Approximation of Quadrilaterals by Rational Quadrilaterals in the Plane

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Abstract. Many questions about triangles and quadrilaterals with rational sides, diagonals and areas can be reduced to solving certain Diophantine equations. We look at a number of such questions including the question of approximating arbitrary triangles and quadrilaterals by those with rational sides, diagonals and areas. We transform these problems into questions on the existence of infinitely many rational solutions on a two parameter family of quartic curves. This is further transformed to a two parameter family of elliptic curves to deduce our main result concerning density of points on a line which are at a rational distance from three collinear points (Theorem 4). We deduce from this a new proof of density of rational quadrilaterals in the space of all quadrilaterals (Theorem 39). The other main result (Theorem 3) of this article is on the density of rational triangles which is related to analyzing rational points on the unit circle. Interestingly, this enables us to deduce that parallelograms with rational sides and area are dense in the class of all parallelograms.

We also give a criterion for density of certain sets in topological spaces using local product structure and prove the density Theorem 6 in the appendix section. An application of this proves the density of rational points as stated in Theorem 31.

1. Introduction

Throughout this article, we call a polygon *rational* if its sides, diagonals and area are all rational numbers. Interest in the theory of rational triangles goes back to the time of Leonhard Euler. Euler found formulae expressing proportions of the sides of a rational right-angled triangle and a general rational triangle. For the latter, he proved:

Theorem 1 (Euler [**10**]). *The sides of a general rational triangle ΔABC with sides $AB = c, BC = a, AC = b$ with rational area satisfy the proportion*

$$a : b : c = \frac{r^2 + s^2}{rs} : \frac{(ps \pm rq)(pr \mp qs)}{pqrs} : \frac{p^2 + q^2}{pq}$$

for some integers $p > q, r \geq s$.

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H.F. Blichfeldt [**5**] and D.N. Lehmer [**15**] have independently derived formulae for the sides of a rational triangle. Lehmer also characterized integral triangles i.e. triangles with integer sides and area in the plane in [**15**].

E.E. Kummer [**14**] obtained a characterization of rational quadrilaterals in the plane. He reduced the problem of finding rational quadrilaterals to the problem of finding rational solutions to the equation.

$$\frac{(\xi+c)^2-1}{2\xi}.\frac{(x-c)^2-1}{2x}=\frac{(\nu-c)^2-1}{2\nu}.\frac{(y+c)^2-1}{2y} \tag{2}$$

in rationals ξ,ν,x,y,c with $|c|<1$.

L.E. Dickson [**8**] also derived expressions for rational quadrilaterals similar to Kummer's. In the conclusion of his paper [**8**] he mentions that some questions about triangles and quadrilaterals reduce to deciding whether certain quartic functions can be written in terms of rational squares.

In what follows, we say that a polygon of n sides is rationally approximable if there are rational polygons of n sides whose vertices are arbitrarily close to the given one. I.J. Shoenberg posed the general question:

Is every $n-$ sided polygon rationally approximable?

A.S. Besicovitch [**4**] answered this question for the special cases of right-angled triangles and parallelograms.

D.E. Daykin [**6**] answered Shoenberg's question affirmatively in the class of quadrilaterals, parallelograms, and some classes of hexagons. We can deduce a new proof of rational approximability for quadrilaterals as a consequence of our Theorem 4.

Open questions regarding integral and rational distances and rational approximation have attracted many other mathematicians such as John Isbell, John Leech, Harborth, Kemnitz, Richard Guy, N.H. Anning, Paul Erdös, J.H.J Almering, T.K. Sheng and T.G. Berry. (See the list of references).

1.1. The Two Main Results. In this paper, we prove two results. To this end we need some definitions. Let us call a set $\{A,B,C\}$ of three points in the plane a rational 3-set if the lengths AB,AC,BC are rational. Call a set $\{A,B,C,D\}$ of four points on the plane a rational 4-set, if all the six distances are rational. If $K\subset\mathbb{R}^2$ is a compact/finite set and $L\subset\mathbb{R}^2$ is a closed set then we define the distance $d(K,L)=min\{d(x,y)\in\mathbb{R}\mid x\in K,y\in L\}$. We also define for any two finite $n-$sets $K_1=\{A_1,A_2,\ldots,A_n\},K_2=\{A'_1,A'_2,A'_3,\ldots,A'_n\}\subset\mathbb{R}^2$, equipped with a bijection between K_1,K_2, given by $A_i\mapsto A'_i$, the distance $D(K_1,K_2)=max\{d(A_i,A'_i)\in\mathbb{R}\mid 1\le i\le n\}$. As such this definition depends on the bijection $A_i\mapsto A'_i$. However see section 2 definition 8 as we require the distance $D(K_1,K_2)$ when $d(A_i,A'_i)$ is very small for $1\le i\le n$. Now we are ready to state our two results.

The first one is on the density of rational triangles in the space of triangles in the plane.

THEOREM 3. *Let $X=\{A,B,C\}$ represent three vertices of a triangle ΔABC in the Euclidean plane. Then given $\epsilon>0$ there exists a rational 3-set X_ϵ in the plane such that the points in X_ϵ form a rational triangle with rational area and $D(X,X_\epsilon)<\epsilon$.*

More precisely if BC is the largest side of the triangle ΔABC then we can choose the 3-set $X_\epsilon = \{A, B', C'\}$ to also contain the vertex A opposite to the side BC and make the side $BC \parallel B'C'$.

Using this main result on rational approximability of triangles we also deduce the analogous result for parallelograms.

The second main result addresses the density of points on a line which are at a rational distance from three collinear points.

THEOREM 4. *Let A, O, C be three distinct, collinear points in the plane with point O on the line segment AC such that the 3-set $\{A, O, C\}$ is a rational set. Let L be a line passing through O such that the sine and cosine of the angle between L and AC are rational. There exist finite sets $F_\measuredangle \subset [0, \pi]$ and $F_{ratio} \subset \mathbb{R}$ such that*

- *For a fixed angle between L and AC not in the finite set $F_\measuredangle$, the set of all points B on L such that the four set $\{A, B, C, O\}$ is a rational set is dense, except for a finitely many values of the ratio $\frac{AO}{OC}$.*
- *For a fixed ratio $\frac{AO}{OC}$ not in the finite set F_{ratio}, the set of all points B on L such that the four set $\{A, B, C, O\}$ is a rational set is dense, except for a finitely many choices of the angles between L and AC.*

For the proof we use quartic curves as well as a family of cubic curves. From this Theorem 4, we deduce rational approximability of general quadrilaterals.

In 1960 L.J. Mordell [**16**] proved that every quadrilateral in the plane is approximable by quadrilaterals with rational sides and diagonals (with no condition on the area). He used Nagell's theorem on integral points and torsion points on cubic curves.

1.2. The Density Result. We also prove the following useful topological density result and as a consequence we prove density of rational points as stated in Theorem 31.

For this purpose we introduce a definition

DEFINITION 5 (Local Product Structure). Let X, Y be topological spaces and $f : X \longrightarrow Y$ be a surjective continuous map. Let $x_0 \in X$ and $f(x_0) = y_0$. Let F_{x_0} be any topological space. Suppose there exists open sets $O \subset X, U \subset F_{x_0}, V \subset Y$ such that $x_0 \in O, y_0 \in V$ and $O \cong_\psi U \times V$ and such that the following diagram commutes.

$$\big(O \underset{f}{\longrightarrow} V\big) = \big(O \underset{\psi}{\longrightarrow} U \times V \underset{\pi_2}{\longrightarrow} V\big).$$

Then we say that X has the local product structure property at the point x_0 with respect to f, F_{x_0}.

Now we state the theorem.

THEOREM 6. *Let $X_0 \underset{f_1}{\longrightarrow} X_1 \underset{f_2}{\longrightarrow} X_2 \underset{f_3}{\longrightarrow} X_3 \underset{f_4}{\longrightarrow} \ldots \underset{f_n}{\longrightarrow} X_n$ be a sequence of surjective continuous maps of topological spaces such that the local product structure property is satisfied on a dense set Z_i of X_i with respect to the map f_{i+1} for $i = 0, \ldots, n-1$. Then if $B \subset X_n$ is dense then any fibre-wise dense set in the preimage of B in each X_i is dense in X_i for all $0 \leq i \leq n-1$.*

We end this introduction with another question which, similar to Shoenberg's question for $n \geq 5$, is still open [**3**].

- *Does there exist a point in the plane at a rational distance from each of the corners of a unit square?*

2. Definitions

In this article we use the following definitions.

DEFINITION 7. Let $X \subset \mathbb{R}^2$ be a subset of the Euclidean plane. The distance set is defined as $\Delta(X) = \{r \in \mathbb{R} \mid \text{ there exists } p, q \in X \text{ with } r = d(p,q)\}$. We say that the set X is rational if $\Delta(X) \subset \mathbb{Q}$.

DEFINITION 8.

- A finite $n-$subset $X = \{A_1, A_2, \ldots, A_n\} \subset \mathbb{R}^2$ is said to be rational approximable if given $\epsilon > 0$ there exists a rational $n-$subset $X_\epsilon = \{A'_1, A'_2, \ldots, A'_n\} \subset \mathbb{R}^2$ with a bijection $A_i \to A'_i$ such that $max\{d(A_i, A'_i) \mid i = 1, \ldots, n\} = D(X, X_\epsilon) < \epsilon$. Here in the case of rational approximability for a given finite set X and ϵ small enough depending on X, the bijection between X and X_ϵ is unique if such a set X_ϵ exists.
- A polygon of n sides is said to be rational, if its sides, diagonals and area are all rational.

