.17}$$

where $\mathbb{1}$ denotes the unit-element in $\mathrm{CCR}(\mathscr{L})$. □

Corollary 2.8. *Let $CCR(\mathscr{L})$ be the CCR-algebra, generators $a(k)$, $a^*(l)$, $k,l \in \mathscr{L}$, and let $[\cdot,\cdot]$ denote the commutator Lie bracket; then, for all $k,h_1,\cdots,h_n \in \mathscr{L}$, and all $p \in \mathbb{R}[x_1,\cdots,x_n]$ (= the n-variable polynomials over $\mathbb{R}$), we have*

$$\begin{aligned} &[a(k), p(a^*(h_1),\cdots,a^*(h_n))] \\ = &\sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(a^*(h_1),\cdots,a^*(h_n))\,\langle k,h_i\rangle_{\mathscr{L}}. \end{aligned} \tag{2.18}$$

Proof. The verification of (2.18) uses only the axioms for the CCR, i.e., the commutation relations (2.17) above, plus a little combinatorics. □

We shall now return to a stochastic variation of formula (2.18), the so called Malliavin derivative in the direction k. In this, the system $(a^*(h_1),\cdots,a^*(h_n))$ in (2.18) instead takes the form of a multivariate Gaussian random variable.

2.4. **Gaussian Hilbert space.** The literature on Gaussian Hilbert space, white noise analysis, and its relevance to Malliavin calculus is vast; and we limit ourselves here to citing [BØSW04, AJL11, AJ12, VFHN13, AJS14, AJ15, AØ15], and the papers cited there.

Setting and Notation.

$\mathscr{L}$: a fixed *real* Hilbert space
$(\Omega,\mathcal{F},\mathbb{P})$: a fixed probability space
$L^2(\Omega,\mathbb{P})$: the Hilbert space $L^2(\Omega,\mathcal{F},\mathbb{P})$, also denoted by $L^2(\mathbb{P})$
$\mathbb{E}$: the mean or expectation functional, where $\mathbb{E}(\cdots) = \int_\Omega (\cdots)\, d\mathbb{P}$

Definition 2.9. Fix a *real* Hilbert space $\mathscr{L}$ and a given probability space $(\Omega,\mathcal{F},\mathbb{P})$. We say the pair $(\mathscr{L},(\Omega,\mathcal{F},\mathbb{P}))$ is a *Gaussian Hilbert space.*

A *Gaussian field* is a linear mapping $\Phi : \mathscr{L} \longrightarrow L^2(\Omega,\mathbb{P})$, such that

$$\{\Phi(h) \mid h \in \mathscr{L}\}$$

is a Gaussian process indexed by $\mathscr{L}$ satisfying:

(1) $\mathbb{E}(\Phi(h)) = 0$, $\forall h \in \mathscr{L}$;
(2) $\forall n \in \mathbb{N}$, $\forall l_1,\cdots,l_n \subset \mathscr{L}$, the random variable $(\Phi(l_1),\cdots,\Phi(l_n))$ is jointly Gaussian, with

$$\mathbb{E}(\Phi(l_i)\,\Phi(l_j)) = \langle l_i,l_j\rangle, \tag{2.19}$$

i.e., $(\langle l_i,l_j\rangle)_{i=1}^{n}$ = the covariance matrix. (For the existence of Gaussian fields, see the discussion below.)

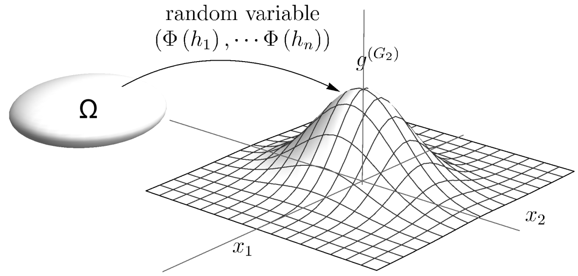


FIGURE 2.1. The multivariate Gaussian $(\Phi(h_1), \cdots \Phi(h_n))$ and its distribution. The Gaussian with Gramian matrix (Gram matrix) G_n, $n = 2$.

Remark 2.10. For all finite systems $\{l_i\} \subset \mathscr{L}$, set $G_n = (\langle l_i, l_j \rangle)_{i,j=1}^n$, called the Gramian. Assume G_n non-singular for convenience, so that $\det G_n \neq 0$. Then there is an associated Gaussian density $g^{(G_n)}$ on $\mathbb{R}^n$,

$$g^{(G_n)}(x) = (2\pi)^{-n/2} (\det G_n)^{-1/2} \exp\left(-\frac{1}{2} \langle x, G_n^{-1} x \rangle_{\mathbb{R}^n}\right) \tag{2.20}$$

The condition in (2.19) assumes that for all continuous functions $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ (e.g., polynomials), we have

$$\mathbb{E}(\underbrace{f(\Phi(l_1), \cdots, \Phi(l_n))}_{\text{real valued}}) = \int_{\mathbb{R}^n} f(x) g^{(G_n)}(x) dx; \tag{2.21}$$

where $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, and $dx = dx_1 \cdots dx_n$ = Lebesgue measure on $\mathbb{R}^n$. See Figure 2.1 for an illustration.

In particular, for $n = 2$, $\langle l_1, l_2 \rangle = \langle k, l \rangle$, and $f(x_1, x_2) = x_1 x_2$, we then get $\mathbb{E}(\Phi(k)\Phi(l)) = \langle k, l \rangle$, i.e., the inner product in $\mathscr{L}$.

For our applications, we need the following facts about Gaussian fields.

Fix a Hilbert space $\mathscr{L}$ over $\mathbb{R}$ with inner product $\langle \cdot, \cdot \rangle_{\mathscr{L}}$. Then (see [Hid80, AØ15, Gro70]) there is a *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$, depending on $\mathscr{L}$, and a *real* linear mapping $\Phi : \mathscr{L} \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$, i.e., a Gaussian field as specified in Definition 2.9, satisfying

$$\mathbb{E}\left(e^{i\Phi(k)}\right) = e^{-\frac{1}{2}\|k\|^2}, \quad \forall k \in \mathscr{L}. \tag{2.22}$$

It follows from the literature (see also [JT14]) that $\Phi(k)$ may be thought of as a generalized Itō-integral. One approach to this is to select a nuclear Fréchet space $\mathcal{S}$ with dual $\mathcal{S}'$ such that

$$\mathcal{S} \hookrightarrow \mathscr{L} \hookrightarrow \mathcal{S}' \tag{2.23}$$

forms a Gelfand triple. In this case we may take $\Omega = \mathcal{S}'$, and $\Phi(k)$, $k \in \mathscr{L}$, to be the extension of the mapping

$$\mathcal{S}' \ni \omega \longrightarrow \omega(\varphi) = \langle \varphi, \omega \rangle \tag{2.24}$$

defined initially only for $\varphi \in \mathcal{S}$, but, with the use of (2.24), now extended, via (2.22), from $\mathcal{S}$ to $\mathscr{L}$. See also Example 2.12 below.

Example 2.11. Fix a measure space $(X, \mathcal{B}, \mu)$. Let $\Phi : L^2(\mu) \longrightarrow L^2(\Omega, \mathbb{P})$ be a Gaussian field such that

$$\mathbb{E}(\Phi_A \Phi_B) = \mu(A \cap B), \quad \forall A, B \in \mathcal{B}$$

where $\Phi_E := \Phi(\chi_E)$, $\forall E \in \mathcal{B}$; and χ_E denotes the characteristic function. In this case, $\mathscr{L} = L^2(X, \mu)$.

Then we have $\Phi(k) = \int_X k(x)\, d\Phi$, i.e., the Itō-integral, and the following holds:

$$\mathbb{E}(\Phi(k)\Phi(l)) = \langle k, l \rangle = \int_X k(x)\, l(x)\, d\mu(x) \tag{2.25}$$

for all $k, l \in \mathscr{L} = L^2(X, \mu)$. Eq. (2.25) is known as the Itō-isometry.

Example 2.12 (The special case of Brownian motion)**.** There are many ways of realizing a Gaussian probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Two candidates for the sample space:

Case 1. Standard Brownian motion process: $\Omega = C(\mathbb{R})$, $\mathcal{F} = \sigma$-algebra generated by cylinder sets, $\mathbb{P} =$ Wiener measure. Set $B_t(\omega) = \omega(t)$, $\forall \omega \in \Omega$; and $\Phi(k) = \int_{\mathbb{R}} k(t)\, dB_t$, $\forall k \in L^2(\Omega, \mathbb{P})$.

Case 2. The Gelfand triples: $\mathcal{S} \hookrightarrow L^2(\mathbb{R}) \hookrightarrow \mathcal{S}'$, where

$S =$ the Schwartz space of test functions;

$\mathcal{S}' =$ the space of tempered distributions.

