CUMULANTS OF JACK SYMMETRIC FUNCTIONS AND b-CONJECTURE

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ABSTRACT. Goulden and Jackson (1996) introduced, using Jack symmetric functions, some multivariate generating series $\psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; t, 1+\beta)$ that might be interpreted as a continuous deformation of generating series of rooted hypermaps. They made the following conjecture: the coefficients of $\psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; t, 1+\beta)$ in the power-sum basis are polynomials in β with nonnegative integer coefficients (by construction, these coefficients are rational functions in β).

We prove partially this conjecture, nowadays called *b-conjecture*, by showing that coefficients of $\psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; t, 1+\beta)$ are polynomials in β with rational coefficients. A key step of the proof is a strong factorization property of Jack polynomials when α tends to 0, that may be of independent interest.

1. INTRODUCTION

1.1. **Jack symmetric functions.** Jack [Jac71] introduced a family of symmetric polynomials — which are now known as *Jack polynomials* $J_\pi^{(\alpha)}$ — indexed by a partition and a deformation parameter α. From the contemporary point of view, probably the main motivation for studying Jack polynomials comes from the fact that they are a special case of the celebrated *Macdonald polynomials* which *"have found applications in special function theory, representation theory, algebraic geometry, group theory, statistics and quantum mechanics"* [GR05]. Indeed, some surprising features of Jack polynomials [Sta89] have led in the past to the discovery of Macdonald polynomials [Mac95], and Jack polynomials have been regarded as a relatively easy case, which later allowed the understanding of the more difficult case of Macdonald polynomials (the series of paper [LV95, LV97] illustrates this very well). A brief overview of Macdonald polynomials and their relationship to Jack polynomials is given in [GR05]. Jack polynomials are also interesting on their own, for instance in the context of Selberg integrals [Kad97] and in theoretical physics [FJMM02, BH08].

Finally, according to Goulden and Jackson [GJ96], Jack polynomials are also related to hypermap enumeration, via specific multivariate generating functions. This relation is still partially a conjecture, and the main goal of the paper is to make a step forward to its resolution.

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1.2. **b-conjecture and our main result.** Let $J_\lambda^{(\alpha)}(\boldsymbol{x})$ be the Jack symmetric function indexed by a partition λ in the infinite alphabet $\boldsymbol{x}$. Let us denote by $h_\alpha(\lambda)$ and $h'_\alpha(\lambda)$ the α hook-polynomials (these are combinatorial factors that appears often in Jack polynomial theory; see Section 2.1 for the definition). We also use $\mathcal{P}$ for the set of all integer partitions and $|\lambda|$ for the size of a partition λ. In their article [GJ96], Goulden and Jackson defined a family of coefficients $h^\tau_{\mu,\nu}(\alpha-1)$ by the following formal series identity:

$$\log\left(\sum_{\lambda\in\mathcal{P}} \frac{J_\lambda^{(\alpha)}(\boldsymbol{x})\, J_\lambda^{(\alpha)}(\boldsymbol{y})\, J_\lambda^{(\alpha)}(\boldsymbol{z})\, t^{|\lambda|}}{h_\alpha(\tau)\, h'_\alpha(\tau)}\right) = \sum_{n\geq 1} \frac{t^n}{\alpha\, n} \left(\sum_{\mu,\nu,\tau\vdash n} h^\tau_{\mu,\nu}(\alpha-1)\, p_\mu(\boldsymbol{x})\, p_\nu(\boldsymbol{y})\, p_\tau(\boldsymbol{z})\right), \tag{1}$$

where $\mu,\nu,\tau \vdash n$ means that μ, ν and τ are three partitions of n and p_μ is the power-sum symmetric function associated with μ.

This rather involved definition is motivated by the following combinatorial interpretations for particular values of α; see [GJ96, Section 1.1] and references therein.

- In the case $\alpha = 1$, the quantity $h^\tau_{\mu,\nu}(0)$ enumerates connected hypergraphs embedded into oriented surfaces with vertex-, edge- and face-degree distributions given by μ, ν and τ.
- In the case $\alpha = 2$, the quantity $h^\tau_{\mu,\nu}(1)$ enumerates connected hypergraphs embedded into non-oriented surfaces with the same degree conditions.

Connected hypergraphs embedded into surfaces are usually called *maps* and are a classical topic in enumerative combinatorics related to the computation of matrix integrals or the study of moduli spaces of curves, as explained in detail in the book [LZ04]. The logarithm in Eq. (1) is present because we only want to count connected objects.

Note that $h^\tau_{\mu,\nu}(\alpha-1)$ is *a priori* a quantity depending on the parameter α, and describing it as a quantity depending on a different parameter $\beta := \alpha - 1$ might seem be artificial. However, it turned out that this shift seems to be a right one for finding a combinatorial interpretation of $h^\tau_{\mu,\nu}(\beta)$, as suggested by Goulden and Jackson [GJ96] in the following conjecture.

Conjecture 1.1 (b-conjecture)**.** *For all partitions $\tau,\mu,\nu \vdash n \geq 1$, the quantity $h^\tau_{\mu,\nu}(\beta)$ is a polynomial in β with nonnegative, integer coefficients. Moreover, there exists a statistics η on maps such that*

$$h^\tau_{\mu,\nu}(\beta) = \sum_{\mathcal{M}} \beta^{\eta(\mathcal{M})}, \tag{2}$$

where the summation index runs over all rooted, bipartite maps $\mathcal{M}$ with face distribution τ, black vertex distribution μ and white vertex distribution ν, and $\eta(\mathcal{M})$ is a nonnegative integer equals to 0 if and only if $\mathcal{M}$ is orientable.

This conjecture is still open. The thesis of La Croix [LaC09] gives a number of evidences for it, and gives a good account of what is known so far. In particular, some constructions for a candidate statistics η have been given, establishing particular cases of the conjecture [BJ07, LaC09, KV14]. However, there is not much known about the structure of $h^{\tau}_{\mu,\nu}(\beta)$ for arbitrary partitions $\tau, \mu, \nu \vdash n$. Strictly from the construction they are rational functions in β with rational coefficients. Our main result in this paper is a proof of the polynomiality of $h^{\tau}_{\mu,\nu}(\beta)$ for all partitions $\tau, \mu, \nu \vdash n \geq 1$.

Theorem 1.2. *For all partitions $\tau, \mu, \nu \vdash n \geq 1$ quantity $h^{\tau}_{\mu,\nu}(\beta)$ is a polynomial in β of degree $2 + n - \ell(\tau) - \ell(\mu) - \ell(\nu)$ with rational coefficients.*

Unfortunately, the nonnegativity and the integrality of the coefficients seem out of reach with our approach. However, the polynomiality could be useful in the investigation of 1.1. In particular, the first author has recently found a combinatorial description of the top-degree part of $h^{(n)}_{\mu,\nu}(\beta)$, which will be presented in the forthcoming paper [Doł16]. Theorem 1.2 is one of the ingredients of the proof.