3. Rational Approximability of Triangles and Parallelograms

In this section, we prove that triangles are rationally approximable. Towards that, we quote the following Lemmas 9, and 10.

LEMMA 9.
Let C be a circle centered at the origin in $\mathbb{R}^2$ whose radius is rational. Then the set $\mathbb{P} = \{(x,y) \in C \mid x, y \in \mathbb{Q}\}$ of points with rational coordinates on the circle is dense in C.

The proof is straightforward. See the proof given by Paul D. Humke and Lawrence L. Krajewski [**12**] for a characterization of circles in the plane whose rational points are dense in their respective circles.

The following lemma addresses the density question for angles.

LEMMA 10.

(1) Let $\mathbb{Q}_{tan} = \{\theta \in \mathbb{R} \mid tan(\theta)$ is rational or undefined $\}$.

(2) Let $\mathbb{Q}_{tan2} = \{\theta \in \mathbb{R} \mid tan(\theta) = \frac{q}{p}, gcd(p,q) = 1, p^2 + q^2$ is a square or $tan(\theta)$ is undefined $\}$.

(3) Let $C_{\mathbb{Q}} = \{\theta \in \mathbb{R} \mid Cos(\theta), Sin(\theta)$ are rational $\}$.

Then

(i) Then the set $\mathbb{Q}_{tan2}$ is dense in $\mathbb{R}$

(ii) $\mathbb{Q}_{tan2} \subset \mathbb{Q}_{tan}$ and $\mathbb{Q}_{tan2} = 2\mathbb{Q}_{tan} = C_{\mathbb{Q}}$.

(iii) $\mathbb{Q}_{tan}, \mathbb{Q}_{tan2}$ are additive subgroups of $\mathbb{R}$

Proof. First we observe that for every integer $k \in \mathbb{Z}$ the function

$$Tan_k : (\pi k - \frac{\pi}{2}, \pi k + \frac{\pi}{2}) \longrightarrow \mathbb{R}, \theta \mapsto Tan(\theta)$$

is a homeomorphism. Hence the set $\mathbb{Q}_{tan}$ is dense in $\mathbb{R}$.

Now we use some elementary geometry. Let C be a circle with center O of unit radius. Let A, B be two points on the circle such that the $arcAB$ subtends an angle 2θ at the center. Extend OA to meet the circle again at P. Then the $\measuredangle APB = \theta$.

We prove (ii) first.

Now we prove $\mathbb{Q}_{tan2} = C_{\mathbb{Q}}$. Let $\theta \in C_{\mathbb{Q}}$; then $Cos(\theta) = \frac{r}{s}, Sin(\theta) = \frac{u}{v}$ for some relatively prime integers r, s and u, v. So we have $\frac{r^2v^2+u^2s^2}{s^2v^2} = 1$ i.e. $r^2v^2 + u^2s^2 = s^2v^2$. If $Cos(\theta) = 0$ then $\theta \in \mathbb{Q}_{tan2}$. Let $Cos(\theta) \neq 0$. Now we observe that $Tan(\theta)$ is rational and if for some q, p relatively prime integers $\frac{q}{p} = Tan(\theta) = \frac{us}{rv}$. Then there exists an integer t such that $us = tq$ and $rv = tp$ so $t^2(p^2+q^2) = r^2v^2+u^2s^2 = s^2v^2$. So $t^2 \mid s^2v^2 \Rightarrow t \mid sv$ and $p^2 + q^2 = \left(\frac{sv}{t}\right)^2$ a perfect square. So $\theta \in \mathbb{Q}_{tan2}$. The converse is also clear; i.e. if $\theta \in \mathbb{Q}_{tan2}$ then $Cos(\theta), Sin(\theta)$ are rational.

Now we prove $\mathbb{Q}_{tan2} = 2\mathbb{Q}_{tan}$. Let $\theta \in \mathbb{Q}_{tan}$ and if $Tan(\theta)$ is undefined then θ is an odd multiple of $\frac{\pi}{2}$. So 2θ is an integer multiple of π. So $Tan(2\theta) = 0$ and $2\theta \in \mathbb{Q}_{tan2}$. If $Tan(\theta) = 0$ then $Tan(2\theta) = 0$ so $2\theta \in \mathbb{Q}_{tan2}$. If $Tan(\theta) = \frac{q}{p}$ with $gcd(q, p) = 1$ then $Tan(2\theta) = \frac{2Tan(\theta)}{1-Tan^2(\theta)} = \frac{2pq}{p^2-q^2}$. We observe that $(p^2 - q^2)^2 + 4p^2q^2 = (p^2 + q^2)^2$ a perfect square. Hence if $Tan(2\theta) = \frac{u}{v}$ with $gcd(u, v) = 1$ then also $u^2 + v^2$ is a perfect square because there exists an integer t such that $2pq = tu, p^2 - q^2 = tv$. So $2\theta \in \mathbb{Q}_{tan2}$. Conversely it is also clear that if $2\theta \in \mathbb{Q}_{tan2}$ then $Tan(\theta)$ is rational. i.e. $\theta \in \mathbb{Q}_{tan}$.

Now we prove (iii). We observe that $Tan(0) = 0, Tan(-\theta) = -Tan(\theta)$ and if $\theta_1 + \theta_2 \neq (2k+1)\frac{\pi}{2}$ for some $k \in \mathbb{Z}$ then $Tan(\theta_1 + \theta_2) = \frac{Tan(\theta_1)+Tan(\theta_2)}{1-Tan(\theta_1)Tan(\theta_2)}$. So $\mathbb{Q}_{tan}$ is an additive subgroup. Hence $\mathbb{Q}_{tan2} = 2\mathbb{Q}_{tan}$ is also an additive subgroup.

Now to prove (i) we observe that any finite index additive subgroup of a dense additive subgroup of reals is also dense in reals. □

We note that for a right angled triangle with rational sides, the area is rational. From the lemma above, we deduce the following density theorem for right angled triangles which we mention below without proof as it is straightforward.

Theorem 11. *Let $X = \{A, B, C\}$ represent three vertices of a right-angled triangle ΔABC in the Euclidean plane. Then given $\epsilon > 0$ there exists a $3-$set X_ϵ in the plane such that the points in X_ϵ form a rational right-angled triangle and $D(X, X_\epsilon) < \epsilon$. In fact we can choose X_ϵ such that it has any one of the points of X in common.*

The general case of triangles is also a straightforward consequence of the right-angled triangles case. Now we prove Theorem 3 here.

Proof. Given the triangle ΔABC in the plane, let a be a largest side among a, b, c. Drop a perpendicular AD from the vertex A to the opposite side BC with intersection point $D = AD \cap BC$. Now $\measuredangle BAD = \alpha, \measuredangle CAD = \beta$. Choose a point D' on AD such that AD' is rational and $d(D', D) < \delta$. Choose by Lemma 10, α_1, β_1 such that $0 < \alpha_1, \beta_1 < \frac{\pi}{2}$ and $Cos(\alpha_1), Sin(\alpha_1), Cos(\beta_1), Sin(\beta_1)$ are rational and $d(\alpha, \alpha_1) < \delta, d(\beta, \beta_1) < \delta$. Consider the right-angled triangles $\Delta AD'B'$ and $\Delta AD'C'$, both having right-angles at the vertex D'. Hence the line $B'C'$ is parallel to BC and $a' \overset{def^n}{=} B'C' = AD'(Tan(\alpha_1) + Tan(\beta_1))$ which is rational. We also observe that $AB' = AD'Sec(\alpha_1), AC' = AD'Sec(\beta_1)$ which are rational. Finally the area of the triangle $\Delta AB'C'$ is $\frac{1}{2}a'(AD')$ which is rational. Next choose δ such that $D(X, \{A, B', C'\}) < \epsilon$ and take $X_\epsilon = \{A, B', C'\}$. Here again we observe that the vertex A is unchanged in the approximant X_ϵ and $BC \parallel B'C'$.

In the case when the three points lie on a line then the proof is straight forward. $\square$

We now use the above theorem on triangles to deduce the analogous result in the class of parallelograms.

Theorem 12. *Let $X = \{A, B, C, D\}$ represent the vertices of a parallelogram $\square ABCD$ in the Euclidean plane. Then given $\epsilon > 0$ there exists a rational 4-set X_ϵ in the plane such that the points in X_ϵ form a rational parallelogram with rational area and $D(X, X_\epsilon) < \epsilon$.*

Proof. Let AC, BD be the diagonals of the parallelogram such that $AC \geq BD$. Then AC is the largest side of the congruent triangles ΔABC and ΔADC because $\measuredangle ABC = \measuredangle ADC$ is obtuse or just right. Using Theorem 3 we get an approximant $\Delta A'B'C'$ such that $B' = B$ and $A'C' \parallel AC$. Now we parallel translate $\Delta A'B'C'$ so that line determined by the line segment $A'C'$ coincides with that of AC. Then we complete this to a parallelogram $\square A'B'C'D'$ with $A'C'$ and $B'D'$ as diaagonals. In this procedure while approximating we make sure $D(\{A, B, C\}, \{A', B', C'\}) < \epsilon$. So that by symmetry of parallelograms we obtain an ϵ-approximant rational parallelogram $\square A'B'C'D'$ to parallelogram $\square ABCD$. Moreover it has rational area as the area is twice the area of the rational triangle $\Delta A'B'C'$. $\square$

4. Rational Points on a Hyperbola

The proof of our general results on quadrilaterals in section 7 requires some analysis of rational points on hyperbolae. Indeed, we prove:

Theorem 13.
Let P, Q be two points in the plane at a rational distance from each other. Let L be a line passing through P such that the cosine of the angle between L and PQ is rational. Then the set of all points on L which are at a rational distance from P and Q are dense in L. Conversely if there exists a point on L which is at a rational distance from P and Q then the cosine of the angle between L and PQ is rational.

