A NOTE ON INDUCED RAMSEY NUMBERS

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ABSTRACT. The induced Ramsey number $r_{\mathrm{ind}}(F)$ of a k-uniform hypergraph F is the smallest natural number n for which there exists a k-uniform hypergraph G on n vertices such that every two-coloring of the edges of G contains an induced monochromatic copy of F. We study this function, showing that $r_{\mathrm{ind}}(F)$ is bounded above by a reasonable power of $r(F)$. In particular, our result implies that $r_{\mathrm{ind}}(F) \leqslant 2^{2^{ct}}$ for any 3-uniform hypergraph F with t vertices, mirroring the best known bound for the usual Ramsey number. The proof relies on an application of the hypergraph container method.

1. INTRODUCTION

The *Ramsey number* $r(F; q)$ of a k-uniform hypergraph F is the smallest natural number n such that every q-coloring of the edges of $K_n^{(k)}$, the complete k-uniform hypergraph on n vertices, contains a monochromatic copy of F. In the particular case when $q = 2$, we simply write $r(F)$. The existence of $r(F; q)$ was established by Ramsey in his foundational paper [16] and there is now a large body of work studying the Ramsey numbers of graphs and hypergraphs. For a recent survey, we refer the interested reader to [4].

In this paper, we will be concerned with a well-known refinement of Ramsey's theorem, the induced Ramsey theorem. We say that a k-uniform hypergraph F is an *induced subgraph* of another k-uniform hypergraph G if $V(F) \subset V(G)$ and any k vertices in F form an edge if and only if they also form an edge in G. The *induced Ramsey number* $r_{\mathrm{ind}}(F; q)$ of a k-uniform hypergraph F is then the smallest natural number n for which there exists a k-uniform hypergraph G on n vertices such that that every q-coloring of the edges of G contains an induced monochromatic copy of F. Again, in the particular case when $q = 2$, we simply write $r_{\mathrm{ind}}(F)$.

For graphs, the existence of induced Ramsey numbers was established independently by Deuber [5], Erdős, Hajnal, and Pósa [7], and Rödl [17], while for k-uniform hypergraphs with $k \geqslant 3$ their existence was shown independently by Nešetřil and Rödl [15] and Abramson and Harrington [1]. The bounds that these original proofs gave on $r_{\mathrm{ind}}(F; q)$ were enormous. However, at that time it was noted by Rödl (unpublished) that for bipartite graphs F the induced Ramsey numbers are exponential. Moreover, it was conjectured by Erdős [6] that there exists a constant c such that every graph F with t vertices satisfies $r_{\mathrm{ind}}(F) \leqslant 2^{ct}$. If true, the complete graph would show that this is best possible up to the constant c. A result of Conlon, Fox, and Sudakov [3], building on earlier work by Kohayakawa, Prömel, and Rödl [12], comes close to establishing this conjecture, showing that

$$r_{\mathrm{ind}}(F) \leqslant 2^{ct \log t}.$$

However, the method used to prove this estimate only works in the 2-color case. For $q \geqslant 3$, the best known bound, due to Fox and Sudakov [10], is $r_{\mathrm{ind}}(F; q) \leqslant 2^{ct^3}$, where c depends only on q.

In this note, we study the analogous question for hypergraphs, showing that the induced Ramsey number is never significantly larger than the usual Ramsey number. Our main result is the following.

Theorem 1. *Let F be a k-uniform hypergraph with t vertices and ℓ edges. Then there are positive constants c_1, c_2, and c_3 such that*

$$r_{\mathrm{ind}}(F; q) \leqslant 2^{c_1 k \ell^3 \log(qt\ell)} R^{c_2 k \ell^2 + c_3 t\ell},$$

where $R = r(F; q)$ is the classical q-color Ramsey number of F.