1.3. **Strong factorization of Jack polynomials.** A key step in our proof is a strong factorization property for Jack polynomials when α tends to zero. To state it, let us introduce a few notations. If λ^1 and λ^2 are partitions, we denote $\lambda^1 \oplus \lambda^2$ their entry-wise sum; see Section 2.1. If $\lambda^1, \cdots, \lambda^r$ are partitions and I a subset of $[r] := \{1, \cdots, r\}$, then we denote

$$\lambda^I := \bigoplus_{i \in I} \lambda^i.$$

Theorem 1.3. *Let $\lambda^1, \cdots, \lambda^r$ be partitions. Then*

$$\prod_{I \subset [r]} \left(J^{(\alpha)}_{\lambda^I}\right)^{(-1)^{|I|}} = 1 + O(\alpha^{r-1}), \tag{3}$$

where symbol $O(\alpha^r)$ is defined in Definition 3.2.

The exponent $(-1)^{|I|}$ may be a bit disturbing so let us unpack the notation for small values of r.

- For $r = 2$, Eq. (3) writes as
$$\frac{J^{(\alpha)}_{\lambda^1 \oplus \lambda^2}}{J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2}} = 1 + O(\alpha).$$
In other terms, this means that for $\alpha = 0$, one has the factorization property $J^{(0)}_{\lambda^1 \oplus \lambda^2} = J^{(0)}_{\lambda^1} J^{(0)}_{\lambda^2}$. This is indeed true and follows from an explicit expression for $J^{(0)}_{\lambda}$ given by Stanley; see [Sta89, Proposition 7.6] or Eq. (12) in this paper. Thus, in this case, our theorem does not give anything new.
- For $r = 3$, Eq. (3) writes as
$$\frac{J^{(\alpha)}_{\lambda^1 \oplus \lambda^2 \oplus \lambda^3} J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2} J^{(\alpha)}_{\lambda^3}}{J^{(\alpha)}_{\lambda^1 \oplus \lambda^2} J^{(\alpha)}_{\lambda^1 \oplus \lambda^3} J^{(\alpha)}_{\lambda^2 \oplus \lambda^3}} = 1 + O(\alpha^2).$$

Using the case $r = 2$, it is easily seen that the left-hand side is $1 + O(\alpha)$. But our theorem says more and asserts that it is $1 + O(\alpha^2)$, which is not trivial at all.

This explains the terminology *strong factorization property*.

The theorem has an equivalent form that uses the notion of *cumulants of Jack polynomials* — see Section 3 for comments on the terminology. For partitions $\lambda^1, \cdots, \lambda^r$, we denote

$$\kappa^J(\lambda^1, \cdots, \lambda^r) = \sum_{\pi \in \mathcal{P}([r])} \mu(\pi, \{H\}) \prod_{B \in \pi} J_{\lambda^B}.$$

Here, the sum is taken over set partitions π of $[r]$ and μ stands for the Möbius function of the set partition lattice; see Section 2.4 for details. For example

$$\begin{aligned} \kappa^J(\lambda^1, \lambda^2) &= J^{(\alpha)}_{\lambda^1 \oplus \lambda^2} - J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2}, \\ \kappa^J(\lambda^1, \lambda^2, \lambda^3) &= J^{(\alpha)}_{\lambda^1 \oplus \lambda^2 \oplus \lambda^3} - J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2 \oplus \lambda^3} \\ &\quad - J^{(\alpha)}_{\lambda^2} J^{(\alpha)}_{\lambda^1 \oplus \lambda^3} - J^{(\alpha)}_{\lambda^3} J^{(\alpha)}_{\lambda^1 \oplus \lambda^2} + 2 J^{(\alpha)}_{\lambda^1} J^{(\alpha)}_{\lambda^2} J^{(\alpha)}_{\lambda^3}. \end{aligned}$$

We then have the following estimate for cumulants of Jack polynomials

Theorem 1.4. *For any partitions $\lambda^1, \dots, \lambda^r$, one has*

$$\kappa^J(\lambda^1, \cdots, \lambda^r) = O(\alpha^{r-1}). \tag{4}$$

Theorem 1.4 is in fact equivalent to Eq. (3), as shown (in a more general setting) by Proposition 3.3 (we need here the fact that J_λ has a non-zero limit when α tends to 0 [Sta89, Proposition 7.6]; this ensures that $J_\lambda = O(1)$ and $J_\lambda^{-1} = O(1)$). We prove Theorem 1.4 in Section 4.

We noticed, using computer simulations, that a similar property seems to hold for Macdonald polynomials $J_\lambda^{(q,t)}$. Unfortunately, we were unable to prove it and we state it here as a conjecture. Similarly to the Jack case, we define

$$\kappa^M(\lambda^1, \cdots, \lambda^r) = \sum_{\pi \in \mathcal{P}([r])} \mu(\pi, \{H\}) \prod_{B \in \pi} J^{(q,t)}_{\lambda^B}.$$

Conjecture 1.5. *For any partitions $\lambda^1, \dots, \lambda^r$, one has:*

- *the strong factorization property of Macdonald polynomials when q goes to 1, i.e.*

$$\prod_{I \subset [r]} \left(J^{(q,t)}_{\lambda^I}\right)^{(-1)^{|I|}} = 1 + O((q-1)^{r-1}); \tag{5}$$

- *the following estimates on cumulants of Macdonald polynomials*

$$\kappa^M(\lambda^1, \cdots, \lambda^r) = O\big((q-1)^{r-1}\big). \tag{6}$$

As in the Jack case, the two items are equivalent from Proposition 3.3. Note that the case $r = 2$ of both items says that

$$J^{(1,t)}_{\lambda^1 \oplus \lambda^2} = J^{(1,t)}_{\lambda^1} J^{(1,t)}_{\lambda^2},$$

which follows from the explicit expression for $J^{(1,t)}_\lambda$ given in [Mac95, Chapter VI, Remark (8.4), item (iii)]. Finally, we mention that Conjecture 1.5 implies Theorem 1.4 as a special case by substitution $q = t^\alpha$ and taking a limit $t \to 1$ since one has (see [Mac95, Chapter VI, Eq. (10.23)]):

$$\lim_{t \to 1} (1-t)^{-|\lambda|} J^{(t^\alpha,t)}_\lambda(\boldsymbol{x}) = J^{(\alpha)}_\lambda(\boldsymbol{x}).$$

1.4. **Related problems.** We finish this section, mentioning two similar problems. First, a very similar conjecture to Conjecture 1.1 (without logarithm in Equation (1)) was also stated by Goulden and Jackson [GJ96]. The series so obtained is conjecturally a multivariate generating function of *matchings*, where the exponent of β is some combinatorial integer-valued statistics. The conjecture is still open, while some special cases have been solved by Goulden and Jackson in their original article [GJ96] and recently by Kanunnikov and Vassilieva [KV14]. The polynomiality was proven by the authors of this paper [DF14] and is used here. Indeed, together with a simple argument given in Section 2.3, it reduces the proof of Theorem 1.2 to checking that there is no singularity in $\alpha = 0$.

A second related problem is the investigation of *Jack characters*, that is suitably normalized coefficients of the power-sum expansion of Jack polynomials. In a series of paper [Las08, Las09], Lassalle made some polynomiality and positivity conjectures suggesting that a combinatorial description of these objects might exist. Although these conjectures are not fully resolved, it was proven by us together with Śniady [DFŚ14] that in some special cases indeed, such combinatorial setup exists. Moreover, similarly to Conjecture 1.1, these special cases involve hypermaps and some statistics that "measures their non-orientability".

We cannot resist to state that there must be a deep connection between all these problems, and understanding it would be of great interest.