Proof. Assume without loss of generality the line PQ represents x-axis with P as the origin and Q is at a rational distance r from the origin. Let the equation of the line L be $y = Tan(\theta)x$ where $Cos(\theta)$ is rational which implies $Sin^2(\theta), Cos^2(\theta)$, $Tan^2(\theta)$ are rational if $\theta \neq (2k+1)\frac{\pi}{2}, k \in \mathbb{Z}$ with $Tan(\theta), Sin(\theta)$ need not be rational. Let R be a point on L at a distance q from the origin and at a distance p from Q i.e. ΔPQR is a triangle with $PQ = r, QR = p, RP = q$. We see by computing distances, if we set $s^2 = p^2 - r^2 Sin^2(\theta)$ then $q = rCos(\theta) + s$. If s is rational then q is rational since $r, Cos(\theta)$ are rational.

To proceed with the proof, we need the following observation.

Observation 14. *Let $H_a = \{(x, y) \in \mathbb{R}^2 \mid x^2 - y^2 = a\}$ where a is a non-zero rational representing a hyperbola H_a in the plane $\mathbb{R}^2$. Then the set $H_a(\mathbb{Q}) = \{(x, y) \in H_a \mid x, y \in \mathbb{Q}\}$ of rational points is dense in H_a.*

PROOF. (of Observation 14) Let $u \in \mathbb{R}^*$ and set $x = \frac{u+\frac{a}{u}}{2}, y = \frac{u-\frac{a}{u}}{2}$. We see immediately that $(x, y) \in H_a(\mathbb{Q})$ if $u \in \mathbb{Q}^*$ since a is rational and the isomorphism $\mathbb{R}^* \longrightarrow H_a$ taking u to (x, y) establishes the Claim 14. □

Continuing with the proof of the Theorem, now consider the part of the hyperbola $H^+_{r^2 Sin^2(\theta)}$ corresponding to $p = x$-coordinate being positive and the following isomorphism

$$\underbrace{u}_{\mathbb{R}^+} \longrightarrow \underbrace{(p = \frac{(u + \frac{r^2 Sin^2(\theta)}{u})}{2}, s = \frac{(u - \frac{r^2 Sin^2(\theta)}{u})}{2})}_{H^+_{r^2 Sin^2(\theta)}} \longrightarrow$$
$$\longrightarrow \underbrace{R = ((rCos(\theta) + s)Cos(\theta), (rCos(\theta) + s)Sin(\theta))}_{L}$$

The inverse map being

$$\underbrace{R = (x, xTan(\theta))}_{L} \longrightarrow \underbrace{(p = \sqrt{r^2 - 2xr + \frac{x^2}{Cos^2(\theta)}}, s = \frac{x}{Cos(\theta)} - rCos(\theta))}_{H^+_{r^2 Sin^2(\theta)}} \longrightarrow$$
$$\longrightarrow \underbrace{u = \frac{p+s}{2}}_{\mathbb{R}^+}$$

In the maps defined above we note that the co-ordinates of R need not be rational if $Tan(\theta), Sin(\theta)$ are not rational, however the distances PR, QR are rational if $Cos(\theta)$ is rational when we have a rational u. From the observation 14 we establish the density of points at a rational distance from P and Q on the line L. Conversely by cosine rule if the distances PR, QR are rational for some point R on L then by cosine rule the cosine of the angle between L and PQ is rational. Hence Theorem 13 follows. □

5. Rational Points on Families of Quartic and Cubic Curves

As mentioned in the introduction, our results on quadrilaterals will proceed by re-expressing the problems in terms of rational points on families of some quartic and cubic curves. Firstly, we reformulate the density question for the quadrilateral as a question about rational points on a two parameter family of quartic curves.

LEMMA 15. *Let A, O, C be three collinear points in the plane with point O on the line segment AC such that the 3-set $\{A, O, C\}$ is a rational set. Let L be a line passing through O such that the sine and cosine of the angle θ between and L and AC are rational. Let $\frac{OC}{AO} = Cot(\beta)$. Then any rational point (x, y) to the equation*

$$\begin{aligned} &y^2 = x^4 + p(m,n)x^3 + q(m,n)x^2 + r(m,n)x + 1 \text{ where} \\ &p = 4(1+n)m = 4Cot(\theta)(1 + Cot(\beta)) \\ &q = 4(1+n)^2m^2 + 4n^2 - 2 = 4(1 + Cot(\beta))^2 Cot^2(\theta) + 4Cot^2(\beta) - 2 \\ &r = -4(1+n)m = -4Cot(\theta)(1 + Cot(\beta)) \end{aligned} \tag{16}$$

where $m = Cot(\theta), n = Cot(\beta)$ gives rise to a point B on the line L such that the distances AB, CB, OB are all rational and conversely any such point B gives rise to a rational point (x, y) on the quartic curve.

PROOF. By rotation and translation if necessary we can assume that the line AOC is the x-axis, O is the origin and A is to the left of O with coordinates $(-a, 0)$ and C is a point to the right of O with coordinates $(c, 0)$. Let the line L make an angle $\theta \neq \frac{\pi}{2}$ with respect to x-axis at the origin. The case $\theta = \frac{\pi}{2}$ can be considered separately.

Consider two families $\mathcal{C}, \mathcal{A}$ of lines passing through the point C and the point A respectively. Let m_C and m_A denote the slopes of any two lines one representing a line in the family $\mathcal{C}$ and one representing a line in family $\mathcal{A}$ respectively. The equations of the lines are given by

$$Y = m_C(X - c) \ldots \text{ family } \mathcal{C}$$
$$Y = m_A(X + a) \ldots \text{ family } \mathcal{A}$$
$$Y = Tan(\theta)X \ldots \text{ Line } L$$

Any intersection point $B = (X, Y)$ in the plane of two lines one from family $\mathcal{C}$ and one from family $\mathcal{A}$ are given by

$$X = \frac{cm_C + am_A}{m_C - m_A}$$
$$Y = \frac{(a+c)m_C m_A}{m_C - m_A}$$

The distances BO, BA, BC are given by

$$BO = \sqrt{\frac{(cm_C + am_A)^2 + (a+c)^2 m_C^2 m_A^2}{(m_C - m_A)^2}}$$
$$BA = \frac{(a+c)m_C\sqrt{1+m_C^2}}{\sqrt{(m_C - m_A)^2}}$$
$$BC = \frac{(a+c)m_A\sqrt{1+m_A^2}}{\sqrt{(m_C - m_A)^2}}$$

Suppose B lies on the line L then we have

$$m_A = \frac{cm_C Tan(\theta)}{(a+c)m_C - aTan(\theta)}$$
$$BO = \frac{(cm_C + am_A)\sqrt{Sec^2(\theta)}}{\sqrt{(m_C - m_A)^2}}$$

So for such a point B on the line L, BO, BA, BC are rational if the following happens.

(17)
m_C is rational.

$1 + m_C^2$ is a square of a rational.

$1 + m_A^2$ is a square of a rational which is equivalent to

$((a+c)m_C - aTan(\theta))^2(1 + m_A^2) = c^2 m_C^2 Tan^2(\theta) + ((a+c)m_C - aTan(\theta))^2$

being a square of a rational.

Substituting

$$\begin{aligned} \frac{a}{c} &= Tan(\beta) \\ m_C &= \frac{2Tan(\gamma)}{1 - Tan^2(\gamma)} \end{aligned} \tag{18}$$

the rationality conditions 17 are satisfied if

$$\begin{aligned} &Tan(\gamma) \text{ is rational.} \\ &Tan^4(\gamma) + 4Cot(\theta)\big(1 + Cot(\beta)\big)Tan^3(\gamma) + \\ &\big(4Cot^2(\beta) - 2 + 4\big(1 + Cot(\beta)\big)^2 Cot^2(\theta)\big)Tan^2(\gamma) \\ &- 4Cot(\theta)\big(1 + Cot(\beta)\big)Tan(\gamma) + 1 \\ &\text{is square of a rational.} \end{aligned} \tag{19}$$

The above rationality condition 19 gives rise to a rational solution to the following equation (21) and conversely any rational solution (x, y) to the following equation (21) gives rise to a value of $Tan(\gamma) = x$ and hence the slope m_C with other rationality conditions 17 and 19 being satisfied.

The case $\theta = \frac{\pi}{2}$ is similar. □

As a second step we transform two parameter family of quartic curves into a family of cubic curves over rationals.