Set $\Omega = \mathcal{S}'$, $\mathcal{F} = \sigma$-algebra generated by cylinder sets of $\mathcal{S}'$, and define

$$\Phi(k) := \widehat{k}(\omega) = \langle k, \omega \rangle, \ k \in L^2(\mathbb{R}), \ \omega \in \mathcal{S}'.$$

Note Φ is defined by extending the duality $\mathcal{S} \longleftrightarrow \mathcal{S}'$ to $L^2(\mathbb{R})$. The probability measure $\mathbb{P}$ is defined from

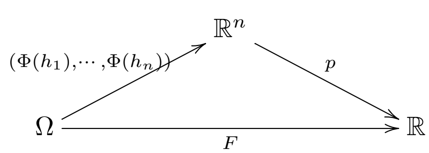
$$\mathbb{E}\big(e^{i\langle k, \cdot \rangle}\big) = \int_{\mathcal{S}'} e^{i\widehat{k}(\omega)} d\mathbb{P}(\omega) = e^{-\frac{1}{2}\|k\|^2_{L^2(\mathbb{R})}},$$

by Minlos' theorem [Hid80, AØ15].

Definition 2.13. Let $\mathscr{D} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ be the dense subspace spanned by functions F, where $F \in \mathscr{D}$ iff $\exists n \in \mathbb{N}$, $\exists h_1, \cdots, h_n \in \mathscr{L}$, and $p \in \mathbb{R}[x_1, \cdots, x_n] =$ the polynomial ring, such that

$$F = p(\Phi(h_1), \cdots, \Phi(h_n)) : \Omega \longrightarrow \mathbb{R}.$$

(See the diagram below.) The case of $n = 0$ corresponds to the constant function $\mathbb{1}$ on Ω. Note that $\Phi(h_i) \in L^2(\Omega, \mathbb{P})$.



Lemma 2.14. *The polynomial fields $\mathscr{D}$ in Def. 2.13 form a dense subspace in* $L^2(\Omega, \mathbb{P})$.

Proof. The easiest argument below takes advantage of the isometric isomorphism of $L^2(\Omega, \mathbb{P})$ with the symmetric Fock space

$$\Gamma_{sym}(\mathscr{L}) = \underbrace{\mathscr{H}_0}_{1\text{ dim}} \oplus \sum_{n=1}^{\infty} \underbrace{(\mathscr{L} \otimes \cdots \otimes \mathscr{L})}_{n\text{-fold symmetric}}.$$

For $k_i \in \mathscr{L}$, $i = 1, 2$, there is a unique vector $e^{k_i} \in \Gamma_{sym}(\mathscr{L})$ such that

$$\left\langle e^{k_1}, e^{k_2} \right\rangle_{\Gamma_{sym}(\mathscr{L})} = \sum_{n=0}^{\infty} \frac{\langle k_1, k_2 \rangle^n}{n!} = e^{\langle k_1, k_2 \rangle_{\mathscr{L}}}.$$

Moreover,

$$\Gamma_{sym}(\mathscr{L}) \ni e^k \xrightarrow{W_0} e^{\Phi(k) - \frac{1}{2}\|k\|^2_{\mathscr{L}}} \in L^2(\Omega, \mathbb{P})$$

extends by linearity and closure to a unitary isomorphism $\Gamma_{sym}(\mathscr{L}) \xrightarrow{W} L^2(\Omega, \mathbb{P})$, mapping onto $L^2(\Omega, \mathbb{P})$. Hence $\mathscr{D}$ is dense in $L^2(\Omega, \mathbb{P})$, as *span* $\left\{e^k \mid k \in \mathscr{L}\right\}$ is dense in $\Gamma_{sym}(\mathscr{L})$. □

Lemma 2.15. *Let $\mathscr{L}$ be a real Hilbert space, and let $(\Omega, \mathcal{F}, \mathbb{P}, \Phi)$ be an associated Gaussian field. For $n \in \mathbb{N}$, let $\{h_1, \cdots, h_n\}$ be a system of linearly independent vectors in $\mathscr{L}$. Then, for polynomials $p \in \mathbb{R}[x_1, \cdots, x_n]$, the following two conditions are equivalent:*

$$p(\Phi(h_1), \cdots, \Phi(h_n)) = 0 \quad \text{a.e. on } \Omega \text{ w.r.t } \mathbb{P}\text{; and} \tag{2.26}$$

$$p(x_1, \cdots, x_n) \equiv 0, \quad \forall (x_1, \cdots, x_n) \in \mathbb{R}^n. \tag{2.27}$$

Proof. Let $G_n = (\langle h_i, h_j \rangle)_{i,j=1}^n$ be the Gramian matrix. We have $\det G_n \neq 0$. Let $g^{(G_n)}(x_1, \cdots, x_n)$ be the corresponding Gaussian density; see (2.20), and Figure 2.1. Then the following are equivalent:

(1) Eq. (2.26) holds;
(2) $p(\Phi(h_1), \cdots, \Phi(h_n)) = 0$ in $L^2(\Omega, \mathcal{F}, \mathbb{P})$;
(3) $\mathbb{E}\left(|p(\Phi(h_1), \cdots, \Phi(h_n))|^2\right) = \int_{\mathbb{R}^n} |p(x)|^2 g^{(G_n)}(x)\, dx = 0$;
(4) $p(x) = 0$ a.e. x w.r.t. the Lebesgue measure in $\mathbb{R}^n$;
(5) $p(x) = 0$, $\forall x \in \mathbb{R}^n$; i.e., (2.27) holds.

□

3. The Malliavin derivatives

Below we give an application of the closability criterion for linear operators T between different Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, but having dense domain in the first Hilbert space. In this application, we shall take for T to be the so called Malliavin derivative. The setting for it is that of the Wiener process. For the Hilbert space $\mathscr{H}_1$ we shall take the L^2-space, $L^2(\Omega, \mathbb{P})$ where $\mathbb{P}$ is generalized Wiener measure. Below we shall outline the basics of the Malliavin derivative, and we shall specify the two Hilbert spaces corresponding to the setting of Theorem 2.6. We also stress that the literature on Malliavin calculus and its applications is vast, see e.g., [BØSW04, AØ15].

Settings. It will be convenient for us to work with the *real* Hilbert spaces.

Let $(\Omega, \mathcal{F}, \mathbb{P}, \Phi)$ be as specified in Definition 2.9, i.e., we consider the Gaussian field Φ. Fix a *real* Hilbert space $\mathscr{L}$ with $\dim \mathscr{L} = \aleph_0$. Set $\mathscr{H}_1 = L^2(\Omega, \mathbb{P})$, and $\mathscr{H}_2 = L^2(\Omega \to \mathscr{L}, \mathbb{P}) = L^2(\Omega, \mathbb{P}) \otimes \mathscr{L}$, i.e., vector valued random variables.

For $\mathscr{H}_1$, the inner product $\langle \cdot, \cdot \rangle_{\mathscr{H}_1}$ is

$$\langle F, G \rangle_{\mathscr{H}_1} = \int_{\Omega} FG\, d\mathbb{P} = \mathbb{E}(FG); \tag{3.1}$$

where $\mathbb{E}(\cdots) = \int_{\Omega} (\cdots)\, d\mathbb{P}$ is the mean or expectation functional.

On $\mathscr{H}_2$, we have the tensor product inner product: If $F_i \in \mathscr{H}_1$, $k_i \in \mathscr{L}$, $i = 1, 2$, then

$$\begin{aligned}\langle F_1 \otimes k_1, F_2 \otimes k_2 \rangle_{\mathscr{H}_2} &= \langle F_1, F_2 \rangle_{\mathscr{H}_1} \langle k_1, k_2 \rangle_{\mathscr{L}} \\ &= \mathbb{E}\left(F_1 F_2\right) \langle k_1, k_2 \rangle_{\mathscr{L}} . \end{aligned} \tag{3.2}$$

Equivalently, if $\psi_i : \Omega \longrightarrow \mathscr{L}$, $i = 1, 2$, are measurable functions on Ω, we set

$$\langle \psi_1, \psi_2 \rangle_{\mathscr{H}_2} = \int_\Omega \langle \psi_1(\omega), \psi_2(\omega) \rangle_{\mathscr{L}} \, d\mathbb{P}(\omega) ; \tag{3.3}$$

where it is assumed that

$$\int_\Omega \left\| \psi_i(\omega) \right\|_{\mathscr{L}}^2 d\mathbb{P}(\omega) < \infty, \quad i = 1, 2. \tag{3.4}$$

Remark 3.1. In the special case of standard Brownian motion, we have $\mathscr{L} = L^2(0, \infty)$, and set $\Phi(h) = \int_0^\infty h(t) \, d\Phi_t$ (= the Itō-integral), for all $h \in \mathscr{L}$. Recall we then have

$$\mathbb{E}\left(\left| \Phi(h) \right|^2 \right) = \int_0^\infty \left| h(t) \right|^2 dt, \tag{3.5}$$

or equivalently (the Itō-isometry),

$$\left\| \Phi(h) \right\|_{L^2(\Omega, \mathbb{P})} = \left\| h \right\|_{\mathscr{L}}, \quad \forall h \in \mathscr{L}. \tag{3.6}$$

The consideration above also works in the context of general Gaussian fields; see Section 2.4.