1.5. **Organization of the paper.** We describe all necessary definitions and background in Section 2, and in Section 3 we discuss cumulants and their relation with strong factorization. Section 4 is devoted to the proof of the strong factorization property of Jack polynomials, while Section 5 presents the proof of the main result, that is the polynomiality in b-conjecture.

2. PRELIMINARIES

2.1. **Partitions.** We call $\lambda := (\lambda_1, \lambda_2, \dots, \lambda_l)$ *a partition* of n if it is a weakly decreasing sequence of positive integers such that $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Then n is called *the size* of λ while l is *its length*. As usual we use the notation $\lambda \vdash n$, or

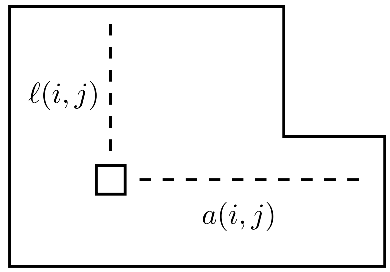

Figure 1. Arm and leg length of boxes in Young diagrams.

$|\lambda| = n$, and $\ell(\lambda) = l$. We denote the set of partitions of n by $\mathcal{Y}_n$ and we define a partial order on $\mathcal{Y}_n$, called *dominance order*, in the following way:

$$\lambda \leq \mu \iff \sum_{i \leq j} \lambda_i \leq \sum_{i \leq j} \mu_i \text{ for any positive integer } j.$$

For any two partitions $\lambda \in \mathcal{Y}_n$ and $\mu \in \mathcal{Y}_m$ we can construct two new partitions $\lambda \oplus \mu, \lambda \cup \mu \in \mathcal{Y}_{n+m}$, where $\lambda \oplus \mu := (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \dots)$, and $\lambda \cup \mu$ is obtained by merging parts of λ and μ and ordering them in a decreasing fashion. Moreover, there exists a canonical involution on the set $\mathcal{Y}_n$, which associate with a partition λ its *conjugate partition* λ^t. By definition, the j-th part λ^t_j of the conjugate partition is the number of positive integers i such that $\lambda_i \geq j$. Notice that for any two partitions λ, μ, we have $(\lambda \cup \mu)^t = \lambda^t \oplus \mu^t$. A partition λ is identified with some geometric object, called *Young diagram*, that can be defined as follows (using *French convention*):

$$\lambda = \{(i,j) : 1 \leq i \leq \lambda_j, 1 \leq j \leq \ell(\lambda)\}.$$

For any box $\Box := (i,j) \in \lambda$ from Young diagram we define its *arm-length* by $a(\Box) := \lambda_j - i$ and its *leg-length* by $\ell(\Box) := \lambda^t_i - j$ (the same definitions as in [Mac95, Chapter I]), see Fig. 1.

There are many combinatorial quantities associated with partitions that we will use extensively through this paper, so let us define them. First, set

$$z_\lambda := \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!, \tag{7}$$

where $m_i(\lambda)$ denotes the number of parts of λ equal to i. We also define *α-hook polynomials* $h_\alpha(\lambda)$ and $h'_\alpha(\lambda)$ by the following equations

$$h_\alpha(\lambda) := \prod_{\Box \in \lambda} \left(\alpha a(\Box) + \ell(\Box) + 1\right), \tag{8}$$

$$h'_\alpha(\lambda) := \prod_{\Box \in \lambda} \left(\alpha a(\Box) + \ell(\Box) + \alpha\right). \tag{9}$$

Finally, we consider a partition binomial given by

$$b(\lambda) := \sum_i \binom{\lambda_i}{2}. \tag{10}$$

2.2. **Jack polynomials and Laplace-Beltrami operator.** Jack polynomials are a classical one-parameter deformation of Schur symmetric functions that can be defined in several different ways. To our purpose, we will use a characterization via *Laplace-Beltrami operators*, suggested by Stanley in his seminal paper [Sta89, note p. 85]. Since this is now a well-established theory, results of this section are given without proofs but with explicit references to the literature (mostly to Stanley's paper [Sta89]).

First, consider the vector space Sym^N of symmetric polynomials in N variables over $\mathbb{Q}(\alpha)$. The following differential operators act on this space:

$$D_1 = \sum_{i \le N} \sum_{i \ne j} \frac{x_i^2}{x_i - x_j} \frac{\partial}{\partial x_i}, \qquad D_2 = \frac{1}{2} \sum_{i \le N} x_i^2 \frac{\partial^2}{\partial x_i^2}.$$

Then the *Laplace-Beltrami operator* D_α is defined as $D_\alpha = D_1 + \alpha D_2$.

Proposition 2.1. *There exists a unique family* $J_\lambda^{(\alpha)}$ *(indexed by partitions* λ *of length at most* N*) in* Sym^N *that satisfy:*

(C1) $J_\lambda^{(\alpha)}(x_1, \dots, x_N)$ *is an eigenvector of* D_α *with eigenvalue*

$$ev(\lambda) = \big(\alpha b(\lambda) - b(\lambda^t) + (N-1)|\lambda|\big)\,;$$

(C2) the monomial expansion of $J_\lambda^{(\alpha)}$ *is given by*

$$J_\lambda = h_\alpha(\lambda) m_\lambda + \sum_{\nu < \lambda} a_\nu^\lambda m_\nu, \text{ where } a_\nu^\lambda \in \mathbb{Q}(\alpha). \tag{11}$$

(Recall that we use the dominance order on partitions.)

These polynomials are called Jack polynomials.

This is not the definition of Jack polynomials used by Stanley, but the fact that Jack polynomials indeed satisfy these properties can be found in [Sta89]; see Theorem 3.1 and Theorem 5.6. The uniqueness is an easy linear algebra exercise when one has observed that $\mathrm{ev}(\lambda) = \mathrm{ev}(\mu)$ and $|\lambda| = |\mu|$ imply that λ and μ are either equal or incomparable for the dominance order [Sta89, Lemma 3.2]. A deep result of Knop and Sahi [KS97] asserts that a_ν^λ lies in fact in $\mathbb{N}[\alpha]$. In particular, Jack polynomials depend polynomially on α.

With the definition above, the Jack polynomial $J_\lambda^{(\alpha)}$ depends on the number N of variables. However, it is easy to see that it satisfies the compatibility relation $J_\lambda^{(\alpha)}(x_1, \dots, x_N, 0) = J_\lambda^{(\alpha)}(x_1, \dots, x_N)$ and thus $J_\lambda^{(\alpha)}$ can be seen as a symmetric function. In the sequel, when working with differential operators, we sometimes confuse a symmetric function f with its restriction $f(x_1, \dots, x_N, 0, 0, \dots)$ to N variables.

Stanley also established the following specialization formula at $\alpha = 0$:

$$J_\lambda^{(0)} = \left(\prod_i \lambda_i^t! \right) e_{\lambda^t}, \tag{12}$$

where e_λ is the *elementary symmetric function* associated with λ [Sta89, Proposition 7.6]. A key point in his proof, that will be also important in the present paper, is the following proposition.

Proposition 2.2. *For any partition $\lambda \vdash n$,*

(1) the elementary symmetric function e_λ is an eigenvector of the operator D_1:

$$D_1 e_\lambda = ((N-1)|\lambda| - b(\lambda))\, e_\lambda;$$

(2) for any partition $\mu \vdash n$ such that $b(\lambda) = b(\mu)$ either $\lambda = \mu$ or $\lambda \not\leq \mu$.