LEMMA 20. *There exists a $(x, y) - (U, W)$ transformation which transforms the given two parameter family of quartic curves*

$$\begin{aligned} &y^2 = x^4 + p(m,n)x^3 + q(m,n)x^2 + r(m,n)x + 1 \ \textit{where} \\ &p = 4(1+n)m = 4Cot(\theta)(1 + Cot(\beta)) \\ &q = 4(1+n)^2m^2 + 4n^2 - 2 = 4(1 + Cot(\beta))^2Cot^2(\theta) + 4Cot^2(\beta) - 2 \\ &r = -4(1+n)m = -4Cot(\theta)(1 + Cot(\beta)) \end{aligned} \tag{21}$$

where $m = Cot(\theta), n = Cot(\beta)$

into a two parameter family of cubic curves given by

$$\begin{aligned} W^2 &= U^3 + \frac{3p^2 - 8q}{16}U^2 + \frac{3p^4 - 16p^2q + 16q^2 + 16pr - 64}{256}U + \frac{(p^3 - 4pq + 8r)^2}{4096} \\ &= U^3 + AU^2 + BU + C \end{aligned}$$

where

$$\begin{aligned}
A &= \frac{3p^2-8q}{16} = 1-2Cot^2(\beta)+\big(1+Cot(\beta)\big)^2Cot^2(\theta)\\
&= 1-2n^2+\big(1+n\big)^2m^2\\
B &= \frac{3p^4-16p^2q+16q^2+16pr-64}{256}\\
&= -Cot^2(\beta)\big(1+Cot(\beta)\big)\Bigg(\big(1-Cot(\beta)\big)+2\big(Cot(\beta)+1\big)Cot^2(\theta)\Bigg)\\
&= -n^2\big(1+n\big)\Bigg(\big(1-n\big)+2\big(n+1\big)m^2\Bigg)\\
C &= \left(\frac{p^3-4pq+8r}{64}\right)^2\\
&= \big(-Cot(\beta)^2\big(1+Cot(\beta)\big)Cot(\theta)\big)^2\\
&= \big(n^4\big(1+n\big)^2m^2\big)
\end{aligned}$$

Proof. Let

$$P(x) = x^4+px^3+qx^2+rx+1$$

Notice that the equation (21) can be written as follows by completing the squares

$$y^2 = \left(x^2+\frac{p}{2}x-\frac{p^2-4q}{8}\right)^2+\left(\frac{p(p^2-4q)}{8}+r\right)x+1-\left(\frac{p^2-4q}{8}\right)^2 \tag{22}$$

Now to get rid of the fourth power in x we substitute $y = U'+\Sigma$ where $\Sigma = x^2+\frac{p}{2}x-\frac{p^2-4q}{8}$ and also substitute $x=\frac{V'}{U'}$. So y becomes $y=(U'+(\frac{V'}{U'})^2+\frac{p}{2}(\frac{V'}{U'})-\frac{p^2-4q}{8})$ and now multiplying by U' on both sides of equation (22) we get

$$\begin{aligned}
U'^3-\left(\frac{p^2-4q}{4}\right)U'^2+&\left(\frac{p^2-4q-8}{8}\right)\left(\frac{p^2-4q+8}{8}\right)U'\\
&= -2V'^2-pU'V'+\left(\frac{p^3-4pq+8r}{8}\right)V'
\end{aligned}$$

Replacing U' by $-2U$ and V' by $-2V$ and dividing by 8 we get

$$\begin{aligned}
U^3+\left(\frac{p^2-4q}{8}\right)U^2+&\frac{(p^2-4q-8)(p^2-4q+8)}{256}U\\
&= V^2+\frac{p}{2}UV+\frac{p^3-4pq+8r}{32}V
\end{aligned}$$

Now to get rid of the $UV-$term we substitute $W=V+\frac{p}{4}U+\frac{p^3-4pq+8r}{64}$ and eliminating V we get the following two parameter family of cubic curves

$$\begin{aligned}
W^2 &= U^3+\frac{3p^2-8q}{16}U^2+\frac{3p^4-16p^2q+16q^2+16pr-64}{256}U+\frac{(p^3-4pq+8r)^2}{4096}\\
&= U^3+AU^2+BU+C
\end{aligned}$$

where

$$A = \frac{3p^2 - 8q}{16}$$
$$B = \frac{3p^4 - 16p^2q + 16q^2 + 16pr - 64}{256}$$
$$C = \left(\frac{p^3 - 4pq + 8r}{64}\right)^2$$

The $(x, y) - (U - V)$ transformation in this case of the quartic equation (21) is given by

$$\begin{aligned} U &= -\frac{1}{2}\left(y - x^2 - 2\big(1 + Cot(\beta)\big)Cot(\theta)x + 1 - 2Cot^2(\beta)\right) \\ &= -\frac{1}{2}\left(y - x^2 - 2(1+n)mx + (1 - 2n^2)\right) \\ V &= -\frac{1}{2}\left(x\big(y - x^2 - 2\big(1 + Cot(\beta)\big)Cot(\theta)x + 1 - 2Cot^2(\beta)\big)\right) \\ &= -\frac{1}{2}\left(x\big(y - x^2 - 2(1+n)mx + (1 - 2n^2)\big)\right) \\ x &= \frac{V}{U} \\ y &= -2U + \left(\frac{V}{U}\right)^2 + 2\big(1 + Cot(\beta)\big)Cot(\theta)\left(\frac{V}{U}\right) + 2Cot^2(\beta) - 1 \\ &= -2U + \left(\frac{V}{U}\right)^2 + 2\big(1+n\big)m\left(\frac{V}{U}\right) + 2n^2 - 1 \end{aligned} \tag{23}$$

and

$$\begin{aligned} A(m,n) &= \frac{3p^2 - 8q}{16} = 1 - 2Cot^2(\beta) + \big(1 + Cot(\beta)\big)^2 Cot^2(\theta) \\ &= 1 - 2n^2 + \big(1+n\big)^2 m^2 \\ B(m,n) &= \frac{3p^4 - 16p^2q + 16q^2 + 16pr - 64}{256} \\ &= -Cot^2(\beta)\big(1 + Cot(\beta)\big)\Big(\big(1 - Cot(\beta)\big) + 2\big(Cot(\beta) + 1\big)Cot^2(\theta)\Big) \\ &= -n^2\big(1+n\big)\Big(\big(1-n\big) + 2\big(n+1\big)m^2\Big) \\ C(m,n) &= \left(\frac{p^3 - 4pq + 8r}{64}\right)^2 \\ &= \big(-Cot(\beta)^2\big(1 + Cot(\beta)\big)Cot(\theta)\big)^2 \\ &= \big(n^4\big(1+n\big)^2 m^2\big) \end{aligned} \tag{24}$$

This proves the lemma on transformation from quartics to cubics. □

We observe that the cubic polynomial $Q(U) = U^3 + AU^2 + BU + C$ has the following factorization into a linear and a quadratic factor.

$$\begin{aligned}Q(U) &= U^3 + \big(1 - 2n^2 + \big(1+n\big)^2 m^2\big)U^2 + \\ &\big(-n^2(1+n)\big((1-n) + 2(n+1)m^2\big)\big)U + \big(n^4(1+n)^2m^2\big) \\ &= (U - n^2)(U^2 + (m^2(1+n)^2 - n^2 + 1)U - m^2n^2(1+n)^2)\end{aligned} \tag{25}$$

The discriminant of $Q(U)$ is given by

$$disc(Q(U)) = n^4(1+n)^2(1+m^2)((1-n)^2 + (1+n)^2m^2)$$

Now we figure the points $(m,n) \in \mathbb{C}^2$ where the cubic polynomial fails to have three distinct factors and hence these points $(m,n) \in \mathbb{C}^2$ represent a singular cubic.

We mention the following Lemma 26 without proof as it is straight forward.

Lemma 26 (Repeated Factors). *Let $m,n \in \mathbb{C}$. The cubic polynomial $Q(U)$ has repeated factors if and only if the discriminant of $Q(U)$ is zero if and only if*

- $n = 0, -1$
- $m = \pm\imath$
- *For any value of* $n \neq -1$, $m = \pm\imath\frac{n-1}{n+1}$

We quote the lemma below by sketching its proof.

Lemma 27. *Let $\tau(x) = x^3 + Ax^2 + Bx + C \in \mathbb{R}[x]$ be a cubic polynomial. Then $\tau(x)$ has three distinct real roots if and only if*

- $A^2 - 3B > 0$
- $\tau(\frac{-A+\sqrt{A^2-3B}}{3})\tau(\frac{-A-\sqrt{A^2-3B}}{3})$
  $= \frac{1}{27}\Bigg(-A^2B^2 + 4B^3 + 4A^3C - 18ABC + 27C^2\Bigg) < 0$

Proof. Between two distinct reals roots of a polynomial function there is a root of its derivative as a consequence of Rolle's Theorem. The above lemma follows by observing that there are two real roots of the derivative of the cubic where the values of the cubic itself have different signs. $\square$

In the case of the polynomial

$$Q(U) = (U - n^2)(U^2 + (n+1)(m^2(n+1) - n + 1)U - m^2n^2(1+n)^2)$$

we note that for real values of m, n

- $A^2 - 3B = \Bigg(1 - n^2 + n^4 + 2m^2(1+n)^2(1+n^2) + m^4(1+n)^4\Bigg) > 0$
  for all $m, n \in \mathbb{R}$
- The discriminant of $Q(U)$ is
  $$\big((1+m^2)n^4(1+n)^2\big)((-1+n)^2 + m^2(1+n)^2\big) > 0$$
  except in the cases $n = 0, -1$ or in the case where $m = 0$ and $n = 1$ in which the value is zero.

Now we quote the following lemma without proof.(cf. [**13**] Chapter 0, Section 7 Proposition 7.2&7.3 Chapter 9 Section 4 Theorem 4.3)

LEMMA 28.
Let $\tau(x) = x^3 + Ax^2 + Bx + C \in \mathbb{R}[x]$ be a cubic polynomial. Suppose $\tau(x)$ has three distinct real roots. Let E be the elliptic curve defined by $y^2 = x^3 + Ax^2 + Bx + C$. Then the real locus $E(\mathbb{R})$ is isomorphic to $S^1 \times \mathbb{Z}/2\mathbb{Z}$.