Define the tower function $t_k(x)$ by $t_1(x) = x$ and, for $i \geqslant 1$, $t_{i+1}(x) = 2^{t_i(x)}$. A seminal result of Erdős and Rado [8] says that

$$r(K_t^{(k)}; q) \leqslant t_k(ct),$$

where c depends only on k and q. This yields the following immediate corollary of Theorem 1.

Corollary 1. *For any natural numbers $k \geqslant 3$ and $q \geqslant 2$, there exists a constant c such that if F is a k-uniform hypergraph with t vertices, then*

$$r_{\mathrm{ind}}(F; q) \leqslant t_k(ct).$$

A result of Erdős and Hajnal (see, for example, Chapter 4.7 in [11]) says that

$$r(K_t^{(k)}; 4) \geqslant t_k(c't),$$

where c' depends only on k. Therefore, the Erdős–Rado bound is sharp up to the constant c for $q \geqslant 4$. By taking $F = K_t^{(k)}$, this also implies that Corollary 1 is tight up to the constant c for $q \geqslant 4$. Whether it is also sharp for $q = 2$ and 3 depends on whether $r(K_t^{(k)}) \geqslant t_k(c't)$, though determining if this is the case is a famous, and seemingly difficult, open problem.

The proof of Theorem 1 relies on an application of the hypergraph containers method of Saxton and Thomason [19] and Balogh, Morris, and Samotij [2]. In Ramsey theory, the use of this method was pioneered by Nenadov and Steger [13] and developed further by Rödl, Ruciński, and Schacht [18] in order to give an exponential-type upper bound for Folkman numbers. Our modest results are simply another manifestation of the power of this beautiful method.

2. Proof of Theorem 1

In order to state the result we first need some definitions. Recall that the degree $d(\sigma)$ of a set of vertices σ in a hypergraph H is the number of edges of H containing σ, while the average degree is the average of $d(v) := d(\{v\})$ over all vertices v.

Definition 2. Let H be an ℓ-uniform hypergraph of order N with average degree d. Let $\tau > 0$. Given $v \in V(H)$ and $2 \leqslant j \leqslant \ell$, let

$$d^{(j)}(v) = \max\{d(\sigma) \,:\, v \in \sigma \subset V(H), |\sigma| = j\}. \tag{1}$$

If $d > 0$, define δ_j by the equation

$$\delta_j \tau^{j-1} N d = \sum_v d^{(j)}(v). \tag{2}$$

The *codegree function* $\delta(H, \tau)$ is then defined by

$$\delta(H, \tau) = 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \delta_j. \tag{3}$$

If $d = 0$, define $\delta(H, \tau) = 0$.

The precise lemma we will need is a slight variant of Corollary 3.6 from Saxton and Thomason's paper [19]. A similar version was already used in the work of Rödl, Ruciński, and Schacht [18] and we refer the interested reader to that paper for a thorough discussion.

Lemma 3. *Let H be an ℓ-uniform hypergraph on N vertices with average degree d. Let $0 < \varepsilon < 1/2$. Suppose that τ satisfies $\delta(H, \tau) \leqslant \varepsilon/12\ell!$ and $\tau \leqslant 1/144\ell!^2\ell$. Then there exists a collection $\mathcal{C}$ of subsets of $V(H)$ such that*

1. *for every set $I \subset V(H)$ such that $e(H[I]) \leqslant \varepsilon\tau^\ell e(H)$, there is $C \in \mathcal{C}$ with $I \subset C$,*
2. *$e(H[C]) \leqslant \varepsilon e(H)$ for all $C \in \mathcal{C}$,*
3. *$\log |\mathcal{C}| \leqslant 1000\ell!^3\ell \log(1/\varepsilon)N\tau\log(1/\tau)$.*

Before we give the proof of Theorem 1, we first describe the ℓ-uniform hypergraph H to which we will apply Lemma 3.