Here is an easy corollary, that will be useful for us.

Corollary 2.3. *Let $f \in \mathrm{Sym}$ be a homogeneous symmetric function with an expansion in the monomial basis of the following form:*

$$f = \sum_{\boldsymbol{\mu} < \boldsymbol{\lambda}} d_\mu m_\mu.$$

If, for any number N of variables, $D_1 f = \big((N-1)|\lambda| - b(\lambda^t)\big)\, f$ then $f = 0$.

Proof. Since $D_1 f = \big((N-1)|\lambda| - b(\lambda^t)\big) f$, we know from Proposition 2.2 (1) and (2) that the expansion of f in the elementary basis has the following form:

$$f = c_\lambda e_{\lambda^t} + \sum_{\lambda^t \not\leq \rho^t} c_\rho e_{\rho^t}.$$

Recall that $\lambda^t \not\leq \rho^t$ is equivalent to $\rho \not\leq \lambda$. Moreover, it is easy to see that the expansion of the elementary symmetric function e_{λ^t} in the monomial basis involves only elements m_μ indexed by partitions $\mu \leq \lambda$:

$$e_{\lambda^t} = m_\lambda + \sum_{\mu<\lambda} b_\mu^\lambda m_\mu.$$

Combining these two facts we know that the expansion of f in the monomial basis has the following form:

$$f = c_\lambda (m_\lambda + \sum_{\mu<\lambda} b_\mu^\lambda m_\mu) + \sum_{\rho \not\leq \lambda} c_\rho (m_\rho + \sum_{\mu<\rho} b_\mu^\rho m_\mu).$$

But we assumed that

$$f = \sum_{\mu<\lambda} d_\mu m_\mu,$$

which implies that $c_\lambda = 0$ and $c_\rho = 0$ for all $\rho \not\leq \lambda$, thus $f = 0$ as claimed. □

2.3. **Goulden and Jackson's conjectures.** Following Goulden and Jackson, we define

$$\phi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; t, \alpha) := \sum_{n \geq 0} t^n \sum_{\lambda \vdash n} \frac{J_\lambda^{(\alpha)}(\boldsymbol{x}) J_\lambda^{(\alpha)}(\boldsymbol{y}) J_\lambda^{(\alpha)}(\boldsymbol{z})}{\left\langle J_\lambda^{(\alpha)}, J_\lambda^{(\alpha)} \right\rangle_\alpha}. \tag{13}$$

$$\psi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; t, \alpha) := \alpha t \frac{\partial}{\partial_t} \log \phi(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; t, \alpha), \tag{14}$$

We then consider their power-sum expansion, *i.e.* the two families of coefficients $h^{\tau}_{\mu,\nu}$ and $c^{\tau}_{\mu,\nu}$ defined by

$$\psi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};t,\alpha) = \sum_{n\geq 1} t^n \sum_{\mu,\nu,\tau\vdash n} h^{\tau}_{\mu,\nu}(\alpha-1) p_\tau(\boldsymbol{x}) p_\mu(\boldsymbol{y}) p_\nu(\boldsymbol{z}); \tag{15}$$

$$\phi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};t,\alpha) = \sum_{n\geq 1} t^n \sum_{\mu,\nu,\tau\vdash n} \frac{c^{\tau}_{\mu,\nu}(\alpha-1)}{\alpha^{\ell(\tau)} z_\tau} p_\tau(\boldsymbol{x}) p_\mu(\boldsymbol{y}) p_\nu(\boldsymbol{z}), \tag{16}$$

The definition of the coefficients $h^{\tau}_{\mu,\nu}(\alpha-1)$ was already given in Section 1.2, we recall it here to emphasize the similarity with $c^{\tau}_{\mu,\nu}(\alpha-1)$. Goulden and Jackson conjecture that all these coefficients are polynomials in $\beta = \alpha - 1$ with non-negative integer coefficients and some combinatorial interpretations. The polynomiality of $c^{\tau}_{\mu,\nu}(\beta)$ with rational coefficients was recently proven by the authors of this paper:

Theorem 2.4. [DF14, Proposition B.2] *For any positive integer n and for any partitions $\mu,\nu,\tau \vdash n$, the quantity $c^{\tau}_{\mu,\nu}(\beta)$ is a polynomial in β (or equivalently in α).*

Recall, from Eq. (14), that

$$\psi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};t,\alpha)/\alpha = t\frac{\partial}{\partial_t}\log\phi(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z};t,\alpha).$$

Therefore, the coefficients of the power-sum expansion of the left-hand side — that are $h^{\tau}_{\mu,\nu}(\beta)/\alpha$ — can be expressed as polynomials in terms of the coefficients of the power-sum expansion of ϕ — that are $|\lambda| c^{\tau}_{\mu,\nu}(\beta)/(\alpha^{\ell(\tau)} z_\lambda)$. In particular, an immediate corollary of the above theorem is the following:

Corollary 2.5. *For any positive integer n and for any partitions $\mu,\nu,\tau \vdash n$, the coefficient $h^{\tau}_{\mu,\nu}(\beta)$ is a rational function in α with only possible pole at $\alpha = 0$.*

Showing that there is in fact no pole at $\alpha = 0$, as claimed in Theorem 1.2, requires a great deal of work and is the main result of this paper.

2.4. **Set partitions.** The combinatorics of set partitions is central in the theory of cumulants and will be important in this article. We recall here some well-known facts about them.

A *set partition* of a set S is a (non-ordered) family of non-empty disjoint subsets of S (called parts of the partition), whose union is S. In the following, we always assume that S is finite.

Denote $\mathcal{P}(S)$ the set of set partitions of a given set S. Then $\mathcal{P}(S)$ may be endowed with a natural partial order: the *refinement* order. We say that π is *finer* than π' (or π' *coarser* than π) if every part of π is included in a part of π'. We denote this by $\pi \leq \pi'$.

Endowed with this order, $\mathcal{P}(S)$ is a complete lattice, which means that each family F of set partitions admits a join (the finest set partition which is coarser than all set partitions in F; we denote the join operator by $\vee$) and a meet (the coarsest set partition which is finer than all set partitions in F; we denote the meet operator by $\wedge$). In particular, the lattice $\mathcal{P}(S)$ has a maximum $\{S\}$ (the partition in only one part) and a minimum $\{\{x\}, x \in S\}$ (the partition in singletons).