Definition 3.2. Let $\mathscr{D}$ be the dense subspace in $\mathscr{H}_1 = L^2(\Omega, \mathbb{P})$ as in Definition 2.13. The operator $T : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ (= Malliavin derivative) with $dom(T) = \mathscr{D}$ is specified as follows:

For $F \in \mathscr{D}$, i.e., $\exists n \in \mathbb{N}$, $p(x_1, \cdots, x_n)$ a polynomial in n real variables, and $h_1, h_2, \cdots, h_n \in \mathscr{L}$, where

$$F = p\left(\Phi(h_1), \cdots, \Phi(h_n)\right) \in L^2(\Omega, \mathbb{P}) . \tag{3.7}$$

Set

$$T(F) = \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} p \right) \left(\Phi(h_1), \cdots, \Phi(h_n)\right) \otimes h_j \in \mathscr{H}_2. \tag{3.8}$$

In the following two remarks we outline the argument for why the expression for $T(F)$ in (3.8) is independent of the chosen representation (3.7) for the particular F. Recall that F is in the domain $\mathscr{D}$ of T. Without some careful justification, it is not even clear that T, as given, defines a linear operator on its dense domain $\mathscr{D}$. The key steps in the argument to follow will be the result (3.12) in Theorem 3.8 below, and the discussion to follow.

There is an alternative argument, based instead on Corollary 2.8; see also Section 5 below.

Remark 3.3. It is non-trivial that the formula in (3.8) defines a linear operator. Reason: On the LHS in (3.8), the representation of F from (3.7) is not unique. So we must show that $p\left(\Phi(h_1), \cdots, \Phi(h_n)\right) = 0 \Longrightarrow \mathrm{RHS}_{(3.8)} = 0$ as well. (The dual pair analysis below (see Def. 3.6) is good for this purpose.)

Suppose $F \in \mathscr{D}$ has two representations corresponding to systems of vectors $h_1, \cdots, h_n \in \mathscr{L}$, and $k_1, \cdots, k_m \in \mathscr{L}$, with polynomials $p \in \mathbb{R}[x_1, \cdots, x_n]$, and $q \in \mathbb{R}[x_1, \cdots, x_m]$, where

$$F = p(\Phi(h_1), \cdots, \Phi(h_n)) = q(\Phi(k_1), \cdots, \Phi(k_m)). \tag{3.9}$$

We must then verify the identity:

$$\sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(\Phi(h_1), \cdots, \Phi(h_n)) \otimes h_i = \sum_{i=1}^{m} \frac{\partial q}{\partial x_i}(\Phi(k_1), \cdots, \Phi(k_m)) \otimes k_i. \tag{3.10}$$

The significance of the next result is the implication (3.9) $\Longrightarrow$ (3.10), valid for all choices of representations of the same $F \in \mathscr{D}$. The conclusion from (3.12) in Theorem 3.8 is that the following holds for all $l \in \mathscr{L}$:

$$\mathbb{E}\left(\left\langle \mathrm{LHS}_{(3.10)}, l \right\rangle\right) = \mathbb{E}\left(\left\langle \mathrm{RHS}_{(3.10)}, l \right\rangle\right) = \mathbb{E}\left(F\Phi(l)\right).$$

Moreover, with a refinement of the argument, we arrive at the identity

$$\left\langle \mathrm{LHS}_{(3.10)} - \mathrm{RHS}_{(3.10)}, G \otimes l \right\rangle_{\mathscr{H}_2} = 0,$$

valid for all $G \in \mathscr{D}$, and all $l \in \mathscr{L}$.

But *span* $\{G \otimes l \mid G \in \mathscr{D},\ l \in \mathscr{L}\}$ is dense in $\mathscr{H}_2 \left(= L^2(\mathbb{P}) \otimes \mathscr{L}\right)$ w.r.t. the tensor-Hilbert norm in $\mathscr{H}_2$ (see (3.2)); and we get the desired identity (3.10) for any two representations of F.

Remark 3.4. An easy case where (3.9) $\Longrightarrow$ (3.10) can be verified "by hand":

Let $F = \Phi(h)^2$ with $h \in \mathscr{L} \setminus \{0\}$ fixed. We can then pick the two systems $\{h\}$ and $\{h, h\}$ with $p(x) = x^2$, and $q(x_1, x_2) = x_1 x_2$. A direct calculus argument shows that $\mathrm{LHS}_{(3.10)} = \mathrm{RHS}_{(3.10)} = 2\Phi(h) \otimes h \in \mathscr{H}_2$.

We now resume the argument for the general case.

Definition 3.5 (symmetric pair)**.** For $i = 1, 2$, let $\mathscr{H}_i$ be two Hilbert spaces, and suppose $\mathscr{D}_i \subset \mathscr{H}_i$ are given dense subspaces.

We say that a pair of operators (S, T) forms a *symmetric pair* if $dom(T) = \mathscr{D}_1$, and $dom(S) = \mathscr{D}_2$; and moreover,

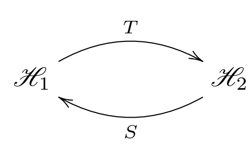
$$\langle Tu, v \rangle_{\mathscr{H}_2} = \langle u, Sv \rangle_{\mathscr{H}_1} \tag{3.11}$$

holds for $\forall u \in \mathscr{D}_1$, $\forall v \in \mathscr{D}_2$.

It is immediate that (3.11) may be rewritten in the form of containment of graphs:

$$T \subset S^*, \quad S \subset T^*.$$

In that case, both S and T are *closable*. We say that a symmetric pair is *maximal* if $\overline{T} = S^*$ and $\overline{S} = T^*$.



We will establish the following two assertions:

(1) Indeed T from Definition 3.2 is a well-defined linear operator from $\mathscr{H}_1$ to $\mathscr{H}_2$.
(2) Moreover, (S, T) is a maximal symmetric pair (see Definitions 3.5, 3.6).

Definition 3.6. Let $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$ be the Malliavin derivative with $\mathscr{D}_1 = dom\,(T)$, see Definition 3.2. Set $\mathscr{D}_2 = \mathscr{D}_1 \otimes \mathscr{L}$ = algebraic tensor product, and on $dom\,(S) = \mathscr{D}_2$, set

$$S\,(F \otimes k) = -\left\langle T\,(F)\,, k\right\rangle + M_{\Phi(k)}F,\ \forall F \otimes k \in \mathscr{D}_2,$$

where $M_{\Phi(k)}$ = the operator of multiplication by $\Phi\,(k)$.

Note that both operators S and T are linear and well defined on their respective dense domains, $\mathscr{D}_i \subset \mathscr{H}_i$, $i = 1, 2$. For density, see Lemma 2.14.

It is a "modern version" of ideas in the literature on analysis of Gaussian processes; but we are adding to it, giving it a twist in the direction of multi-variable operator theory, representation theory, and especially to representations of infinite-dimensional algebras on generators and relations. Moreover our results apply to more general Gaussian processes than covered so far.

Lemma 3.7. *Let (S, T) be the pair of operators specified above in Definition 3.6. Then it is a symmetric pair, i.e.,*

$$\left\langle Tu, v\right\rangle_{\mathscr{H}_2} = \left\langle u, Sv\right\rangle_{\mathscr{H}_1}, \quad \forall u \in \mathscr{D}_1,\ \forall v \in \mathscr{D}_2.$$

Equivalently,

$$\left\langle T\,(F)\,, G \otimes k\right\rangle_{\mathscr{H}_2} = \left\langle F, S\,(G \otimes k)\right\rangle_{\mathscr{H}_1}, \quad \forall F, G \in \mathscr{D},\ \forall k \in \mathscr{L}.$$

In particular, we have $S \subset T^*$, and $T \subset S^*$ (containment of graphs.) Moreover, the two operators $S^*\overline{S}$ and $T^*\overline{T}$ are selfadjoint. (For the last conclusion in the lemma, see Theorem 2.6.)

Theorem 3.8. *Let $T : \mathscr{H}_1 \longrightarrow \mathscr{H}_2$ be the Malliavin derivative, i.e., T is an unbounded closable operator with dense domain $\mathscr{D}$ consisting of the span of all the functions F from (3.7). Then, for all $F \in dom\,(T)$, and $k \in \mathscr{L}$, we have*

$$\mathbb{E}\left(\left\langle T\,(F)\,, k\right\rangle_{\mathscr{L}}\right) = \mathbb{E}\left(F\Phi\,(k)\right). \tag{3.12}$$

Proof. We shall prove (3.12) in several steps. Once (3.12) is established, then there is a recursive argument which yields a dense subspace in $\mathscr{H}_2$, contained in $dom\,(T^*)$; and so T is closable.