Construction 4. Given a k-uniform hypergraph F with ℓ edges, we construct an auxiliary hypergraph H by taking

- $V(H) = \binom{[n]}{k}$
- $E(H) = \left\{ E \in \binom{V(H)}{\ell} : E \cong F \right\}$

In other words, the vertices of H are the k-tuples of $[n]$ and the edges of H are copies of F in $\binom{[n]}{k}$.

Proof of Theorem 1. Recall that $R = r(F; q)$, the q-color Ramsey number of F and suppose that F has t vertices and ℓ edges. Let us fix the following numbers:

$$\tau = n^{-\frac{1}{2\ell}}, \qquad p = 1000R^k q\alpha, \qquad \alpha = n^{-\frac{1}{2\ell} + \frac{1}{4\ell(\ell+1)}}$$

$$\varepsilon = 1/(2qR^t), \qquad n = \ell^{40\ell^2(\ell+1)}(1000q)^{8\ell(\ell+1)}R^{4k\ell(\ell+1)+4t\ell}\binom{t}{k}^{4\ell}. \tag{4}$$

Remark 5. Note that n is bounded above by an expression of the form

$$2^{c_1 k\ell^3 \log(qt\ell)}R^{c_2 k\ell^2 + c_3 t\ell},$$

as required. Obviously, $R \geqslant t$ and one can check that p and n satisfy the following conditions, which we will make use of during the course of the proof:

$$p \leqslant 1 \tag{5}$$

$$n \geqslant (24 \cdot 2^{\binom{\ell}{2}}t^t q\ell!R^t)^2 \tag{6}$$

$$n > (144\ell!^2\ell)^{2\ell} \tag{7}$$

$$n > \ell^{40\ell^2(\ell+1)} \tag{8}$$

$$n > (1000q)^{8\ell(\ell+1)}R^{4k\ell(\ell+1)+4t\ell}\binom{t}{k}^{4\ell} \tag{9}$$

We will show that, with positive probability, a random hypergraph $G \in \mathbb{G}^{(k)}(n, p)$ has the property that every q-coloring of its edges contains an induced monochromatic copy of F. The proof proceeds in two stages. First, we use Lemma 3 to show that, with probability $1 - o(1)$, G has the property that any q-coloring of its edges yields many monochromatic copies of F. Then we show that some of these monochromatic copies must be induced.

More formally, let X be the event that there is a q-coloring of the edges of G which contains at most

$$M := \varepsilon\tau^\ell(n)_t/\operatorname{Aut}(F)$$

monochromatic copies of F in each color, and let Y be the event that G contains at least M noninduced copies of F. Note that if $\overline{X} \cap \overline{Y}$ happens, then, in any q-coloring, there are more monochromatic copies of F in one of the q colors than there are noninduced copies of F in G. Hence, that color class must contain an induced copy of F.

We now proceed to show that the probability $\mathbf{P}(X)$ tends to zero as n tends to infinity. In order to apply Lemma 3, we need to check that τ and ε satisfy the requisite assumptions with respect to the ℓ-uniform hypergraph H defined in Construction 4. Let $\sigma \subset V(H)$ be arbitrary and define

$$V_\sigma = \bigcup_{v \in \sigma} v \subset [n].$$

For a set $W \subset [n] \backslash V_\sigma$ with $|W| = t - |V_\sigma|$, let $\mathrm{EMB}_F(\sigma)$ denote the set of all copies $\widetilde{F}$ of F with $V(\widetilde{F}) = W \cup V_\sigma$ and $\sigma \subset E(\widetilde{F})$. Since there are clearly $\binom{n-|V_\sigma|}{t-|V_\sigma|}$ choices for the set W, we arrive at the following claim.