Moreover, this lattice is ranked: the rank $\mathrm{rk}(\pi)$ of a set partition π is $|S| - \#(\pi)$, where $\#(\pi)$ denotes the number of parts of π. The rank is compatible with the lattice structure in the following sense: for any two set partitions π and π',

$$\mathrm{rk}(\pi \vee \pi') \leq \mathrm{rk}(\pi) + \mathrm{rk}(\pi'). \tag{17}$$

Lastly, denote μ the Möbius function of the partition lattice $\mathcal{P}(S)$. In this paper, we only use evaluations of μ at pairs $(\pi, \{S\})$ (that is the second argument is the one-part partition of S, which is the maximum of $\mathcal{P}(S)$). In this case, the value of the Möbius function is given by:

$$\mu(\pi, \{S\}) = (-1)^{\#(\pi)-1}(\#(\pi) - 1)!. \tag{18}$$

3. CUMULANTS

3.1. **Partial cumulants.**

Definition 3.1. Let $(u_I)_{I \subseteq J}$ be a family of elements in a field, indexed by subsets of a finite set J. Then its *partial cumulant* is defined as follows. For any non-empty subset H of J, set

$$\kappa_H(\boldsymbol{u}) = \sum_{\pi \in \mathcal{P}(H)} \mu(\pi, \{H\}) \prod_{B \in \pi} u_B. \tag{19}$$

The terminology comes from probability theory. Let $J = [r]$, and let $X_1, \cdots, X_r$ be random variables with finite moments defined on the same probability space. Then define $u_I = \mathbb{E}(\prod_{i \in I} X_i)$, where $\mathbb{E}$ denotes the expectation of this probability space. The quantity $\kappa_{[r]}(\boldsymbol{u})$ as defined above, is known as the joint (or mixed) cumulant of the random variables $X_1, \cdots, X_r$. Also, $\kappa_H(\boldsymbol{u})$ is the joint/mixed cumulant of the smaller family $\{X_h, h \in H\}$.

Joint/mixed cumulants have been studied by Leonov and Shiryaev in [LS59] — see also an older note of Schützenberger [Sch47], where they are introduced under the French name *déviation d'indépendence*. They now appear in random graph theory [JŁR00, Chapter 6] and have inspired a lot of work in noncommutative probability theory [NŚ11].

Even if this probabilistic interpretation of cumulants is not relevant here, we will use several lemmas that have been discovered by the second author in a probabilistic context [Fér13].

A classical result – see, *e.g.*, [JŁR00, Proposition 6.16 (vi)] – is that relation (19) can be inverted as follows: for any non-empty subset H of J,

$$u_H = \sum_{\pi \in \mathcal{P}(H)} \prod_{B \in \pi} \kappa_B(\boldsymbol{u}). \tag{20}$$

3.2. **A multiplicative criterion for small cumulants.** Let R be a ring and α a formal parameter. Denote $R(\alpha)$ the field of rational functions in α with coefficients in R. In all applications in this paper, α is the Jack parameter.

Definition 3.2. We use the following notation: for $r \in R(\alpha)$ and an integer k, we write $r = O(\alpha^k)$ if the rational function $r \cdot \alpha^{-k}$ has no pole in 0.

As above, we consider a family $\boldsymbol{u} = (u_I)_{I \subseteq [r]}$ of elements of $R(\alpha)$ indexed by subsets of $[r]$. Throughout this section, we also assume that these elements are **non-zero** and $u_\emptyset = 1$.

In addition to partial cumulants, we also define the *cumulative factorization error terms* $T_H(\boldsymbol{u})$ of the family $\boldsymbol{u}$. The quantities $T_H(\boldsymbol{u})_{H \subseteq [r], |H| \geq 2}$ are inductively defined by: for any subset G of $[r]$ of size at least 2,

$$u_G = \prod_{g \in G} u_{\{g\}} \cdot \prod_{\substack{H \subseteq G \\ |H| \geq 2}} (1 + T_H(\boldsymbol{u})). \tag{21}$$

Using inclusion-exclusion principle, a direct equivalent definition is the following: for any subset H of $[r]$ of size at least 2, set

$$T_H(\boldsymbol{u}) = \prod_{G \subseteq H} u_G^{(-1)^{|H|}} - 1. \tag{22}$$

We have the following result.

Proposition 3.3. *Using the notation above, the following statements are equivalent:*

I. Strong factorization property: *for any subset $H \subseteq [r]$ of size at least* 2*, one has*

$$T_H(\boldsymbol{u}) = O(\alpha^{|H|-1}). \tag{23}$$

II. Small cumulant property: *for any subset $H \subseteq [r]$ of size at least* 2*, one has*

$$\kappa_H(\boldsymbol{u}) = \left(\prod_{h \in H} u_h \right) O(\alpha^{|H|-1}). \tag{24}$$

This proposition is a reformulation of [Fér13, Lemma 2.2]. However, the context and notation are quite different: in [Fér13], we are interested in sequences of random variables, while here, we consider rational functions in α. Thus, we prefer to copy the proof here, adapting it to our context.

Proof. We first assume that $u_{\{i\}} = 1$ for all i in $[r]$.

Let us first show that Item I implies Item II. Assume that $T_H(\boldsymbol{u}) = O(\alpha^{|H|-1})$, for any $H \subseteq [r]$ of size at least 2. The goal is to prove that $\kappa_{[r]} = O(\alpha^{r-1})$. This corresponds only to the case $H = [r]$ of Item II, but the same proof will work for any $H \subseteq [r]$.

Fix a set partition $\pi \in \mathcal{P}(r)$. For a block B of π, one has, expanding the second product in Eq. (21):

$$u_B = \sum_{H_1, \dots, H_m} T_{H_1} \dots T_{H_m},$$

where the sum runs over all finite lists of distinct (but not necessarily disjoint) subsets of B of size at least 2 (in particular, the length m of the list is not fixed). Therefore,

$$\prod_{B\in\pi} u_B = \sum_{H_1,\dots,H_m} T_{H_1}\dots T_{H_m},$$

where the sum runs over all lists of distinct subsets of $[r]$ of size at least 2 such that each H_i is contained in a block of π. In other terms, for each $i \in [m]$, π must be coarser than the partition $\Pi(H_i)$, which, by definition, has H_i and singletons as blocks. Finally, from Eq. (19)

$$\kappa_{[r]}(\boldsymbol{u}) = \sum_{\substack{H_1,\dots,H_m\subseteq[r]\\ \text{distinct}}} T_{H_1}\dots T_{H_m} \left(\sum_{\substack{\pi\in\mathcal{P}([r])\\ \forall i,\ \pi\geq\Pi(H_i)}} \mu(\pi,\{[r]\}) \right). \tag{25}$$

The condition on π can be rewritten as

$$\pi \geq \Pi(H_1)\vee\dots\vee\Pi(H_m).$$

Hence, by definition of the Möbius function, the sum in the parenthesis is equal to 0, unless $\Pi(H_1)\vee\dots\vee\Pi(H_m) = \{[r]\}$ (in other terms, unless the hypergraph with edges $(H_i)_{1\leq i\leq m}$ is connected). On the one hand, by Eq. (17), it may happen only if:

$$\sum_{i=1}^m \mathrm{rk}\left(\Pi(H_i)\right) = \sum_{i=1}^m (|H_i|-1) \geq \mathrm{rk}([r]) = r-1.$$

On the other hand, one has

$$T_{H_1}\dots T_{H_m} = O\left(\alpha^{\sum_{i=1}^m(|H_i|-1)}\right).$$

Hence only summands of order of magnitude $O(\alpha^k)$ for $k > r-1$ survive and one has

$$\kappa_{[r]}(\boldsymbol{u}) = O(\alpha^{r-1}),$$

as wanted.

Let us now consider the converse statement. We proceed by induction on r and we assume that, for all r' smaller than a given $r\geq 2$, the proposition holds.