PROOF. This proof is standard as given in the reference. □

So here for $m, n \in \mathbb{R}$ the real locus of the elliptic curve $E(m,n) = \{(U,W) \in \mathbb{R}^2 \mid W^2 = U^3 + A(m,n)U^2 + B(m,n)U + C(m,n)\}$ is isomorphic to $S^1 \times \mathbb{Z}/2\mathbb{Z}$ except for the cases $n = 0, n = -1$ and the case $m = 0, n = 1$ in which the curve is singular.

We quote this lemma without proof as it is straight forward.

LEMMA 29. *Let $r_i \in \mathbb{C}, i = 1,2,3$ be distinct complex numbers. Then the j-invariant of the elliptic curve $y^2 = (x-r_1)(x-r_2)(x-r_3) = x^3 + ax^2 + bx + c$ is given by*

$$\begin{aligned} j - invariant &= const\frac{(a^2-3b)^3}{a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2} \\ &= const\frac{(r_1^2 + r_2^2 + r_3^2 - r_1r_2 - r_2r_3 - r_3r_1)^3}{(r_1-r_2)^2(r_2-r_3)^2(r_3-r_1)^2} \end{aligned} \tag{30}$$

In the current case the non-constant j-invariant function $j(m,n) \in \mathbb{Q}(m,n)$ given by

$$j(m,n) = const * \frac{\left(1 - n^2 + n^4 + 2m^2(1+n)^2(1+n^2) + m^4(1+n)^4\right)^3}{\left((1+m^2)n^4(1+n)^2)((-1+n)^2 + m^2(1+n)^2\right)}$$

Hence the elliptic variety $V = \{E(m,n)(\mathbb{C}) \mid (m,n) \in \mathbb{C}\} \longrightarrow \{(m,n) \in \mathbb{C}^2\}$ has a nonconstant $j-$invariant as a function of (m,n). In Lemma 27 we have given a condition for a real cubic to have three distinct roots in terms of the numerator and denominator polynomials of the j-invariant appearing in equation (30).

6. Density of Rational Points on a Family of Elliptic Curves

Consider the variety $V(m,n)(\mathbb{K})$ for a field $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ defined by the equation

$$V(m,n)(\mathbb{K}) = \{(U,W) \in \mathbb{K}^2 \mid W^2 = U^3 + A(m,n)U^2 + B(m,n)U + C(m,n)\}$$

and define $V(\mathbb{K}) = \bigcup_{(m,n)\in\mathbb{K}} V(m,n)(\mathbb{K})$

with A, B, C as in the previous section.

We know that $C(m,n)$ is a square in $\mathbb{Q}(m,n)$ from equation (24). We establish the density theorem:

THEOREM 31. *The set*

$$\begin{aligned} \mathcal{D}_{\mathbb{A}} = \{kP_1(m,n) \in V(\mathbb{Q}) \mid k \in \mathbb{Z}, (m,n) \in \mathbb{A}^2_{\mathbb{Q}} - \textit{(discriminant locus)-}\{m{=}0\} \\ \subset \mathbb{A}^2_{\mathbb{R}} - \textit{(discriminant locus)-}\{m{=}0\}\} \end{aligned}$$

is dense in $V(\mathbb{R})$ in both Zariski and usual topologies on $V(\mathbb{R})$.

Towards the proof, we start by noting that

$$\begin{aligned}(W,U) &= (\pm\frac{(p^3-4pq+8r)}{64},0)\\ &= (\pm n^2(1+n)m,0)\\ &= (\pm Cot^2(\beta)(1+Cot(\beta))Cot(\theta),0)\end{aligned}$$

are two polynomial points on the elliptic variety defined by $V(m,n)$ over $\mathbb{Q}(m,n)$. Since $y=\pm 1, x=0$ is a solution to the equation (21) we obtain

$$(W,U) = (\frac{\pm(r-p)}{8},\frac{-(p^2-4q+8)}{16})$$
$$(W,U) = (\frac{\pm(r+p)}{8},\frac{-(p^2-4q-8)}{16})$$

as the polynomial points on V. Consider the polynomial points

$$\begin{aligned}P_1(m,n) &= (\frac{p-r}{8},\frac{-(p^2-4q+8)}{16})\\ &= (m(1+n),n^2-1)\\ &= (Cot(\theta)(1+Cot(\beta)),Cot^2(\beta)-1)\\ P_2(m,n) &= (\frac{p^3-4pq+8r}{64},0)\\ &= (n^2(1+n)m,0)\\ &= (Cot^2(\beta)(1+Cot(\beta))Cot(\theta),0)\\ P_3(m,n) &= (\frac{p+r}{8},\frac{-(p^2-4q-8)}{16})\\ &= (0,n^2)\\ P_4(m,n) &= (\frac{r-p}{8},\frac{-(p^2-4q+8)}{16})\\ &= (-m(1+n),n^2-1)\\ &= -P_1(m,n)\end{aligned}$$

The point $P_3(m,n)$ corresponding to $y=-1,x=0$ is a 2-torsion polynomial point on V over $\mathbb{Q}(m,n)$. This follows from equation (25) because $U=n^2$ is a root of the polynomial $Q(U)$. The points on the elliptic curve corresponding to the roots of Q and the identity gives rise to the torsion subgroup $\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$.

So we prove that the points P_1,P_4,P_2 are points of infinite order in $X(m,n)$ over $\mathbb{Q}(m,n)$.

Now we state a very important theorem due to Mazur on torsion orders of points on an elliptic curve over $\mathbb{Q}$.

Theorem 32 (Mazur's Theorem). *Let C be a non-singular rational cubic curve over $\mathbb{K}=\mathbb{C}$ or $\bar{\mathbb{Q}}$, and suppose that $C(\mathbb{Q})$ contains a point of finite order m. Then either*

$$1\le m\le 10 \text{ or } m=12$$

We observe that P_1,P_2,P_3 lie on the line $W+(1+n)mU=n^2(1+n)m$. So $P_1+P_2=P_3$ or $P_1+P_2+P_3=O$. It is enough to show that the polynomial point

$P_1 = -P_4$ is of infinite order. Using the computer, we can check that P_1 does not have polynomial torsion order $1, 2, \ldots, 10, 12$.

We will show this by a very simple computation using computer that $kP_1 \neq -P_1$ by explicitly showing that there are points (m, n) where $kP_1(m, n) \neq -P_1(m, n)$ for various $k = 1, 2, 3, \ldots, 10, 11$.

The initial point is given by $P_1(m, n) = (m(1+n), n^2 - 1)$. The elliptic variety is given by the equation

$$W^2 - U^3 - (1 - 2n^2 + m^2(1+n)^2)U^2 - n^2(n+1)((n-1) - 2m^2(n+1))U - m^2n^4(1+n)^2 = 0$$

For a generic point $(m, n) \in \mathbb{A}^2_{\mathbb{K}}$ the tangent at $P_1(m, n)$ meets the elliptic curve $E(m, n)$ at $-2P_1(m, n) = \left(-\frac{(n-1)\left((n-1)^2 + 2m^2(1+n^2)\right)}{8m^3}, \frac{(n-1)^2 + 4m^2n^2)}{4m^2} \right)$.

Let $(x_0, y_0), (x_1, y_1)$ lie on the cubic whose equation is given by $Y^2 = X^3 + \lambda X^2 + \mu X + \nu$. Then the line determined by the points (x_0, y_0) and (x_1, y_1) meets the cubic curve again at the point $(X, -Y)$ whose values are given by

$$X = -\lambda + \left(\frac{y_1 - y_0}{x_1 - x_0}\right)^2 - x_1 - x_0$$
$$Y = -y_0 - (X - x_0)\left(\frac{y_1 - y_0}{x_1 - x_0}\right).$$

Note that under the elliptic curve addition $(x_0, y_0) + (x_1, y_1) = (X, Y)$.

Let $k > 0$ be a positive integer. Let $(x, y) = (x(m, n), y(m, n))$ be a multiple of P_1 say kP_1. Then the X-coordinate $X(m, n)$ and the Y-coordinate $Y(m, n)$ of the multiple $(k+1)P_1$ are given by

$$\begin{aligned} X[x,y] &= -(1 - 2n^2 + m^2(1+n)^2) + \left(\frac{y - m(1+n)}{x - n^2 + 1}\right)^2 - n^2 + 1 - x \\ Y[x,y] &= -m(n+1) - (X[x,y] - n^2 + 1)\left(\frac{y - m(1+n)}{x - n^2 + 1}\right). \end{aligned} \tag{33}$$

Using a computer we can check that the multiples of P_1 as rational functions in $\mathbb{Q}(m, n)$. However we will find a suitable point $(m, n) \in \mathbb{A}^2_{\mathbb{K}}$ such that the value $kP_1(m, n)$ differs from $-P_1(m, n)$ for $k = 1, \ldots, 11$ there by showing that P_1 cannot have polynomial torsion $1, 2, \ldots, 10, 12$.