Claim 1. *For any $\sigma \subset V(H)$,*

$$d(\sigma) = \binom{n - |V_\sigma|}{t - |V_\sigma|} |\mathrm{EMB}_F(\sigma)|. \qquad \square$$

Let us denote by t_j the minimum number of vertices of F which span j edges. From Claim 1, it follows that for any $\sigma \subset V(H)$ with $|\sigma| = j$, we have

$$d(\sigma) = \binom{n - |V_\sigma|}{t - |V_\sigma|} |\mathrm{EMB}_F(\sigma)| \leqslant \binom{n - t_j}{t - t_j} |\mathrm{EMB}_F(\sigma)|.$$

On the other hand, for a singleton $\sigma_1 \subset V(H)$, we have $|V_{\sigma_1}| = k$ and therefore $d = d(\sigma_1)$ is such that

$$\frac{d(\sigma)}{d} \leqslant \frac{\binom{n-t_j}{t-t_j}}{\binom{n-k}{t-k}} \frac{|\mathrm{EMB}_F(\sigma)|}{|\mathrm{EMB}_F(\sigma_1)|} \leqslant \frac{\binom{n-t_j}{t-t_j}}{\binom{n-k}{t-k}} < \left(\frac{n}{t}\right)^{k-t_j}.$$

It then follows from Definition 2 and (4) that

$$\delta_j < \frac{(n/t)^{k-t_j}}{\tau^{j-1}} < t^t n^{k-t_j+(j-1)/(2\ell)}. \tag{10}$$

Since t_j is increasing with respect to j, $t_2 \geqslant k+1$, and $j \leqslant \ell$, it follows that $k - t_j + \frac{j-1}{2\ell} \leqslant -1/2$. Thus, in view of (10), we have

$$\delta_j < t^t n^{k-t_j+(j-1)/(2\ell)} \leqslant t^t n^{-1/2} \tag{11}$$

for all $2 \leqslant j \leqslant \ell$.

Using Definition 2 and inequality (11), we can now bound the codegree function $\delta(H, \tau)$ by

$$\delta(H, \tau) = 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \delta_j \leqslant 2^{\binom{\ell}{2}-1} t^t n^{-1/2} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \leqslant 2^{\binom{\ell}{2}} t^t n^{-1/2}. \tag{12}$$

Since n satisfies (6), inequality (12) implies that

$$\delta(H, \tau) \leqslant 2^{\binom{\ell}{2}} t^t n^{-1/2} \leqslant \frac{\varepsilon}{12\ell!}.$$

That is, $\delta(H, \tau)$ satisfies the condition in Lemma 3.

Finally, (7) implies that τ satisfies the condition

$$\tau = n^{-1/(2\ell)} < \frac{1}{144\ell!^2\ell}.$$

Therefore, the assumptions of Lemma 3 are met and we can let $\mathcal{C}$ be the collection of subsets from $V(H)$ obtained from applying Lemma 3. Denote the elements of $\mathcal{C}$ by $C_1, C_2, \ldots, C_{|\mathcal{C}|}$.

For every choice of $1 \leqslant a_1, \ldots, a_q \leqslant |\mathcal{C}|$ (not necessarily distinct) let $E_{a_1,\ldots,a_q}$ be the event that $G \subseteq C_{a_1} \cup \cdots \cup C_{a_q}$. Next we will show the following claim.

Claim 2.

$$\mathbf{P}(X) \leqslant \mathbf{P}\left(\bigvee_{a_1,\ldots,a_q} E_{a_1,\ldots,a_q} \right) \leqslant \sum_{a_1,\ldots,a_q} \mathbf{P}(E_{a_1,\ldots,a_q}). \tag{13}$$

Proof. Suppose that $G \in X$. By definition, there exists a q-coloring of the edges of G, say with colors $1, 2, \ldots, q$, which contains at most M copies of F in each color. For any color class j, let I_j denote the set of vertices of H which correspond to edges of color j in G. Since each edge in $H[I_j]$ corresponds to a copy of F in color j, we have $e(H[I_j]) \leqslant M$. Note that

$$M = \varepsilon \tau^\ell e(H),$$

which means that each I_j satisfies the condition (1) of Lemma 3. Therefore, for each color class j, there must be a set $C_{a_j} \in \mathcal{C}$ such that $C_{a_j} \supset I_j$. Since $G = \bigcup_j I_j$, this implies that $G \in E_{a_1,\ldots,a_q}$. Since $G \in X$ was arbitrary, the bound (13) follows and the claim is proved. $\square$