Consider some family $(u_I)_{I\subseteq[r]}$ such that Item II holds. By induction hypothesis, for all $H\subsetneq[r]$, one has $T_H(\boldsymbol{u}) = O(\alpha^{|H|-1})$. Note that Eq. (21) then implies $u_H = O(1)$ and $u_H^{-1} = O(1)$ for $H\subsetneq[r]$. It remains to prove that

$$T_{[r]}(\boldsymbol{u}) = \prod_{H\subseteq[r]} (u_H)^{(-1)^{|H|}} - 1 = O(\alpha^{r-1}).$$

Thanks to the estimates above for u_H, this can be rewritten as

$$u_{[r]} = \prod_{H\subsetneq[r]} (u_H)^{(-1)^{r-1-|H|}} + O(\alpha^{r-1}). \tag{26}$$

Define now an auxiliary family $\boldsymbol{v}$:

$$v_G = \begin{cases} u_G & \text{if } G \subsetneq [r]; \\ \prod_{H \subsetneq [r]} (u_H)^{(-1)^{r-1-|H|}} & \text{for } G = [r]. \end{cases}$$

Clearly, since $T_G(\boldsymbol{v}) = T_G(\boldsymbol{u})$ for $G \subsetneq [r]$ and $T_{[r]}(\boldsymbol{v}) = 0$, the family $\boldsymbol{v}$ has the strong factorization property. Thus, using the first part of the proof, it also has the small cumulant property. In particular:

$$\kappa_{[r]}(\boldsymbol{v}) = O(\alpha^{r-1}).$$

But, by hypothesis,

$$\kappa_{[r]}(\boldsymbol{u}) = O(\alpha^{r-1}).$$

As $v_H = u_H$ for $H \subsetneq [r]$, one has:

$$u_{[r]} - v_{[r]} = \kappa_{[r]}(\boldsymbol{u}) - \kappa_{[r]}(\boldsymbol{v}) = O(\alpha^{r-1}),$$

which proves Eq. (26).

The general case follows directly from the case $u_{\{i\}} = 1$ by considering the family $u'_I = u_I / \prod_{i \in I} u_{\{i\}}$. Indeed, if $|H| \geq 2$, then

$$T_H(\boldsymbol{u'}) = T_H(\boldsymbol{u});$$
$$K_H(\boldsymbol{u'}) = K_H(\boldsymbol{u}) / \prod_{h \in H} u_{\{h\}}. \qquad \square$$

A first consequence of this multiplicative criterion for small cumulants is the following stability result.

Corollary 3.4. *Consider two families $(u_I)_{I \subseteq [r]}$ and $(v_I)_{I \subseteq [r]}$ with the small cumulant property. Then their entry-wise product $(u_I v_I)_{I \subseteq [r]}$ and quotient $(u_I / v_I)_{I \subseteq [r]}$ also have the small cumulant property.*

Proof. This is trivial for the strong factorization property and the small cumulant property is equivalent to it. □

3.3. **Hook cumulants.** To illustrate the propositions above and as a preparation for our next results, we show in this section that families constructed from the hook polynomials defined by Eq. (8) and Eq. (9) have the small cumulant properties.

We first consider the case of h_α and start by a technical lemma.

Lemma 3.5. *Fix a positive integer r and a subset K of $[r]$. Let C and $(c_i)_{i \in K}$ be some elements of $R(\alpha)$. Assume that C, C^{-1} and the c_i are $O(1)$. For a subset I of K, we define*

$$v_I = C + \alpha \cdot \sum_{i \in I} c_i$$

Then we have, for any subset H of K,

$$T_H(\boldsymbol{v}) = O(\alpha^{|H|}).$$

This is a reformulation of [Fér13, Lemma 2.4], but, again, as notation is quite different there, we adapt the proof to our context.

Proof. It is enough to prove the statement for $H = K$. Indeed, the case of a general set H follows by considering the same family restricted to subsets of H.

Define R_{ev} (resp. R_{odd}) as

$$\prod_{\delta} \left(C + \alpha \sum_{i \in \delta} c_i \right),$$

where the product runs over subsets of K of even (resp. odd) size. With this notation, $T_K(\boldsymbol{v}) = R_{\text{ev}}/R_{\text{odd}} - 1 = (R_{\text{ev}} - R_{\text{odd}})/R_{\text{odd}}$. Since $R_{\text{odd}}^{-1} = O(1)$ (each term in the product is $O(1)$, as well as its inverse), it is enough to show that $R_{\text{ev}} - R_{\text{odd}} = O(\alpha^{|K|})$.

Expanding the product in the definition of R_{ev}, one gets

$$R_{\text{ev}} = \sum_{m \geq 0} \sum_{\delta_1, \dots, \delta_m} \sum_{i_1 \in \delta_1, \dots, i_m \in \delta_m} \alpha^m c_{i_1} \dots c_{i_m} C^{2^{|K|-1} - m}.$$

The index set of the second summation symbol is the set of lists of m distinct (but not necessarily disjoint) subsets of K of even size. Note that the summand $\alpha^m c_{i_1} \dots c_{i_m} C^{2^{|K|-1}-m}$ is $O(\alpha^m)$. Of course, a similar formula with subsets of odd size holds for R_{odd}.

Let us fix an integer $m < |K|$ and a list $i_1, \dots, i_m$. Denote i_0 the smallest integer in K different from $i_1, \dots, i_m$ (as $m < |K|$, such an integer necessarily exists). Then one has a bijection:

$$\left\{ \begin{array}{c} \text{lists of subsets} \\ \delta_1, \dots, \delta_m \text{ of even size such} \\ \text{that, for all } h \leq m, i_h \in \delta_h \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{lists of subsets} \\ \delta_1, \dots, \delta_m \text{ of odd size such} \\ \text{that, for all } h \leq m, i_h \in \delta_h \end{array} \right\}$$
$$(\delta_1, \dots, \delta_m) \mapsto (\delta_1 \nabla \{i_0\}, \dots, \delta_m \nabla \{i_0\}),$$

where ∇ is the symmetric difference operator. This bijection implies that the summand $\alpha^m c_{i_1} \dots c_{i_m} C^{2^{\ell-2}-m}$ appears as many times in R_{ev} as in R_{odd}. Finally, in the difference $R_{\text{ev}} - R_{\text{odd}}$, terms corresponding to values of m smaller than $|K|$ cancel each other and one has

$$R_{\text{ev}} - R_{\text{odd}} = O\big(\alpha^{|K|}\big). \qquad \square$$

We recall that for a subset I of $[r]$ we set

$$\lambda^I := \bigoplus_{i \in I} \lambda^i.$$

Proposition 3.6. *Fix some partitions λ^1, …, λ^r and for a subset I of $[r]$ set $u_I = h_\alpha\left(\lambda^I\right)$. The family (u_I) has the strong factorization, and hence, the small cumulant properties.*

Proof. As above, it is enough to prove that $T_{[r]}(\boldsymbol{u}) = O(\alpha^{r-1})$.

Fix some subset $I = \{i_1, \cdots, i_t\}$ of $[r]$ with $i_1 < \cdots < i_t$. Observe that the Young diagram λ^I can be constructed by sorting the columns of the diagrams λ^{i_1},

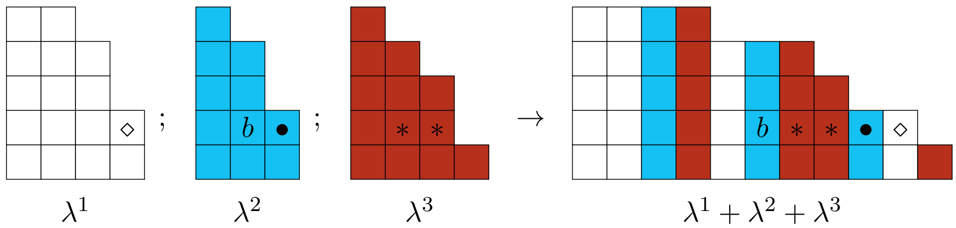

Figure 2. The diagram of an entry-wise sum of partitions.