Moreover, formula (3.12) yields directly the evaluation of $T^* : \mathscr{H}_2 \longrightarrow \mathscr{H}_1$ as follows: If $k \in \mathscr{L}$, set $\mathbb{1} \otimes k \in \mathscr{H}_2$ where $\mathbb{1}$ denotes the constant function "one" on Ω. We get

$$T^*\,(\mathbb{1} \otimes k) = \Phi\,(k) = \int_0^\infty k\,(t)\,d\Phi_t\,(= \text{the Itō-integral.}) \tag{3.13}$$

The same argument works for any Gaussian field; see Definition 2.9. We refer to the literature [BØSW04, AØ15] for details.

The proof of (3.12) works for any Gaussian process $\mathscr{L} \ni k \longrightarrow \Phi\,(k)$ indexed by an arbitrary Hilbert space $\mathscr{L}$ with the inner product $\left\langle k, l\right\rangle_{\mathscr{L}}$ as the covariance kernel.

Formula (3.12) will be established as follows: Let F and $T\,(F)$ be as in (3.7)-(3.8).

Step 1. For every $n \in \mathbb{N}$, the polynomial ring $\mathbb{R}\left[x_1, x_2, \cdots, x_n\right]$ is invariant under matrix substitution $y = Mx$, where M is an $n \times n$ matrix over $\mathbb{R}$.

Step 2. Hence, in considering (3.12) for $\{h_i\}_{i=1}^n \subset \mathscr{L}$, $h_1 = k$, we may diagonalize the $n \times n$ Gram matrix $(\langle h_i, h_j \rangle)_{i,j=1}^n$; thus without loss of generality, we may assume that the system $\{h_i\}_{i=1}^n$ is orthogonal and normalized, i.e., that

$$\langle h_i, h_j \rangle = \delta_{ij},\ \forall i, j \in \{1, \cdots, n\}, \tag{3.14}$$

and we may take $k = h_1$ in $\mathscr{L}$.

Step 3. With this simplification, we now compute the LHS in (3.12). We note that the joint distribution of $\{\Phi(h_i)\}_{i=1}^n$ is thus the standard Gaussian kernel in $\mathbb{R}^n$, i.e.,

$$g_n(x) = (2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^n x_i^2}, \tag{3.15}$$

with $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. We have

$$x_1 g_n(x) = -\frac{\partial}{\partial x_1} g_n(x) \tag{3.16}$$

by calculus.

Step 4. A direct computation yields

$$\begin{aligned}
\text{LHS}_{(3.12)} &= \mathbb{E}\left(\langle T(F), h_1\rangle_{\mathscr{L}}\right) \\
&\underset{\text{by } (3.14)}{=} \mathbb{E}\left(\frac{\partial p}{\partial x_1}\left(\Phi(h_1), \cdots \Phi(h_n)\right)\right) \\
&\underset{\text{by } (3.15)}{=} \int_{\mathbb{R}^n} \frac{\partial p}{\partial x_1}(x_1, \cdots, x_n)\, g_n(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \\
&\underset{\text{int. by parts}}{=} -\int_{\mathbb{R}^n} p(x_1, \cdots, x_n) \frac{\partial g_n}{\partial x_1}(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \\
&\underset{\text{by } (3.16)}{=} -\int_{\mathbb{R}^n} x_1 p(x_1, \cdots, x_n)\, g_n(x_1, \cdots, x_n)\, dx_1 \cdots dx_n \\
&\underset{\text{by } (3.14)}{=} \mathbb{E}\left(\Phi(h_1)\, p\left(\Phi(h_1), \cdots \Phi(h_n)\right)\right) \\
&= \mathbb{E}\left(\Phi(h_1) F\right) = \text{RHS}_{(3.12)},
\end{aligned}$$

which is the desired conclusion (3.12). □

Corollary 3.9. *Let $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$ be as in Theorem 3.8, i.e., T is the Malliavin derivative. Then, for all $h, k \in \mathscr{L} = L^2(0, \infty)$, we have for the closure $\overline{T}$ of T the following:*

$$\overline{T}(e^{\Phi(h)}) = e^{\Phi(h)} \otimes h, \quad \text{and} \tag{3.17}$$

$$\mathbb{E}\big(\langle \overline{T}(e^{\Phi(h)}), k\rangle_{\mathscr{L}}\big) = e^{\frac{1}{2}\|h\|_{\mathscr{L}}^2} \langle h, k\rangle_{\mathscr{L}}. \tag{3.18}$$

Here $\overline{T}$ denotes the graph-closure of T.

Moreover,

$$T^*\overline{T}(e^{\Phi(k)}) = \left(\Phi(k) - \|k\|_{\mathscr{L}}^2\right) e^{\Phi(k)}. \tag{3.19}$$

Proof. Eqs. (3.17)-(3.18) follow immediately from (3.12) and a polynomial approximation to

$$e^x = \lim_{n\to\infty} \sum_0^n \frac{x^j}{j!}, \quad x \in \mathbb{R};$$

see (3.7). In particular, $e^{\Phi(h)} \in dom\left(\overline{T}\right)$, and $\overline{T}\left(e^{\Phi(h)}\right)$ is well defined.

For (3.19), we use the facts for the Gaussians:

$$\mathbb{E}(e^{\Phi(k)}) = e^{\frac{1}{2}\|k\|^2}, \text{ and}$$
$$\mathbb{E}(\Phi(k)\, e^{\Phi(k)}) = \|k\|^2\, e^{\frac{1}{2}\|k\|^2}.$$

□

Example 3.10. Let $F = \Phi(k)^k$, $\|k\| = 1$. We have

$$T\Phi(k)^n = n\Phi(k)^{n-1} \otimes k$$
$$T^*T\Phi(k)^n = -n(n-1)\Phi(k)^{n-2} + n\Phi(k)^n$$

and similarly,

$$\overline{T}e^{\Phi(k)} = e^{\Phi(k)} \otimes k$$
$$T^*\overline{T}e^{\Phi(k)} = e^{\Phi(k)}(\Phi(k) - 1).$$

Let (S, T) be the symmetric pair, we then have the inclusion $\overline{T} \subset S^*$, i.e., containment of the operator graphs, $\mathscr{G}(\overline{T}) \subset \mathscr{G}(S^*)$. In fact, we have

Corollary 3.11. *$\overline{T} = S^*$.*

Proof. We will show that $\mathscr{G}(S^*) \ominus \mathscr{G}(\overline{T}) = 0$, where $\ominus$ stands for the orthogonal complement in the inner product of $\mathscr{H}_1 \otimes \mathscr{H}_2$. Recall that $\mathscr{H}_1 = L^2(\Omega, \mathbb{P})$, and $\mathscr{H}_2 = \mathscr{H}_1 \otimes \mathscr{L}$.

Using (3.17), we will prove that if $F \in dom(S^*)$, and

$$\left\langle \begin{pmatrix} e^{\Phi(k)} \\ e^{\Phi(k)} \otimes k \end{pmatrix}, \begin{pmatrix} F \\ S^*F \end{pmatrix} \right\rangle = 0,\ \forall k \in \mathscr{L} \Longrightarrow F = 0,$$

which is equivalent to

$$\mathbb{E}\left(e^{\Phi(k)}(F + \langle S^*F, k\rangle)\right) = 0,\ \forall k \in \mathscr{L}. \tag{3.20}$$

But it is know that for the Gaussian filed, $span\left\{e^{\Phi(k)} \mid k \in \mathscr{L}\right\}$ is dense in $\mathscr{H}_1$, and so (3.20) implies that $F = 0$, which is the desired conclusion. □

3.1. **A derivation on the algebra $\mathscr{D}$.** The study of unbounded derivations has many applications in mathematical physics; in particular in making precise the time dependence of quantum observables, i.e., the dynamics in the Schrödinger picture; — in more detail, in the problem of constructing dynamics in statistical mechanics. An early application of unbounded derivations (in the commutative case) can be found in the work of Silov [Šil47]; and the later study of unbounded derivations in non-commutative C^*-algebras is outlined in [BR81]. There is a rich in variety unbounded derivations, because of the role they play in applications to dynamical systems in quantum physics.

But previously the theory of unbounded derivations has not yet been applied systematically to stochastic analysis in the sense of Malliavin. In the present section, we turn to this. We begin with the following:

Lemma 3.12 (Leibniz-Malliavin)**.** *Let $\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2$ be the Malliavin derivative from (3.7)-(3.8). Then,*

(1) $dom(T) =: \mathscr{D}$, given by (3.7), is an algebra *of functions on Ω under point-wise product, i.e., $FG \in \mathscr{D}$, $\forall F, G \in \mathscr{D}$.*

(2) $\mathscr{H}_2$ *is a* module *over* $\mathscr{D}$ *where* $\mathscr{H}_2 = L^2(\Omega, \mathbb{P}) \otimes \mathscr{L}$ *(= vector valued* L^2*-random variables.)*

(3) *Moreover,*

$$T(FG) = T(F)G + F\,T(G), \quad \forall F, G \in \mathscr{D}, \tag{3.21}$$

i.e., T *is a module-derivation.*

Notation. The eq. (3.21) is called the Leibniz-rule. By the Leibniz, we refer to the traditional rule of Leibniz for the derivative of a product. And the Malliavin derivative is thus an infinite-dimensional extension of Leibniz calculus.