Owing to Claim 2, we now bound $\mathbf{P}(E_{a_1,\ldots,a_q})$. Recalling the definition of the event $E_{a_1,\ldots,a_q}$ we note that

$$\mathbf{P}(E_{a_1,\ldots,a_q}) = (1-p)^{|V(H)\backslash(C_{a_1} \cup \cdots \cup C_{a_q})|}. \tag{14}$$

Hence, we shall estimate $|V(H)\backslash(C_{a_1} \cup \cdots \cup C_{a_q})|$ to derive a bound for $\mathbf{P}(X)$ by (13).

Claim 3. *For all choices $1 \leqslant a_1, \ldots, a_q \leqslant |\mathcal{C}|$ we have*

$$|V(H)\backslash(C_{a_1} \cup \cdots \cup C_{a_q})| \geqslant \frac{1}{2}\left(\frac{n}{R}\right)^k.$$

Proof. Let $a_1, \ldots, a_q$ be fixed. Recall that $R = r(F; q)$ and consider an R-element subset $A \subset [n]$. We denote by $\varphi(A)$ the set of copies of F in the complete k-uniform hypergraph with vertex set A that are contained in one of the classes $C_{a_1}, \ldots, C_{a_q}$, i.e.,

$$\varphi(A) := \binom{\binom{A}{k}}{\ell} \cap \left(E(H[C_{a_1}]) \cup \cdots \cup E(H[C_{a_q}]) \right).$$

If $\binom{A}{k} \subset C_{a_1} \cup \cdots \cup C_{a_q}$, then, by the definition of R, there is a copy of F in C_{a_j} for some j. Given the definition of H (see Construction 4), this means that $\varphi(A) \neq \varnothing$, i.e.,

$$\binom{A}{k} \subset C_{a_1} \cup \cdots \cup C_{a_q} \implies \varphi(A) \neq \varnothing. \tag{15}$$

Next we show that for most sets A we have $\varphi(A) = \varnothing$ and use this fact to bound $|C_{a_1} \cup \cdots \cup C_{a_q}|$.

Suppose, for the sake of contradiction, that at least $\frac{1}{2}\binom{n}{R}$ of the R-subsets A of $[n]$ are such that $\varphi(A) \neq \varnothing$. Since every edge $e \in E(H)$ corresponds to a copy of F (which has t vertices), the number of R-sets A for which $e \in \binom{\binom{A}{k}}{\ell}$ is at most $\binom{n-t}{R-t}$. In particular, our assumption implies that

$$e(H[C_{a_1}]) + \cdots + e(H[C_{a_q}]) \geqslant |E(H[C_{a_1}]) \cup \cdots \cup E(H[C_{a_q}])|$$

$$\geqslant \frac{1}{2}\frac{\binom{n}{R}}{\binom{n-t}{R-t}} = \frac{1}{2}\frac{(n)_t}{(R)_t} > \frac{(n)_t}{2R^t} \geqslant q\varepsilon e(H),$$

by the choice of ε. However, this contradicts condition (2) of Lemma 3.