..., λ^{i_t} in decreasing order of their length. When several columns have the same lengths, we put first the columns of λ^{i_1}, then those of λ^{i_2} and so on; see Fig. 2 (at the moment, please disregard symbols in boxes). This gives a way to identify boxes of λ^I with boxes of the diagrams λ^{i_s} $(1 \leq s \leq t)$ that we shall use below.

With this identification, if $b = (r, c)$ is a box in λ^g for some $g \in I$, its leg-length in λ^I is the same as in λ^g. We denote it by $\ell(b)$.

At the opposite, the arm length of b in λ^I may be bigger than the one in λ^g. We denote these two quantities by $a_I(b)$ and $a_g(b)$. Let us also define, $a_i(b)$ for $i \neq g$ in I, as

- for $i < g$, $a_i(b)$ is the number of boxes b' in the r-th row of λ^i such that the size of the column of b' is **smaller than** the size of the column of b (*e.g.*, on Fig. 2, for $i = 1$, these are boxes with a diamond);
- for $i > g$, $a_i(b)$ is the number of boxes b' in the r-th row of λ^i such that the size of the column of b' is **at most** the size of the column of b (*e.g.*, on Fig. 2, for $i = 3$, these are boxes with an asterisk).

Looking at Fig. 2, it is easy to see that

$$a_I(b) = \sum_{i \in I} a_i(b). \tag{27}$$

Therefore, for $G \subseteq [r]$, one has:

$$u_G = h_\alpha\left(\bigoplus_{g\in G} \lambda^g\right) = \prod_{g \in G}\left[\prod_{b\in\lambda^g} \ell(b) + 1 + \alpha\left(a_G(b)\right)\right].$$

From the definition of $T_{[r]}(\boldsymbol{u})$, given by Eq. (22), we get:

$$\begin{aligned} 1 + T_{[r]}(\boldsymbol{u}) &= \prod_{G\subseteq[r]}\prod_{g\in G}\left[\prod_{b\in\lambda^g} \ell(b) + 1 + \alpha\left(a_G(b)\right)\right]^{(-1)^{r-|G|}} \\ &= \prod_{g\in[r]}\prod_{b\in\lambda^g}\left[\prod_{\substack{G\subseteq[r]\\ G\ni g}} \left(\ell(b) + 1 + \alpha\left(a_G(b)\right)\right)^{(-1)^{r-|G|}}\right]. \end{aligned} \tag{28}$$

The expression inside the bracket corresponds to $1 + T_{[r]\setminus\{g\}}(\boldsymbol{v}^b)$, where $\boldsymbol{v}^b$ is defined as follows: if I is a subset of $[r] \setminus \{g\}$, then

$$v_I^b = \ell(b) + 1 + \alpha\,(a_{I\cup\{g\}}(b)).$$

From Eq. (27), we observe that v_I^b is as in Lemma 3.5 with the following values of the parameters: $K = [r] \setminus \{g\}$, $C = \ell(b) + 1 + \alpha\, a_g(b)$, and $c_i = a_i(b)$ for $i \neq g$. Therefore we conclude

$$T_{[r]\setminus\{g\}}(\boldsymbol{v}^b) = O(\alpha^{r-1}).$$

Going back to Eq. (28), we have:

$$1 + T_{[r]}(\boldsymbol{u}) = \prod_{g\in[r]} \prod_{b\in\lambda^g} (1 + T_{[r]\setminus\{g\}}(\boldsymbol{v}^b)) = 1 + O(\alpha^{r-1}),$$

which completes the proof. □

Let us now look at the second hook-polynomial h'_α. If we try to follow the same argument as above, we want to apply Lemma 3.5 with $K = [r] \setminus \{g\}$, $C = \ell(b) + \alpha(1 + a_g(b))$, and $c_i = a_i(b)$ for $i \neq g$. Note, however, that if the box b has leg-length 0, then $C = 0$ for $\alpha = 0$, and in this case the hypothesis $C^{-1} = O(1)$ of Lemma 3.5 is not fulfilled. To overcome this difficulty, we define

$$h''_\alpha(\lambda) = \prod_{\substack{\Box\in\lambda\\ \ell(\Box)\neq 0}} (\alpha a(\Box) + \ell(\Box) + \alpha)\,.$$

By definition, the top-most box of each column of a diagram λ has leg-length 0. Moreover λ has $m_i(\lambda^t)$ columns of height i, thus the arm-length of the top-most boxes of these columns are 0, 1,…, $m_i(\lambda^t) - 1$ respectively. Finally

$$\prod_{\substack{\Box\in\lambda\\ \ell(\Box)= 0}} (\alpha a(\Box) + \ell(\Box) + \alpha) = \alpha^{\lambda_1} \prod_i m_i(\lambda^t)!,$$

so that

$$h'_\alpha(\lambda) = \alpha^{\lambda_1} \left(\prod_i m_i(\lambda^t)! \right) h''_\alpha(\lambda). \tag{29}$$

Besides, the exact same proof than for h_α yields the following result:

Proposition 3.7. *Fix some partitions $\lambda^1, \ldots, \lambda^r$ and, for a subset I of $[r]$, set $v_I = h''_\alpha\left(\lambda^I\right)$. The family (v_I) has the strong factorization, and hence, the small cumulant properties.*

4. STRONG FACTORIZATION PROPERTY OF JACK POLYNOMIALS

Let us fix partitions $\lambda^1, \ldots, \lambda^r$, and for any subset $I \subseteq [r]$ we define $u_I := J_{\lambda^I}$. The purpose of this section is to prove Theorem 1.4, namely that $\kappa^J(\lambda^1, \cdots, \lambda^r) = \kappa_{[r]}(\boldsymbol{u}) = O(\alpha^{r-1})$, using above notation. We start with some preliminary results.

4.1. **Preliminary results.**

Proposition 4.1. *For any partitions $\lambda^1, \dots, \lambda^r$ the cumulant of Jack polynomials has a monomial expansion of the following form*

$$\kappa^J(\lambda^1, \cdots, \lambda^r) = \sum_{\mu < \lambda^{[r]}} c_\mu^{\lambda^1,\dots,\lambda^r} m_\mu + O(\alpha^{r-1}),$$

where the coefficients $c_\mu^{\lambda^1,\dots,\lambda^r}$ are polynomials in α.

Proof. First, observe that for any partitions ν^1 and ν^2, one has

$$m_{\nu^1} m_{\nu^2} = m_{\nu^1 \oplus \nu^2} + \sum_{\mu < \nu^1 \oplus \nu^2} b_\mu^{\nu^1,\nu^2} m_\mu,$$

for some integers $b_\mu^{\nu^1,\nu^2}$.