LEMMA 34. *Let $(m_0, n_0) = (1, 2)$. The point $P_1(m_0, n_0) = (3, 3)$ on $E(m_0, n_0)(\mathbb{Q})$ is a point of infinite order.*

PROOF. The two parameter family of cubic curves reduces to the following elliptic curve at the point $(m_0, n_0) = (1, 2)$ given by

$$X(1, 2) : W^2 = (-4 + U)(-36 + 6U + U^2).$$

There are three distinct real values for U where W is zero. They are given by $U = 4, U = 3(-1 - \sqrt{5}), U = 3(-1 + \sqrt{5})$.

Consider the point $P_1(m_0, n_0) = (3, 3)$ on the elliptic curve $E(m_0, n_0)(\mathbb{Q})$. Using Equation 33 we compute the multiples $kP_1(m_0, n_0)$ for

$$k \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}.$$

They are given as follows.

MULTIPLES OF $\boldsymbol{P_1(m_0, n_0)}$

- $P_1(m_0, n_0) = (W = 3, U = 3)$,
- $2P_1(m_0, n_0) = (W = \frac{11}{8}, U = \frac{17}{4})$,
- $3P_1(m_0, n_0) = (W = -\frac{2091}{125}, U = -\frac{189}{25})$,
- $4P_1(m_0, n_0) = (W = -\frac{740943}{85184}, U = \frac{11713}{1936})$,
- $5P_1(m_0, n_0) = (W = -\frac{774296133}{1647212741}, U = \frac{5104323}{1394761})$,
- $6P_1(m_0, n_0) = (W = \frac{27508807641557}{338608873000}, U = \frac{923701649}{48580900})$,
- $7P_1(m_0, n_0) = (W = \frac{3530515935858140877}{285838253719954489}, U = -\frac{61622709117}{433923895441})$,
- $8P_1(m_0, n_0) = (W = -\frac{6468618165127547413697}{2772058650517790438999904}, U = \frac{17006294967389953}{4251429122504256})$,
- $9P_1(m_0, n_0) = (W = -\frac{31336845177587538848825263753 41}{268943929702576480749933159625}, U = \frac{5746975304186971011}{416652984990350502 25})$,
- $10P_1(m_0, n_0) = (W = -\frac{21637825704318812407875118259091920491}{2265672777740131031391895221484029 68}$, $U = \frac{783039511576266851237185 7}{3716477070917706995659 24}$,
- $11P_1(m_0, n_0) = (W = \frac{5993414358099942285341434200757056424938 47013}{11260664008330625138319520397572048744768 41269}$, $U = \frac{3948440455789942950949604475843}{1082370632513496730007602575721})$

We observe that $kP_1(m_0, n_0) \neq -P_1(m_0, n_0)$ for all $k \in \{1, 2, \ldots, 11\}$ as $P_1(m_0, n_0) \neq -P_1(m_0, n_0)$ and none of the values for U simplifies to 3 for $k \geq 2$. $\square$

Now we prove the following lemma.

Lemma 35. *The point $P_1(m, n)$ is a polynomial point of infinite order.*

Proof. We have seen P_1 does not have polynomial torsion $1, 2, \ldots, 10, 12$. Suppose $kP_1 = -P_1$ for some $k > 11$. Fix an $n_0 \in \mathbb{K}$. Let $m \in \mathbb{K}$. We have $kP_1(m, n_0) = -P_1(m, n_0)$. By Barry Mazur's theorem on torsion orders of rational points on elliptic curves over rationals, $lP_1(m, n_0) = -P_1(m, n_0)$ for some $l \in \{1, 2, \ldots, 9, 11\}$. There are finitely many choices for l and infinitely many choices for $m \in \mathbb{K}$. Hence there exists $l \in \{1, 2, \ldots, 9, 11\}$ such that $lP_1(m, n_0) = -P_1(m, n_0)$ for infinitely many $m \in \mathbb{K}$. This means that the equality $lP_1(m, n_0) = -P_1(m, n_0)$ holds as polynomial points in m for the fixed n_0. By a similar argument, again we have for some $l \in \{1, 2, \ldots, 9, 11\}$, $lP_1(m, n_0) = -P_1(m, n_0)$ for infinitely many $n_0 \in \mathbb{K}$ as polynomial points in m. Hence we get $lP_1(m, n) = -P_1(m, n)$ as polynomial points in m, n. However we have verified that $lP_1 \neq -P_1$ arriving at a contradiction. Now Lemma 35 follows. $\square$

The discriminant locus is defined as the set $\{(m, n) \in \mathbb{C}^2 \mid disc(Q(U)) = 0\}$. Now we state the density lemma for an elliptic curve.

LEMMA 36 (Density Lemma for an Elliptic Curve). *The set*

$$\mathcal{D} = \{kP_1(m_0,n_0) \mid k \in \mathbb{Z}, P_1(m_0,n_0) \text{ of infinite order in } E(m_0,n_0)(\mathbb{R}), \\ (m_0,n_0) \in \mathbb{A}^2_{\mathbb{Q}} - \text{(discriminant locus)}\}$$

is dense in both Zariski and usual topologies in the real locus $E(m_0,n_0)(\mathbb{R})$ of the jacobian elliptic variety $V(\mathbb{R})$.

PROOF. This is straight forward because any infinite subgroup of the circle group is dense. However here we need to observe that the following claim holds.

CLAIM 37. *For $(m_0,n_0) \in \mathbb{A}^2_{\mathbb{R}}$ − (the discriminant locus), the points $\pm P_1(m_0,n_0)$, $\pm P_2(m_0,n_0)$ lie on the oval component of the real locus. If $P_1(m_0,n_0)$ is of infinite order in $E(m_0,n_0)(\mathbb{R})$ then the subgroup generated by each P_1 is a dense subgroup of $S^1 \times \mathbb{Z}/2\mathbb{Z}$ in both Zariski and usual topologies on $S^1 \times \mathbb{Z}/2\mathbb{Z}$.*

PROOF. Note that for real m,n, the quadratic factor $(U^2+(n+1)(m^2(n+1)-n+1)U - m^2n^2(1+n)^2)$ of $Q(U)$ evaluated at $U=0, U=n^2-1$ is non-positive as it is a square polynomial in n,m with a negative sign. So each of the values $U=0, U=n^2-1$ lies between the roots of the quadratic factor of $Q(U)$. We also have that the U-coordinates of $\pm P_1(m,n), \pm P_2(m,n)$ is less than or equal to n^2 which is the root of the linear factor $U-n^2$ of $Q(U)$. Hence we observe that the points $\pm P_1(m,n), \pm P_2(m,n)$ lie on the oval. So under the isomorphism of the real locus $E(m,n)(\mathbb{R})$ to $S^1 \times \mathbb{Z}/2\mathbb{Z}$, the points $\pm P_1(m,n), \pm P_2(m,n) \in S^1 \times \{-1\}$ which does not have the identity element of the group $S^1 \times \mathbb{Z}/2\mathbb{Z}$. So the infinite subgroup generated by P_1 is a dense subgroup of $S^1 \times \mathbb{Z}/2\mathbb{Z}$ in both Zariski and usual topologies on $S^1 \times \mathbb{Z}/2\mathbb{Z}$. This proves the claim. □

Hence the lemma follows. □

LEMMA 38 (Density Bijection Lemma for the Quartics and Cubics). *The set $\{(x,y) \in \mathbb{Q}^2, (x,y)$ satisfies equation (21)$\}$ is dense in the quartic defined by the equation (21) in Zariski and usual topologies when $n \neq 0,-1$ and $m \neq 0$.*

PROOF. The case $U=0, V=0$ occurs when $y = x^2 + \frac{p}{2}x - \frac{p^2-4q}{8}$ and if $U=0$ then $W = \pm\frac{(p^3-4pq+8r)}{64} = \mp n^2(1+n)m \neq 0$.

Using $x-y, U-V$ transformations (23) we get that upon removing the points $(W,U) = \pm P_2(m,n) = (\pm\frac{(p^3-4pq+8r)}{64}, 0) = (\pm n^2(1+n)m, 0) = (\pm Cot^2(\beta)(1+Cot(\beta))m, 0)$ from the cubic and the point $(x_0,y_0) = \left(\frac{(p^2-4q)^2-64}{8(p^3-4pq+8r)} = -\frac{n-1}{2m}, \frac{(n-1)^2+4m^2n^2}{4m^2}\right)$ from the quartic we get a bijection between the complex solutions of the cubic and quartic. This bijection restricts to a bijection of their real locus and also rational locus if m,n are rational. Hence we get the density of rational solutions (x,y) of the quartic in its real locus components corresponding under bijection to the real locus components of the cubic. This completes the proof. □

In order to complete the proof of Theorem 31 we give a topological criterion for the density which is proved in section 9. After giving this criterion for the density of sets in arbitrary topological spaces we establish Theorem 31.

7. Main Theorem on Density of Points on a Line at a Rational Distance from Three Collinear Points

Finally we prove Theorem 4 on the density of points on a line which are at a rational distance from three collinear points in the Euclidean plane from which we deduce rational approximability of quadrilaterals in the next section.

Proof. The conditions 17 are satisfied by the rational points on the quartic. In order to get the required density to complete the proof of Theorem 4 we use Mazur's Theorem again in the following way.