Proof. To show that $\mathscr{D} \subset \mathscr{H}_1 = L^2(\Omega, \mathbb{P})$ is an algebra under pointwise multiplication, the following trick is useful. It follows from finite-dimensional Hilbert space geometry.

Let F, G be as in Definition 2.13. Then $\exists p, q \in \mathbb{R}[x_1, \cdots, x_n]$, $\{l_i\}_{i=1}^n \subset \mathscr{L}$, such that

$$F = p(\Phi(l_1), \cdots, \Phi(l_n)), \text{ and } \quad G = q(\Phi(l_1), \cdots, \Phi(l_n)).$$

That is, the same system $l_1, \cdots, l_n$ may be chosen for the two functions F and G.

For the pointwise product, we have

$$FG = (pq)(\Phi(l_1), \cdots, \Phi(l_n)),$$

i.e., the product in $\mathbb{R}[x_1, \cdots, x_n]$ with substitution of the random variable

$$(\Phi(l_1), \cdots, \Phi(l_n)) : \Omega \longrightarrow \mathbb{R}^n.$$

Eq. (3.21) $\Longleftrightarrow$ $\frac{\partial(pq)}{\partial x_i} = \frac{\partial p}{\partial x_i} q + p \frac{\partial q}{\partial x_i}$, which is the usual Leibniz rule applied to polynomials. Note that

$$T(FG) = \sum_{i=1}^n \frac{\partial}{\partial x_i}(pq)(\Phi(l_1), \cdots, \Phi(l_n)) \otimes l_i.$$

□

Remark 3.13. There is an extensive literature on the theory of densely defined unbounded derivations in C^*-algebras. This includes both the cases of abelian and non-abelian $*$-algebras. And moreover, this study includes both derivations in these algebras, as well as the parallel study of module derivations. So the case of the Malliavin derivative is in fact a special case of this study. Readers interested in details are referred to [Sak98], [BJKR84], [BR79], and [BR81].

Definition 3.14. Let $(\mathscr{L}, \Omega, \mathcal{F}, \mathbb{P}, \Phi)$ be a Gaussian field, and T be the Malliavin derivative with *dom* $(T) = \mathscr{D}$. For all $k \in \mathscr{L}$, set

$$T_k(F) := \langle T(F), k \rangle, \quad F \in \mathscr{D}. \tag{3.22}$$

In particular, let $F = p(\Phi(l_1), \cdots, \Phi(l_1))$ be as in (3.7), then

$$T_k(F) = \sum_{i=1}^n \frac{\partial p}{\partial x_i}(\Phi(l_1), \cdots, \Phi(l_1)) \langle l_i, k \rangle.$$

Corollary 3.15. T_k *is a derivative on* $\mathscr{D}$*, i.e.,*

$$T_k(FG) = (T_k F) G + F(T_k G), \quad \forall F, G \in \mathscr{D}, \, \forall k \in \mathscr{L}. \tag{3.23}$$

Proof. Follows from (3.21). □

Corollary 3.16. *Let* $(\mathscr{L}, \Omega, \mathcal{F}, \mathbb{P}, \Phi)$ *be a Gaussian field. Fix* $k \in \mathscr{L}$*, and let* T_k *be the Malliavin derivative in the* k *direction. Then on* $\mathscr{D}$ *we have*

$$T_k + T_k^* = M_{\Phi(k)}, \text{ and} \tag{3.24}$$

$$[T_k, T_l^*] = \langle k, l\rangle_{\mathscr{L}} I_{L^2(\Omega,\mathbb{P})}. \tag{3.25}$$

Proof. For all $F, G \in \mathscr{D}$, we have

$$\begin{aligned}\mathbb{E}(T_k(F)G) + \mathbb{E}(F\,T_k(G)) &\underset{\text{by (3.23)}}{=} \mathbb{E}(T_k(FG)) \\ &\underset{\text{by (3.12)}}{=} \mathbb{E}(\Phi(k)FG)\end{aligned}$$

which yields the assertion in (3.24). Eq. (3.25) now follows from (3.24) and the fact that $[T_k, T_l] = 0$. □

Definition 3.17. Let $(\mathscr{L}, \Omega, \mathcal{F}, \mathbb{P}, \Phi)$ be a Gaussian field. For all $k \in \mathscr{L}$, let T_k be Malliavin derivative in the k-direction (eq. (3.22)). Assume $\mathscr{L}$ is separable, i.e., $\dim \mathscr{L} = \aleph_0$. For every ONB $\{e_i\}_{i=1}^{\infty}$ in $\mathscr{L}$, let

$$N := \sum_i T_{e_i}^* T_{e_i}. \tag{3.26}$$

(N is the CCR number operator. See Section 4 below.)

Example 3.18. $N\mathbb{1} = 0$, since $T_{e_i}\mathbb{1} = 0$, $\forall i$. Similarly,

$$N\Phi(k) = \Phi(k) \tag{3.27}$$

$$N\Phi(k)^2 = -2\|k\|^2\,\mathbb{1} + 2\Phi(k)^2, \quad \forall k \in \mathscr{L}. \tag{3.28}$$

To see this, note that

$$\begin{aligned}\sum_i T_{e_i}^* T_{e_i}\Phi(k) &= \sum_i T_{e_i}^* \langle e_i, k\rangle\,\mathbb{1} \\ &= \sum_i \Phi(e_i)\langle e_i, k\rangle \\ &= \Phi\left(\sum_i \langle e_i, k\rangle e_i\right) = \Phi(k),\end{aligned}$$

which is (3.27). The verification of (3.28) is similar.

Theorem 3.19. *Let* $\{e_i\}$ *be an ONB in* $\mathscr{L}$*, then*

$$T^*\overline{T} = \sum_i T_{e_i}^* T_{e_i} = N. \tag{3.29}$$

Proof. Note the span of $\left\{e^{\Phi(k)} \mid k \in \mathscr{L}\right\}$ is dense in $L^2(\Omega, \mathbb{P})$, and both sides of (3.29) agree on $e^{\Phi(k)}$, $k \in \mathscr{L}$. Indeed, by (3.26),

$$T^*\overline{T}e^{\Phi(k)} = Ne^{\Phi(k)} = \left(\Phi(k) - \|k\|^2\right)e^{\Phi(k)}.$$

□

Corollary 3.20. *Let* $D := T^*\overline{T}$*. Specialize to the case of* $n = 1$*, and consider* $F = f(\Phi(k))$*,* $k \in \mathscr{L}$*,* $f \in C^\infty(\mathbb{R})$*; then*

$$D(F) = -\|k\|_{\mathscr{L}}^2 f''(\Phi(k)) + \Phi(k)f'(\Phi(k)). \tag{3.30}$$

Proof. A direct application of the formulas of $\overline{T}$ and T^*. □

Remark 3.21. If $\|k\|_{\mathscr{L}} = 1$ in (3.30), then the RHS in (3.30) is obtained by a substitution of the real valued random variable $\Phi(k)$ into the deterministic function

$$\delta(f) := -\left(\frac{d}{dx}\right)^2 f + x\left(\frac{d}{dx}\right) f. \tag{3.31}$$

Then eq. (3.30) may be rewritten as

$$D(f(\Phi(k))) = \delta(f) \circ \Phi(k), \quad f \in C^{\infty}(\mathbb{R}). \tag{3.32}$$

Corollary 3.22. *If $\{H_n\}_{n\in\mathbb{N}_0}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\}$, denotes the Hermite polynomials on $\mathbb{R}$, then we get for $\forall k \in \mathscr{L}$, $\|k\|_{\mathscr{L}} = 1$, the following eigenvalues*

$$D(H_n(\Phi(k))) = n\, H_n(\Phi(k)). \tag{3.33}$$

Proof. It is well-known that the Hermite polynomials H_n satisfies

$$\delta(H_n) = n\, H_n, \quad \forall n \in \mathbb{N}_0, \tag{3.34}$$

and so (3.33) follows from a substitution of (3.34) into (3.32). □

Theorem 3.23. *The spectrum of $T^*\overline{T}$, as an operator in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, is as follows:*

$$spec_{L^2(\mathbb{P})}\left(T^*\overline{T}\right) = \mathbb{N}_0 = \{0, 1, 2, \cdots\}.$$

Proof. We saw that the $L^2(\mathbb{P})$-representation is unitarily equivalent to the Fock vacuum representation, and π (Fock-number operator) $= T^*\overline{T}$. □

3.2. **Infinite-dimensional Δ and ∇_Φ.**

Corollary 3.24. *Let $(\mathscr{L}, \Omega, \mathcal{F}, \mathbb{P}, \Phi)$ be a Gaussian field, and let T be the Malliavin derivative, $L^2(\Omega, \mathbb{P}) \xrightarrow{T} L^2(\Omega, \mathbb{P}) \otimes \mathscr{L}$. Then, for all $F = p(\Phi(h_1), \cdots, \Phi(h_n)) \in \mathscr{D}$ (see Definition 3.2), we have*

$$T^*T(F) = \underbrace{-\sum_{i=1}^{n} \frac{\partial^2 p}{\partial x_i}(\Phi(h_1), \cdots, \Phi(h_n))}_{\Delta F} + \underbrace{\sum_{i=1}^{n} \Phi(h_i) \frac{\partial p}{\partial x_i}(\Phi(h_1), \cdots, \Phi(h_n))}_{\nabla_\Phi F},$$

which is abbreviated

$$T^*T = -\Delta + \nabla_\Phi. \tag{3.35}$$

(For the general theory of infinite-dimensional Laplacians, see e.g., [Hid03].)