Consequently, at least $\frac{1}{2}\binom{n}{R}$ of the R-subsets A of $[n]$ are such that $\varphi(A) = \varnothing$. It follows from (15) that $\binom{A}{k} \not\subset C_{a_1} \cup \cdots \cup C_{a_q}$. Hence, $V(H)\backslash(C_{a_1} \cup \cdots \cup C_{a_q})$ intersects $\binom{A}{k}$. Since an element of $V(H)$ can appear in at most $\binom{n-k}{R-k}$ sets A, it follows that there are at least

$$\frac{1}{2}\binom{n}{R} \bigg/ \binom{n-k}{R-k} \geqslant \frac{1}{2}\left(\frac{n}{R}\right)^k$$

elements in $V(H)\backslash(C_{a_1} \cup \cdots \cup C_{a_q})$, as required. $\square$

In view of Claim 3, our choice of $p = 1000R^k q\alpha$, where $\alpha = n^{-1/2\ell+1/4\ell(\ell+1)}$, and (14), we have, for any $C_{a_1}, \ldots, C_{a_q} \in \mathcal{C}$,

$$
\begin{aligned}
\mathbf{P}(E_{a_1,\ldots,a_q}) &\leqslant (1-p)^{(n/R)^k/2} \\
&\leqslant \exp\left(-pn^k/2R^k\right) \leqslant \exp\left(-(1000R^k q\alpha)n^k/2R^k\right) \\
&= e^{-500q\alpha n^k} \leqslant e^{-1000q\alpha N},
\end{aligned}
\tag{16}
$$

where, in the last step, we used $N = \binom{n}{k} \leqslant \frac{n^k}{2}$. Therefore, (13) and (16) together with the bound on $|\mathcal{C}|$ given by Lemma 3-3 imply that

$$
\begin{aligned}
\mathbf{P}(X) &\leqslant \sum_{C_{a_1},\ldots,C_{a_q}\in\mathcal{C}} \mathbf{P}(E_{a_1,\ldots,a_q}) \leqslant |\mathcal{C}|^q e^{-1000q\alpha N} \\
&\leqslant \exp\left(1000q\ell!^3\ell\log(1/\varepsilon)N\tau\log(1/\tau) - 1000q\alpha N\right) \\
&= \exp\left(1000qN\tau(\ell!^3\ell\log(1/\varepsilon)\log(1/\tau) - \alpha/\tau)\right) \\
&\leqslant \exp\left(1000qN\tau(\ell!^3\log^2 n - n^{1/(4\ell(\ell+1))})\right) \leqslant 1/4,
\end{aligned}
$$

where we used that n satisfies (8).

Now, by Markov's inequality, with probability at most $1/2$, the number of noninduced copies of F in G will be at most twice the expected number of copies, which is fewer than

$$
\begin{aligned}
2p^{\ell+1}\frac{(n)_t}{\operatorname{Aut} F}\binom{t}{k} &= 2(1000q)^{\ell+1}R^{k(\ell+1)}n^{-1/2-1/(4\ell)}\frac{(n)_t}{\operatorname{Aut} F}\binom{t}{k} \\
&< \frac{1}{2qR^t}(n^{-1/(2\ell)})^\ell\frac{(n)_t}{\operatorname{Aut} F} = \varepsilon\tau^\ell\frac{(n)_t}{\operatorname{Aut} F} = M,
\end{aligned}
$$

where the inequality above follows from (9). In other words, $\mathbf{P}(\overline{Y}) \geqslant 1/2$ and, therefore, $\mathbf{P}(\overline{X}\cap\overline{Y}) \geqslant 1/4$, so there exists a graph G such that $\overline{X} \cap \overline{Y}$ holds. By our earlier observations, this completes the proof.

3. Concluding remarks

Beginning with Fox and Sudakov [9], much of the recent work on induced Ramsey numbers for graphs has used pseudorandom rather than random graphs for the target graph G. The results of this paper rely very firmly on using random hypergraphs. It would be interesting to know whether comparable bounds could be proved using pseudorandom hypergraphs.

It would also be interesting to prove comparable bounds for the following variant of the induced Ramsey theorem, first proved by Nešetřil and Rödl [14]: for every graph F, there exists a graph G such that every q-coloring of the triangles of G contains an induced copy of F all of whose triangles receive the same color. By taking $F = K_t$ and $q = 4$, we see that $|G|$ may need to be double exponential in $|F|$. We believe that a matching double-exponential upper bound should also hold.

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