First we observe that if for an $m_0 \in \mathbb{Q}$, $kP_1(m,n) = -P_1(m,n)$ for all n then $lP_1(m_0,n) = -P_1(m_0,n)$ for all n for some $l \in \{1,\ldots,9,11\}$. There are finitely many such $m_0 \in \mathbb{Q}$ for $l \in \{1,\ldots,9,11\}$. Let F_m denote the finite set of such elements $m_0 \in \mathbb{Q}$. Given any rational value m_0 for m apart from a finite subset $F_m \subset \mathbb{Q}$ with $F_{\measuredangle} = Cot^{-1}(F_m) = \{arcCot(m) \mid m \in F_m\}$, there exist finitely many $n \in \mathbb{Q}$ such that $kP_1(m_0,n) = -P_1(m_0,n)$ for any integer k because we need to check only for the finitely many possible torsion order values for k by Mazur's Theorem. Similarly given any rational value n_0 for n apart from a finite subset $F_n \subset \mathbb{Q}$ with $F_{ratio} = \frac{1}{F_n}$, there exist finitely many $m \in \mathbb{Q}$ such that $kP_1(m,n_0) = -P_1(m_0,n)$ for any integer k. Hence for a given rational value $m_0 \notin F_m$ for m, $P_1(m_0,n)$ is a point of infinite order for all but finitely many n and for a given rational value $n_0 \notin F_n$ for n, $P_1(m,n_0)$ is a point of infinite order for all but finitely many m.

This proves the Theorem. □

8. Rational Approximability of Quadrilaterals

Theorem 39. *Let $X = \{A,B,C,D\}$ represent four vertices of a quadrilateral in the Euclidean plane. Then given $\epsilon > 0$ there exists a rational quadrilateral X_ϵ in the plane such that $D(X,X_\epsilon) < \epsilon$.*

Proof. We prove this theorem in a few steps. We rename the vertices of the quadrilateral such that if ΔABD is the triangle formed by three out of four vertices such that if the quadrilateral is concave then the point C is in the interior of ΔABD or on the ΔABD and if the quadrilateral is convex then it lies in the exterior.

We give a slightly elaborate proof in the convex quadrilateral case and give a less elaborate but similar proof in the concave case.

So consider the case of a convex quadrilateral $\square ABCD$.

***Step* : 1**

Let the diagonals AC,BD meet at a point O. We assume that $\measuredangle AOB$ is greater than or equal to $\frac{\pi}{2}$ (i.e. just right or obtuse) by renaming the vertices $\{A,B,C,D\}$ of the quadrilateral so that in the ΔAOB, the side AB is the largest side. Now we approximate ΔAOB using Theorem 3 by a rational triangle with rational area $\Delta A'OB'$ such that $D(\{A,O,B\},\{A',O,B'\}) < \delta_1$. Given a $\delta_2 > 0$, we note that by suitably choosing smaller δ_1 if necessary we can assume that $D(\{A,O,B\},\{A',O,B'\}), < \delta_2$ and $d(C, \text{ Line } A'O) < \delta_2$. Since $\Delta A'OB'$ is rational with rational area we have that both the sine and the cosine of the angles $\measuredangle A'OB', \measuredangle OA'B', \measuredangle OB'A'$ are rational.

While obtaining an approximant $\Delta A'OB'$ for the triangle ΔAOB by using Theorem 3, we make sure that the mentioned angles in Theorem 3 $\measuredangle OA'B' = \frac{\pi}{2} - \alpha_1, \measuredangle OB'A' = \frac{\pi}{2} - \beta_1$ are so chosen that $\alpha_1, \beta_1, \alpha_1+\beta_1, \frac{\pi}{2}-\alpha_1, \frac{\pi}{2}-\beta_1, \pi-(\alpha_1+\beta_1)$

are not in $F_\measuredangle \cup (\pi - F_\measuredangle)$ which is a finite set where this finite set $F_\measuredangle$ arises in Theorem 4. Hence $\measuredangle A'OB' \notin F_\measuredangle \cup (\pi - F_\measuredangle)$ where $F_\measuredangle$ is the set described in Theorem 4.

***Step* : 2**

Using Lemma 13 again by suitably choosing δ_1, δ_2 we can find a point C' on the line $A'O$ such that $D(\{A, B, C\}, \{A', B', C'\}) < \delta_3$. We also make sure the choice of C' on the line $A'O$ is such that the ratio $\frac{A'O}{OC'}$ is not one of those finitely many ratios in F_{ratio} corresponding to the angle $\measuredangle A'OB'$ as described in the Theorem 4 which may not give density as per Theorem 4.

Now we also have that $\Delta B'OC'$ is a rational triangle because the 3-set $\{B', O, C'\}$ is rational and $Sin(\measuredangle B'OC')$ is rational which follows because $Sin(\measuredangle A'OB')$ is rational and hence the $\Delta A'B'C'$ is a rational triangle with rational area.

***Step* : 3**

Again by suitably choosing $\delta_1, \delta_2, \delta_3 < \epsilon$ we can assume that $d(D, \text{ Line } B'O) < \epsilon$. Since the ratio $\frac{A'O}{OC'} \notin F_{ratio}$ using Theorem 4 because of density we can find a point D' on the line $B'O$ such that the set $D(\{A, B, C, D\}, \{A', B', C', D'\}) < \epsilon$ and the set $\{A', O', C', D'\}$ is a rational set. So the triangles

$$\Delta A'OD', \Delta B'OD', \Delta C'OD', \Delta A'OD'$$

are rational triangles with rational area. Hence by taking $X_\epsilon = \{A', B', C', D'\}$ we have that

- X_ϵ is a rational set.
- Area of the quadrilateral $\Box A'B'C'D'$ is rational.
- $D(X, X_\epsilon) < \epsilon$.

In the case when B, C, D are collinear with C in between B, D and A is outside the line of B, C, D one of the angles $\measuredangle BCA, \measuredangle DCA$ is greater than or equal to $\frac{\pi}{2}$ (i.e. just right or obtuse). So we use Theorems 3, 4 to find a rational $4-$set $\{A', C, B', D'\}$ with the point C in common with X which gives the density.

If all four points $\{A, B, C, D\}$ are collinear then we use the density of rationals in reals for the conclusion.

If the $4-$ set $\{A, B, C, D\}$ form a concave quadrilateral then we let C be in the interior of the ΔABD and let AC intersect BD at O'. Assume $\measuredangle AO'B$ is obtuse or just right without loss of generality so that $\measuredangle ACB$ is also obtuse and we apply Theorem 3 with vertex C opposite to the largest side AB so that C is in the approximating rational 3-set. Then we find O close to O' by density and now in the proof of Theorem 4 we run through the argument with the coordinates of the point $C = (-c, 0)$ with $a > c > 0$. Here again we conclude density similar to the convex case by replacing n with $-n$ and n values with $-n$ values. It is the same argument about choosing proper angles for density purposes not in the associated finite set $F_\measuredangle$ and the distances also such that the ratio is not from the finite set F_{ratio} one we fix an angle. Hence Theorem 39 is proved. $\Box$

9. Appendix

LEMMA 40. *Let X, Y be topological spaces and $f : X \longrightarrow Y$. Suppose f is a closed surjective map and if $A \subset X$ is saturated then $\bar{A}$ is saturated. Let $B \subset Y$ be such that $\bar{B} = Y$. Let $A = f^{-1}(B)$. Then $\overline{A} = X$.*

PROOF. Let $C \subset X$ be any closed set containing $A \subset X$. Then $f(C) \supset f(A) = B$ and it is closed. So $f(C) = Y$. If C is saturated then $C = X$. Consider $C = \bar{A}$. By the hypothesis of this lemma $\bar{A}$ is saturated. Hence $\overline{A} = X$. $\square$

EXAMPLE 41.
(1) Let $X = [0,1] \times [0,1]$ and $Y = [0,1]$. Let $f = \pi$ be the first projection.
(2) Let $X = \bigg([0,1] \times [0,1]\bigg) \sqcup [2,3]$. Let $f = \pi$ be the first projection on $[0,1] \times [0,1]$ and $f(t) = 1$ if $t \in [2,3]$.
(3) Let $X = \bigg([0,1] \times [0,1]\bigg) \cup [1,2] \times \{0\}$. Let $f = \pi$ be the first projection on $[0,1] \times [0,1]$ and $f(t) = 1$ if $t \in [1,2] \times \{0\}$.
(4) Let $X = [0,1], Y = \{0,1\}$ with topology $\{\emptyset, \{0\}, \{0,1\}\}$. Define $f : X \longrightarrow Y$ as $f(t) = 0$ for $0 \leq t < \frac{1}{2}$ and $f(t) = 1$ for $\frac{1}{2} \leq t \leq 1$. Then f is a surjective continuous map. Take $B = \{0\}$. Then $A = [0, \frac{1}{2})$ which is not dense in X.

The cases $2,3$ give examples where we have even if f is a closed surjective map, if A is saturated then $\overline{A}$ need not be saturated. Take $A = [0,1) \times [0,1]$. Here in one case X is disconnected and in other case X is connected.

LEMMA 42 (Local Product Structure Lemma). *Let X, Y be topological spaces. Let $f : X \longrightarrow Y$ be topological spaces. Suppose X has the local product structure property with respect to f on a dense subset $Z \subset X$. Let $B \subset Y$ be such that $\bar{B} = Y$. Let $A = f^{-1}(B)$. Then $\overline{A} = X$.*

PROOF. Let $x \in Z$. Let $(x \in O \subset X, U \subset F_x, y = f(x) \in V \subset Y, O \cong_\psi U \times V)$ be a local product structure at x. Let $\psi(x) = (u,y) \in U \times V$. Since V is open we have $V \cap B$ is dense in V. So $\{u\} \times (V \cap B)$ is dense in $\{u\} \times V$ and we also have $U \times (V \cap B)$ is dense in $U \times V$ and $\psi^{-1}(U \times (V \cap B)) = A \cap O = f^{-1}(V \cap B) \cap O$. Hence $x \in O \subset \overline{A}$ which implies $Z \subset \overline{A}$. So $\overline{A} = X$ and the lemma follows. $\square$

9.1. Two Applications. This above lemma can be applied in many instances. In this subsection below we give two applications.