Proof. (Sketch) We may assume the system $\{h_i\}_{i=1}^n \subset \mathscr{L}$ is orthonormal, i.e., $\langle h_i, h_j\rangle = \delta_{ij}$. Hence, for F $F = p(\Phi(h_1), \cdots, \Phi(h_n)) \in \mathscr{D}$, we have

$$TF = \sum_{i=1}^{n} \frac{\partial p}{\partial x_i}(\Phi(h_1), \cdots, \Phi(h_n)) \otimes h_i, \text{ and}$$

$$\begin{aligned} T^*T(F) &= -\sum_{i=1}^{n} \frac{\partial^2 p}{\partial x_i^2}(\Phi(h_1), \cdots, \Phi(h_n)) \\ &\quad + \sum_{i=1}^{n} \Phi(h_i) \frac{\partial p}{\partial x_i}(\Phi(h_1), \cdots, \Phi(h_n)) \end{aligned}$$

which is the assertion. For details, see the proof of Theorem 3.8. □

Definition 3.25. Let $(\mathscr{L}, \Omega, \mathcal{F}, \mathbb{P}, \Phi)$ be a Gaussian field. On the dense domain $\mathscr{D} \subset L^2(\Omega, \mathbb{P})$, we define the Φ-gradient by

$$\nabla_\Phi F = \sum_{i=1}^{n} \Phi(h_i) \frac{\partial p}{\partial x_i} (\Phi(h_1), \cdots, \Phi(h_n)), \tag{3.36}$$

for all $F = p(\Phi(h_1), \cdots, \Phi(h_n)) \in \mathscr{D}$. (Note that ∇_Φ is an unbounded operator in $L^2(\Omega, \mathbb{P})$, and $dom(\nabla_\Phi) = \mathscr{D}$.)

Lemma 3.26. *Let ∇_Φ be the Φ-gradient from Definition 3.25. The adjoint operator ∇_Φ^*, i.e., the Φ-divergence, is given as follows:*

$$\nabla_\Phi^*(G) = \left(\sum_{i=1}^{n} \Phi(h_i)^2 - n \right) G - \nabla_\Phi(G), \quad \forall G \in \mathscr{D}. \tag{3.37}$$

Proof. Fix $F, G \in \mathscr{D}$ as in Definition 3.2. Then $\exists n \in \mathbb{N}$, $p, q \in \mathbb{R}[x_1, \cdots, x_n]$, and $\{h_i\}_{i=1}^n \subset \mathscr{L}$, such that

$$F = p(\Phi(h_1), \cdots, \Phi(h_n))$$
$$G = q(\Phi(h_1), \cdots, \Phi(h_n)).$$

Further assume that $\langle h_i, h_j \rangle = \delta_{ij}$.

In the calculation below, we use the following notation: $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, $dx = dx_1 \cdots dx_n$ = Lebesgue measure, and $g_n = g^{G_n}$ = standard Gaussian distribution in $\mathbb{R}^n$, see (3.15).

Then, we have

$$\begin{aligned}
& \mathbb{E}((\nabla_\Phi F) G) \\
= & \sum_{i=1}^{n} \mathbb{E}\left(\Phi(h_i) \frac{\partial p}{\partial x_i} (\Phi(h_1), \cdots, \Phi(h_n)) \, q(\Phi(h_1), \cdots, \Phi(h_n)) \right) \\
= & \sum_{i=1}^{n} \int_{\mathbb{R}^n} x_i \frac{\partial p}{\partial x_i}(x) \, q(x) \, g_n(x) \, dx \\
= & -\sum_{i=1}^{n} \int_{\mathbb{R}^n} p(x) \frac{\partial}{\partial x_i} (x_i q(x) g_n(x)) \, dx \\
= & -\sum_{i=1}^{n} \int_{\mathbb{R}^n} p(x) \left(q(x) + x_i \frac{\partial q}{\partial x_i}(x) - q(x) x_i^2 \right) g_n(x) \, dx \quad \left(\frac{\partial g_n}{\partial x_i} = -x_i g_n \right) \\
= & \sum_{i=1}^{n} \mathbb{E}\left(FG \Phi(h_i)^2 \right) - n \mathbb{E}(FG) - \mathbb{E}(F \nabla_\Phi G) \\
= & \mathbb{E}\left(FG \left(\sum\nolimits_{i=1}^{n} \Phi(h_i)^2 - n \right) \right) - \mathbb{E}(F \nabla_\Phi G),
\end{aligned}$$

which is the desired conclusion in (3.37). □

Remark 3.27. Note T_k^* is *not* a derivation. In fact, we have

$$T_k^*(FG) = T_k^*(F) G + F T_k^*(G) - \Phi(k) FG,$$

for all $F, G \in \mathscr{D}$, and all $k \in \mathscr{L}$.

However, the divergence operator ∇_Φ does satisfy the Leibniz rule, i.e.,

$$\nabla_\Phi(FG) = (\nabla_\Phi F) G + F(\nabla_\Phi G), \quad \forall F, G \in \mathscr{D}.$$

3.3. **Realization of the operators.**

Theorem 3.28. *Let ω_{Fock} be the Fock state on $CCR(\mathscr{L})$, see (2.10)-(2.11), and let π_F denote the corresponding (Fock space) representation, acting on $\Gamma_{sym}(\mathscr{L})$, see Lemma 2.14. Let $W : \Gamma_{sym}(\mathscr{L}) \longrightarrow L^2(\Omega, \mathbb{P})$ be the isomorphism given by*

$$W\left(e^k\right) := e^{\Phi(k) - \frac{1}{2}\|k\|^2_{\mathscr{L}}}, \quad k \in \mathscr{L}. \tag{3.38}$$

Here $L^2(\Omega, \mathbb{P})$ denotes the Gaussian Hilbert space corresponding to $\mathscr{L}$; see Definition 2.9. For vectors $k \in \mathscr{L}$, let T_k denote the Malliavin derivative in the direction k; see Definition 3.2.

We then have the following realizations:

$$T_k = W\pi_F(a(k))W^*, \text{ and} \tag{3.39}$$

$$M_{\Phi(k)} - T_k = W\pi_F(a^*(k))W^*; \tag{3.40}$$

valid for all $k \in \mathscr{L}$, where the two identities (3.39)-(3.40) hold on the dense domain $\mathscr{D}$ from Lemma 2.14.

Remark 3.29. The two formulas (3.39)-(3.40) take the following form, see Figs 3.1-3.2.

In the proof of the theorem, we make use of the following:

Lemma 3.30. *Let $\mathscr{L}$, $CCR(\mathscr{L})$, and ω_F (= the Fock vacuum state) be as above. Then, for all $n, m \in \mathbb{N}$, and all $h_1, \cdots, h_n$, $k_1, \cdots, k_m \in \mathscr{L}$, we have the following identity:*

$$\begin{aligned} & \omega_F(a(h_1)\cdots, a(h_n)\, a^*(k_m)\cdots a(k_1)) \\ = \; & \delta_{n,m} \sum_{s \in S_n} \left\langle h_1, k_{s(1)}\right\rangle_{\mathscr{L}} \left\langle h_2, k_{s(2)}\right\rangle_{\mathscr{L}} \cdots \left\langle h_n, k_{s(n)}\right\rangle_{\mathscr{L}} \end{aligned} \tag{3.41}$$

where the summation on the RHS in (3.41) is over the symmetric group S_n of all permutations of $\{1, 2, \cdots, n\}$. (In the case of the CARs, the analogous expression on the RHS will instead be a determinant.)

Proof. We leave the proof of the lemma to the reader; it is also contained in [BR81]. □

Remark 3.31. In physics-lingo, we say that the vacuum-state ω_F is determined by its *two-point functions*

$$\omega_F(a(h)\, a^*(k)) = \langle h, k\rangle_{\mathscr{L}}, \text{ and}$$
$$\omega_F(a^*(k)\, a(h)) = 0, \quad \forall h, k \in \mathscr{L}.$$

Proof of Theorem 3.28. We shall only give the details for formula (3.39). The modifications needed for (3.40) will be left to the reader.