LEMMA 43 (First Application: Existence of Local Product Structure on Dense Set). *Let $X = \{(x_1, x_2, \ldots, x_n, y) \in \mathbb{R}^n \mid y^2 - P[x_1, x_2, \ldots, x_n] = 0$ where $P[x_1, x_2, \ldots, x_n] \in \mathbb{R}[x_1, x_2, \ldots, x_{n-1}][x_n]$ a monic polynomial in x_n with coefficients in $\mathbb{R}[x_1, x_2, \ldots, x_{n-1}]$. Let the map $f : X \longrightarrow \mathbb{R}^{n-1}$ given by $f(x_1, x_2, \ldots, x_n, y) = (x_1, x_2, \ldots, x_{n-1})$ be a sujective map. Then X has the local product structure property with respect to f on a dense subset of X.*

PROOF. Consider the set

$$Z = X \backslash \{(x_1, x_2, x_3, \ldots, x_n, y) \in \mathbb{R}^{n+1} \mid P[x_1, x_2, x_3, \ldots, x_n] \neq 0\}.$$

Then we prove that $Z \subset X$ has local product structure property at every point. For this purpose let $(x_1^0, x_2^0, \ldots, x_n^0, y^0) \in Z$. Since $P[x_1^0, x_2^0, x_3^0, \ldots, x_n^0] \neq 0$ we have $y^0 \neq 0$. Hence there exist an open set $V \subset \mathbb{R}^{n-1}$ such that $(x_1^0, x_2^0, x_3^0, \ldots, x_{n-1}^0) \in V$ and $\epsilon > 0$ such that $P[x_1, x_2, x_3, \ldots, x_n] \neq 0$ for all $(x_1, x_2, x_3, \ldots, x_n) \in V \times (x_n^0 - \epsilon, x_n^0 + \epsilon)$. Choosing the space $F_{(x_1^0, x_2^0, x_3^0, \ldots, x_n^0, y^0)} = U = (x_n^0 - \epsilon, x_n^0 + \epsilon), O = \{(x_1, x_2, x_3, \ldots, x_n, y) \in X \mid (x_1, x_2, x_3, \ldots, x_n) \in V \times U, sign(y) = sign(y_0)\}$ and we define a map $\psi : O \longrightarrow V \times U$ given by $\psi(x_1, x_2, x_3, \ldots, x_n, y) =$

$((x_1, x_2, x_3, \ldots, x_{n-1}), x_n)$. Clearly O is open as the sign condition can be treated as open condition over the reals. Now the lemma follows. □

Lemma 44 (First Application: Density). *Let $X = \{(x_1, x_2, \ldots, x_n, y) \in \mathbb{R}^n \mid y^2 - P[x_1, x_2, \ldots, x_n] = 0$ where $P[x_1, x_2, \ldots, x_n] \in \mathbb{R}[x_1, x_2, \ldots, x_{n-1}][x_n]$ a monic polynomial in x_n with coefficients in $\mathbb{R}[x_1, x_2, \ldots, x_{n-1}]$. Let the map $f : X \longrightarrow \mathbb{R}^{n-1}$ given by $f(x_1, x_2, \ldots, x_n, y) = (x_1, x_2, \ldots, x_{n-1})$ be a sujective map. Let $B \subset \mathbb{R}^{n-1}$ be a dense set. Then $A = f^{-1}(B)$ is dense in X.*

Proof. Using the previous two lemmas 42, 43 this lemma follows. This also proves Theorem 31. □

Lemma 45 (Second Application: Existence of Local Product Structure on Dense Set). *Let $X = \{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \in \mathbb{R}^{n+m} \mid F_i[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m] = 0$ for $1 \leq i \leq m$ where F_i is a polynomial function.$\}$ Suppose $f : X \longrightarrow \mathbb{R}^n$ given by $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \longrightarrow (x_1, x_2, \ldots, x_n)$ is a surjective map. Then f has the local product structure property.*

Proof. Let $Z = \{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \mid det\big((\frac{\partial F_i}{\partial y_j})_{i=1,j=1}^{m,m}\big)\} \neq 0$. Let $(x_1^0, x_2^0, \ldots, x_n^0, y_1^0, y_2^0, \ldots, y_m^0) \in Z$ then there exists open sets $V \subset \mathbb{R}^n, O \subset X$ and map $\phi : V \longrightarrow \mathbb{R}^m$ such that $F_i\big(x_1, x_2, \ldots, x_n, (y_1, y_2, \ldots, y_m) = \phi(x_1, x_2, \ldots, x_n)\big) = 0$ for all $(x_1, x_2, \ldots, x_n) \in V$ for all $1 \leq i \leq m$ and $graph(\phi) = O \subset X$ by implicit function theorem. So $O = \{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \mid (x_1, x_2, \ldots, x_n) \in V, (y_1, y_2, \ldots, y_m) = \phi(x_1, x_2, \ldots, x_n)\}$. Now take the space $F_{(x_1^0, x_2^0, \ldots, x_n^0, y_1^0, y_2^0, \ldots, y_m^0)} = U = \{a\}$ to be a singleton topological space and define a map $\psi : O \cong V \times U$ as $\psi : (x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) = ((x_1, x_2, \ldots, x_n), a)$. Then ψ is a homeomorphism. We see that the following diagram commutes.

$$\big(O \underset{f}{\longrightarrow} V\big) = \big(O \underset{\psi}{\longrightarrow} V \times U \underset{\pi_1}{\longrightarrow} V\big)$$

This proves the lemma. □

Lemma 46 (Second Application: Density). *Let $X = \{(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \in \mathbb{R}^{n+m} \mid F_i[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m] = 0$ for $1 \leq i \leq m$ where F_i is a polynomial function $\}$. Suppose $f : X \longrightarrow \mathbb{R}^n$ given by $(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m) \longrightarrow (x_1, x_2, \ldots, x_n)$ is a surjective map. Let $B \subset \mathbb{R}^n$ be a dense set. Then $A = f^{-1}(B)$ is dense in X.*

Proof. Using the lemma 42 and the previous lemma 45 this lemma follows. □

Lemma 47. *Let $X_0 \underset{f_1}{\longrightarrow} X_1 \underset{f_2}{\longrightarrow} X_2 \underset{f_3}{\longrightarrow} \ldots \underset{f_n}{\longrightarrow} X_n$ be a sequence of surjective continuous maps of topological spaces such that the local product structure property is satisfied on a dense set Z_i in X_i with respect to the map f_{i+1} for $i = 0, \ldots, n-1$. Then if $B \subset X_n$ is dense then the preimage of B in each X_i is dense in X_i for all $0 \leq i \leq n-1$.*

Proof. By a repeated application of the same principle this lemma follows. □

Now we prove Theorem 6.

Proof. Using the previous lemma 47 and the observation that in the closure of fibrewise dense set the entire fibre is there and hence upon its closure we get the whole space. □

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References

[1] J. H. J. Almering. Rational Quadrilaterals. *Indagationes Mathematicae*, 25:192–199, 1963.
[2] N. H. Anning and P. Erdos. Integral Distances. *Bulletin of American Mathematical Society*, 51:598–600, 1945.
[3] T. G. Berry. Points at a Rational Distance from the Vertices of a Triangle. *Acta Arithmetica*, LXII(4), 1992.
[4] A. S. Besicovitch. Rational Polygons. *Mathematika*, 6:98, 1959.
[5] H. F. Blichfeldt. On Triangles with Rational Sides and Having Rational Areas. *Annals of Mathematics*, Volume 11(1/6):57–60, 1896-1897.
[6] D. E. Daykin. Rational Polygons. *Mathematika*, 10:125–131, 1963.
[7] D. E. Daykin. Rational Triangles and Parallelograms. *Mathematics Magazine*, 38:46–47, 1965.
[8] L. E. Dickson. Rational Triangles and Quadrilaterals. *The American Mathematical Monthly*, 28(6/7):244–250, June-July 1921.
[9] L. Euler. *Commentationes Arithmeticae*, Volume I:24–25.
[10] L. Euler. *Commentationes Arithmeticae*, Volume II:648.
[11] R. Hartshorne. *Algebraic Geometry*. Graduate Texts in Mathematics: 52. Springer International.
[12] P. D. Humke and L. L. Krajewski. A Characterization of Circles Which Contain Rational Points. *The American Mathematical Monthly*, 86(4):287–290, April 1979.
[13] D. Husemoller. *Elliptic Curves*. Graduate Texts in Mathematics: 111. Springer International.
[14] E. E. Kummer. Ueber die Vierecke deren Seiten und Diagonalen rational sind. *Crelle*, 37:1–20.
[15] D. N. Lehmer. Rational Triangles and Quadrilaterals. *Annals of Mathematics*, 1(1/4):97–102, 1899-1900.
[16] L. J. Mordell. Rational Quadrilaterals. *Journal of the London Mathematical Society*, 35:277–282, 1960.
[17] I. R. Shafarevich. *Basic Algebraic Geometry Volume I: Varieties in Projective Space*. Springer Verlag.
[18] T. K. Sheng. Rational Polygons. *Journal of the Australian Mathematical Society*, 6:452–459, 1966.

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