Since W in (3.38) is an isomorphic isomorphism, i.e., a unitary operator from $\Gamma_{sym}(\mathscr{L})$ onto $L^2(\Omega, \mathbb{P})$, we may show instead that

$$T_k W = W\pi_F(a(k)) \tag{3.42}$$

holds on the dense subspace of all finite symmetric tensor polynomials in $\Gamma_{sym}(\mathscr{L})$; or equivalently on the dense subspace in $\Gamma_{sym}(\mathscr{L})$ spanned by

$$\Gamma(l) := e^l := \sum_{n=0}^{\infty} \frac{l^{\otimes n}}{\sqrt{n!}} \in \Gamma_{sym}(\mathscr{L}), \; l \in \mathscr{L}; \tag{3.43}$$

see also Lemma 2.14. We now compute (3.42) on the vectors e^l in (3.43):

$$
\begin{aligned}
T_k W\left(e^l\right) &= T_k\left(e^{\Phi(k)-\frac{1}{2}\|k\|_{\mathscr{L}}^2}\right) && \text{(by Lemma 2.14)} \\
&= e^{-\frac{1}{2}\|k\|_{\mathscr{L}}^2} T_k\left(e^{\Phi(k)}\right) \\
&= e^{-\frac{1}{2}\|k\|_{\mathscr{L}}^2} \langle k, l\rangle_{\mathscr{L}} e^{\Phi(l)} && \text{(by Remark 3.3)} \\
&= W \pi_F\left(a\left(k\right)\right)\left(e^l\right),
\end{aligned}
$$

valid for all $k, l \in \mathscr{L}$. □

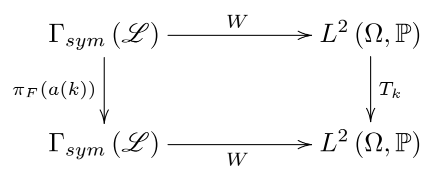


FIGURE 3.1. The first operator.

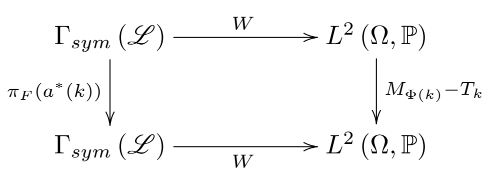


FIGURE 3.2. The second operator.

3.4. **The unitary group.** For a given Gaussian field $(\mathscr{L}, \Omega, \mathcal{F}, \mathbb{P}, \Phi)$, we studied the $\mathrm{CCR}(\mathscr{L})$-algebra, and the operators associated with its Fock-vacuum representation.

From the determination of Φ by

$$\mathbb{E}\left(e^{i\Phi(k)}\right) = e^{-\frac{1}{2}\|k\|_{\mathscr{L}}^2}, \; k \in \mathscr{L}; \tag{3.44}$$

we deduce that $(\Omega, \mathcal{F}, \mathbb{P}, \Phi)$ satisfies the following covariance with respect to the group $\mathrm{Uni}(\mathscr{L}) := G(\mathscr{L})$ of all unitary operators $U : \mathscr{L} \longrightarrow \mathscr{L}$.

We shall need the following:

Definition 3.32. We say that $\alpha \in Aut(\Omega, \mathcal{F}, \mathbb{P})$ iff the following three conditions hold:

(1) $\alpha : \Omega \longrightarrow \Omega$ is defined $\mathbb{P}$ a.e. on Ω, and $\mathbb{P}(\alpha(\Omega)) = 1$.
(2) $\mathcal{F} = \alpha(\mathcal{F})$; more precisely, $\mathcal{F} = \left\{\alpha^{-1}(B) \mid B \in \mathcal{F}\right\}$ where

$$\alpha^{-1}(B) = \{\omega \in \Omega \mid \alpha(\omega) \in B\}. \tag{3.45}$$

(3) $\mathbb{P} = \mathbb{P} \circ \alpha^{-1}$, i.e., α is a measure preserving automorphism.

Note that when (1)-(3) hold for α, then we have the unitary operators U_α in $L^2(\Omega, \mathcal{F}, \mathbb{P})$,

$$U_\alpha F = F \circ \alpha, \tag{3.46}$$

or more precisely,

$$(U_\alpha F)(\omega) = F(\alpha(\omega)), \text{ a.e. } \omega \in \Omega,$$

valid for all $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 3.33.

(1) For every $U \in G(\mathscr{L})$ (= the unitary group of $\mathscr{L}$), there is a unique $\alpha \in Aut(\Omega, \mathcal{F}, \mathbb{P})$ s.t.

$$\Phi(Uk) = \Phi(k) \circ \alpha, \tag{3.47}$$

or equivalently (see (3.46))

$$\Phi(Uk) = U_\alpha(\Phi(k)), \ \forall k \in \mathscr{L}. \tag{3.48}$$

(2) If $T : L^2(\Omega, \mathbb{P}) \longrightarrow L^2(\Omega, \mathbb{P}) \otimes \mathscr{L}$ is the Malliavin derivative from Definition 3.2, then we have:

$$TU_\alpha = (U_\alpha \otimes U) T. \tag{3.49}$$

Proof. The first conclusion in the theorem is immediate from the above discussion, and we now turn to the covariance formula (3.49).

Note that (3.49) involves unbounded operators, and it holds on the dense subspace $\mathscr{D}$ in $L^2(\Omega, \mathbb{P})$ from Lemma 2.14. Hence it is enough to verify (3.49) on vectors in $L^2(\Omega, \mathbb{P})$ of the form $e^{\Phi(k) - \frac{1}{2}\|k\|^2_{\mathscr{L}}}$, $k \in \mathscr{L}$. Using Lemma 2.14, we then get:

$$\begin{aligned}
\text{LHS}_{(3.49)}\left(e^{\Phi(k)-\frac{1}{2}\|k\|^2_{\mathscr{L}}}\right) &= e^{-\frac{1}{2}\|k\|^2_{\mathscr{L}}} T\left(e^{\Phi(Uk)}\right) && \text{(by (3.47))} \\
&= e^{-\frac{1}{2}\|Uk\|^2_{\mathscr{L}}} e^{\Phi(Uk)} \otimes (Uk) && \text{(by Remark 3.3)} \\
&= (U_\alpha \otimes U)\left(e^{\Phi(k)-\frac{1}{2}\|k\|^2_{\mathscr{L}}}\right) \\
&= \text{RHS}_{(3.49)}
\end{aligned}$$

□

4. The Fock-state, and representation of CCR, realized as Malliavin calculus

We now resume our analysis of the representation of the canonical commutation relations (CCR)-algebra induced by the canonical Fock state (see (2.9)). In our analysis below, we shall make use of the following details: Brownian motion, Itō-integrals, and the Malliavin derivative.

The general setting. Let $\mathscr{L}$ be a fixed Hilbert space, and let $\text{CCR}(\mathscr{L})$ be the $*$-algebra on the generators $a(k)$, $a^*(l)$, $k, l \in \mathscr{L}$, and subject to the relations for the CCR-algebra, see Section 2.2:

$$[a(k), a(l)] = 0, \quad \text{and} \tag{4.1}$$

$$[a(k), a^*(l)] = \langle k, l \rangle_{\mathscr{L}} \mathbb{1} \tag{4.2}$$

where $[\cdot, \cdot]$ is the commutator bracket.

A representation π of $\mathrm{CCR}(\mathscr{L})$ consists of a fixed Hilbert space $\mathscr{H} = \mathscr{H}_\pi$ (the representation space), a dense subspace $\mathscr{D}_\pi \subset \mathscr{H}_\pi$, and a $*$-homomorphism $\pi : \mathrm{CCR}(\mathscr{L}) \longrightarrow End(\mathscr{D}_\pi)$ such that

$$\mathscr{D}_\pi \subset dom(\pi(A)), \quad \forall A \in \mathrm{CCR}. \tag{4.3}$$

The representation axiom entails the commutator properties resulting from (4.1)-(4.2); in particular π satisfies

$$[\pi(a(k)), \pi(a(l))] F = 0, \quad \text{and} \tag{4.4}$$

$$\left[\pi(a(k)), \pi(a(l))^*\right] F = \langle k, l\rangle_{\mathscr{L}} F, \tag{4.5}$$

$\forall k, l \in \mathscr{L}$, $\forall F \in \mathscr{D}_\pi$; where $\pi(a^*(l)) = \pi(a(l))^*$.

In the application below, we take $\mathscr{L} = L^2(0,\infty)$, and $\mathscr{H}_\pi = L^2(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ where $(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ is the standard Wiener probability space, and

$$\Phi_t(\omega) = \omega(t), \quad \forall \omega \in \Omega,\ t \in [0,\infty). \tag{4.6}$$

For $k \in \mathscr{L}$, we set

$$\Phi(k) = \int_0^\infty k(t)\, d\Phi_t \text{ (=the Itō-integral.)}$$

The dense subspace $\mathscr{D}_\pi \subset \mathscr{H}_\pi$ is generated by the polynomial fields:

For $n \in \mathbb{N}$, $h_1, \cdots, h_n \in \mathscr{L} = L^2_{\mathbb{R}}(0,\infty)$, $p \in \mathbb{R}^n \longrightarrow \mathbb{R}$ a polynomial in n real variables, set

$$F = p(\Phi(h_1), \cdots, \Phi(h_n)), \quad \text{and} \tag{4.7}$$

$$\pi(a(k)) F = \sum_{j=1}^n \left(\frac{\partial}{\partial x_j} p\right)(\Phi(h_1), \cdots, \Phi(h_n)) \langle h_j, k\rangle. \tag{4.8}$$

It follows from Lemma 3.12 that $\mathscr{D}_\pi$ is an algebra under pointwise product and that

$$\pi(a(k))(FG) = (\pi(a(k))F)G + F(\pi(a(k))G), \tag{4.9}$$

$\forall k \in \mathscr{L}$, $\forall F, G \in \mathscr{D}_\pi$. Equivalently, $T_k := \pi(a(k))$ is a derivation in the algebra $\mathscr{D}_\pi$ (relative to pointwise product.)

Theorem 4.1. *With the operators $\pi(a(k))$, $k \in \mathscr{L}$, we get a $*$-representation $\pi : CCR(\mathscr{L}) \longrightarrow End(\mathscr{D}_\pi)$, i.e., $\pi(a(k))$ = the Malliavin derivative in the direction k,*

$$\pi(a(k)) F = \langle T(F), k\rangle_{\mathscr{L}}, \quad \forall F \in \mathscr{D}_\pi,\ \forall k \in \mathscr{L}. \tag{4.10}$$

Proof. We begin with the following □

Lemma 4.2. *Let π, $CCR(\mathscr{L})$, and $\mathscr{H}_\pi = L^2(\Omega, \mathcal{F}_\Omega, \mathbb{P})$ be as above. For $k \in \mathscr{L}$, we shall identify $\Phi(k)$ with the unbounded multiplication operator in $\mathscr{H}_\pi$:*

$$\mathscr{D}_\pi \ni F \longmapsto \Phi(k) F \in \mathscr{H}_\pi. \tag{4.11}$$

For $F \in \mathscr{D}_\pi$, we have $\pi(a(k))^* F = -\pi(a(k)) F + \Phi(k) F$; or in abbreviated form:

$$\pi(a(k))^* = -\pi(a(k)) + \Phi(k) \tag{4.12}$$

valid on the dense domain $\mathscr{D}_\pi \subset \mathscr{H}_\pi$.

Proof. This follows from the following computation for $F, G \in \mathscr{D}_\pi$, $k \in \mathscr{L}$.

Setting $T_k := \pi(a(k))$, we have

$$\mathbb{E}(T_k(F)G) + \mathbb{E}(F\,T_k(G)) = \mathbb{E}(T_k(FG)) = \mathbb{E}(\Phi(k)\,FG).$$

Hence $\mathscr{D}_\pi \subset dom(T_k^*)$, and $T_k^*(F) = -T_k(F) + \Phi(k)\,F$, which is the desired conclusion (4.12). ☐

Proof of Theorem 4.1 continued. It is clear that the operators $T_k = \pi(a(k))$ form a commuting family. Hence on $\mathscr{D}_\pi$, we have for $k, l \in \mathscr{L}$, $F \in \mathscr{D}_\pi$:

$$\begin{aligned}
[T_k, T_l^*](F) &= [T_k, \Phi(l)](F) && \text{by } (4.12)\\
&= T_k(\Phi(l)\,F) - \Phi(l)(T_k(F)) && \\
&= T_k(\Phi(l))\,F && \text{by } (4.9)\\
&= \langle k, l\rangle_{\mathscr{L}}\,F && \text{by } (4.8)
\end{aligned}$$

which is the desired commutation relation (4.2).

The remaining check on the statements in the theorem are now immediate. ☐

Corollary 4.3. *The state on $CCR(\mathscr{L})$ which is induced by π and the constant function $\mathbb{1}$ in $L^2(\Omega, \mathbb{P})$ is the Fock-vacuum-state, ω_{Fock}.*

Proof. The assertion will follow once we verify the following two conditions:

$$\int_\Omega T_k^* T_k(\mathbb{1})\,d\mathbb{P} = 0 \tag{4.13}$$

and

$$\int_\Omega T_k T_l^*(\mathbb{1})\,d\mathbb{P} = \langle k, l\rangle_{\mathscr{L}} \tag{4.14}$$

for all $k, l \in \mathscr{L}$.

This in turn is a consequence of our discussion of eqs (2.10)-(2.11) above: The Fock state ω_{Fock} is determined by these two conditions. The assertions (4.13)-(4.14) follow from $T_k(\mathbb{1}) = 0$, and $(T_k T_l^*)(\mathbb{1}) = \langle k, l\rangle_{\mathscr{L}}\,\mathbb{1}$. See (3.13). ☐

Corollary 4.4. *For $k \in L^2_{\mathbb{R}}(0, \infty)$ we get a family of selfadjoint multiplication operators $T_k + T_k^* = M_{\Phi(k)}$ on $\mathscr{D}_\pi$ where $T_k = \pi(a(k))$. Moreover, the von Neumann algebra generated by these operators is $L^\infty(\Omega, \mathbb{P})$, i.e., the maximal abelian L^∞-algebra of all multiplication operators in $\mathscr{H}_\pi = L^2(\Omega, \mathbb{P})$.*

Remark 4.5. In our considerations of representations π of $\mathrm{CCR}(\mathscr{L})$ in a Hilbert space $\mathscr{H}_\pi$, we require the following five axioms satisfied:

(1) a dense subspace $\mathscr{D}_\pi \subset \mathscr{H}_\pi$;
(2) $\pi : \mathrm{CCR}(\mathscr{L}) \longrightarrow End(\mathscr{D}_\pi)$, i.e., $\mathscr{D}_\pi \subset \cap_{A \in \mathrm{CCR}(\mathscr{L})} dom(\pi(A))$;
(3) $[\pi(a(k)), \pi(a(l))] = 0$, $\forall k, l \in \mathscr{L}$;
(4) $\left[\pi(a(k)), \pi(a(l))^*\right] = \langle k, l\rangle_{\mathscr{L}}\, I_{\mathscr{H}_\pi}$, $\forall k, l \in \mathscr{L}$; and
(5) $\pi(a^*(k)) \subset \pi(a(k))^*$, $\forall k \in \mathscr{L}$.

Note that in our assignment for the operators $\pi(a(k))$, and $\pi(a^*(k))$ in Lemma 4.2, we have all the conditions (1)-(5) satisfied. We say that π is a *selfadjoint representation.*

If alternatively, we define

$$\rho : \mathrm{CCR}(\mathscr{L}) \longrightarrow End(\mathscr{D}_\pi) \tag{4.15}$$

with the following modification:

$$\begin{cases} \rho\left(a\left(k\right)\right) = T_k, \ k \in \mathscr{L}, \text{ and} \\ \rho\left(a^*\left(k\right)\right) = \Phi\left(k\right) \end{cases} \tag{4.16}$$

then this ρ will satisfy (1)-(3), and

$$\left[\rho\left(a\left(k\right)\right), \rho\left(a^*\left(l\right)\right)\right] = \langle k, l\rangle_{\mathscr{L}} I_{\mathscr{H}_\pi};$$

but then $\rho\left(a\left(k\right)\right) \subsetneqq \rho\left(a\left(k\right)\right)^*$; i.e., non-containment of the respective graphs.

One generally says that the representation π is (formally) selfadjoint, while the second representation ρ is *not.*

5. Conclusions: the general case

Definition 5.1. A representation π of $\mathrm{CCR}\left(\mathscr{L}\right)$ is said to be *admissible* iff (Def.) $\exists\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ as above such that $\mathscr{H}_\pi = L^2\left(\Omega, \mathcal{F}, \mathbb{P}\right)$, and there exists a linear mapping $\Phi : \mathscr{L} \longrightarrow L^2\left(\Omega, \mathcal{F}, \mathbb{P}\right)$ subject to the condition:

For every $n \in \mathbb{N}$, and every $k, h_1, \cdots, h_n \in \mathscr{L}$, the following holds on its natural dense domain in $\mathscr{H}_\pi$: For every $p \in \mathbb{R}\left[x_1, \cdots, x_n\right]$, we have

$$\pi\left(\left[a\left(k\right), p\left(a^*\left(h_1\right), \cdots, a^*\left(h_n\right)\right)\right]\right) = \sum_{i=1}^{n} \langle k, h_i\rangle_{\mathscr{L}} M_{\frac{\partial p}{\partial x_i}\left(\Phi(h_1), \cdots, \Phi(h_n)\right)}, \tag{5.1}$$

with the M on the RHS denoting "multiplication."

Corollary 5.2.

1. *Every admissible representation π of CCR$\left(\mathscr{L}\right)$ yields an associated Malliavin derivative as in (5.1).*
2. *The Fock-vacuum representation π_F is admissible.*

Proof. (1) follows from the definition combined with Corollary 2.8. (2) is a direct consequence of Lemma 3.7 and Theorem 3.8; see also Corollary 4.3. □

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