Fix partitions $\lambda^1, \dots, \lambda^r$ and a set partition $\pi = \{\pi_1, \cdots, \pi_s\} \in \mathcal{P}([r])$. Note that $\lambda^{\pi_1} \oplus \cdots \oplus \lambda^{\pi_s} = \lambda^{[r]}$. Thanks to Eq. (11) and the above observation on products of monomials, there exist coefficients $d_\mu^{\lambda^{\pi_1},\cdots,\lambda^{\pi_s}} \in \mathbb{Q}[\alpha]$ such that:

$$J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} = h_\alpha(\lambda^{\pi_1}) \cdots h_\alpha(\lambda^{\pi_s}) m_{\lambda^{[r]}} + \sum_{\mu < \lambda^{[r]}} d_\mu^{\lambda^{\pi_1},\cdots,\lambda^{\pi_s}} m_\mu.$$

As a consequence, there exist coefficients $c_\mu^{\lambda^1,\dots,\lambda^r} \in \mathbb{Q}[\alpha]$ such that

$$\kappa^J(\lambda^1, \cdots, \lambda^r) = \kappa_{[r]}(\boldsymbol{v}) m_{\lambda^{[r]}} + \sum_{\mu < \lambda^{[r]}} c_\mu^{\lambda^1,\dots,\lambda^r} m_\mu,$$

where $v_I = h_\alpha\left(\lambda^I\right)$. Proposition 3.6 completes the proof. □

For any positive integer r and for any partitions $\lambda^1, \dots, \lambda^r$ we define

$$\mathrm{InEx}(\lambda^1, \dots, \lambda^r) := \sum_{I \subseteq [r]} (-1)^{r-|I|} b\left(\lambda^I\right). \tag{30}$$

Proposition 4.2. *Let $r \geq 3$ be a positive integer. Then, for any partitions $\lambda^1, \dots, \lambda^r$ one has:*

$$\mathrm{InEx}(\lambda^1, \cdots, \lambda^r) = 0.$$

Proof. Expanding the definition and completing partitions with zeros, we have:

$$\mathrm{InEx}(\lambda^1, \dots, \lambda^r) = \sum_{j \geq 1} \sum_{I \subseteq [r]} (-1)^{r-|I|} \binom{\lambda^I_j}{2}.$$

In particular, it is enough to prove that the summand corresponding to any given $j \geq 1$ is equal to 0. In other terms, we can restrict ourselves to the case where $\lambda^i = (\lambda^i_1)$ has only one part.

In this case, $\mathrm{InEx}(\lambda^1, \cdots, \lambda^r)$ is a symmetric polynomial in $\lambda^1_1, \dots \lambda^r_1$ of degree 2 without constant term. Moreover, its coefficients are given by:

$$-\left[\lambda^1_1\right] \mathrm{InEx}(\lambda^1, \cdots, \lambda^r) = \left[\left(\lambda^1_1\right)^2\right] \mathrm{InEx}(\lambda^1, \cdots, \lambda^r) = \sum_{[1] \subseteq I \subseteq [r]} \frac{(-1)^{r-|I|}}{2} = 0,$$

and

$$\left[\lambda^1_1 \cdot \lambda^2_1\right] \mathrm{InEx}(\lambda^1, \cdots, \lambda^r) = \sum_{[2] \subseteq I \subseteq [r]} (-1)^{r-|I|} = 0.$$

This completes the proof. □

Let us now define two functions that will be of great importance in the proof of Theorem 1.4:

$$A_1(\lambda^1, \dots, \lambda^r) := \sum_{\pi \in \mathcal{P}([r])} \left[\mu(\pi, \{[r]\}) \left(\sum_{B \in \pi} b(\lambda^B) \right) \prod_{B \in \pi} J_{\lambda^B} \right], \tag{31}$$

$$A_2(\lambda^1, \dots, \lambda^r) := \sum_{\pi \in \mathcal{P}([r]); \#(\pi) \geq 2} \mu(\pi, \{[r]\}) D_{1,2}\left(J_{\lambda^B} : B \in \pi\right), \tag{32}$$

where $D_{1,2}$ is an operator defined as follows:
(33)

$$D_{1,2}(f_1, \dots, f_k) := \sum_{1 \leq m \leq N} \sum_{1 \leq i < j \leq k} f_1 \cdots \left(x_m \frac{\partial}{\partial x_m} f_i \right) \cdots \left(x_m \frac{\partial}{\partial x_m} f_j \right) \cdots f_k.$$

Lemma 4.3. *For any positive integer $r \geq 2$ and any partitions $\lambda^1, \dots, \lambda^r$, a following equality holds true:*

$$A_1(\lambda^1, \dots, \lambda^r) \;=\; b\left(\lambda^{[r]}\right) \kappa_{[r]}(\boldsymbol{u}) + \frac{1}{2} \sum_{\emptyset \subsetneq I \subsetneq [r]} \mathrm{InEx}\left(\lambda^I, \lambda^{I^c}\right) \kappa_I(\boldsymbol{u}) \kappa_{I^c}(\boldsymbol{u}),$$

where I^c denotes the complement $[r] \setminus I$ of I in $[r]$.

Proof. Let us substitute Eq. (20) into definition of A_1 — Eq. (31) — to obtain the following identity

$$A_1(\lambda^1, \dots, \lambda^r) = \sum_{\sigma \in \mathcal{P}([r])} \left(\sum_{\pi \in \mathcal{P}([r]); \pi \geq \sigma} \mu(\pi, \{[r]\}) \left[\sum_{B \in \pi} b(\lambda^B) \right] \right) \prod_{B \in \sigma} \kappa_B(\boldsymbol{u}).$$

Fix a set partition $\sigma \in \mathcal{P}([r])$. We claim that

$$\sum_{\pi \in \mathcal{P}([r]); \pi \geq \sigma} \mu(\pi, \{[r]\}) \left[\sum_{B \in \pi} b(\lambda^B) \right] = \mathrm{InEx}\left(\lambda^B : B \in \sigma\right). \tag{34}$$

Let us order blocks of σ in some way $\sigma = \{B_1, \dots, B_{\#(\sigma)}\}$. Partitions π coarser than σ are in bijection with partitions of the blocks of σ, that is partitions of $[\#(\sigma)]$. Therefore the left-hand side of (34) can be rewritten as:

$$\sum_{\pi \in \mathcal{P}([r]); \pi \geq \sigma} \mu(\pi, \{[r]\}) \left[\sum_{B \in \pi} b(\lambda^B) \right] = \sum_{\rho \in \mathcal{P}([\#(\sigma)])} \mu(\rho, \{[\#(\sigma)]\}) \left[\sum_{C \in \rho} b\left(\bigoplus_{j \in C} \lambda^{B_j} \right) \right].$$

Fix some subset C of $[\#(\sigma)]$. The coefficient of $b\left(\bigoplus_{j\in C}\lambda^{B_j}\right)$ in the above sum is

$$a_C := \sum_{\substack{\rho\in\mathcal{P}([\#(\sigma)])\\ C\in\rho}} \mu(\rho,\{[\#(\sigma)]\}).$$

Set-partitions ρ of $[\#(\sigma)]$ that have C as a block write uniquely as $C\cup\rho'$, where ρ' is a set partition of $[\#(\sigma)]\setminus C$). Thus

$$\begin{aligned} a_C = &\sum_{\rho'\in\mathcal{P}([\#(\sigma)]\setminus C)} \mu(C\cup\rho',\{[\#(\sigma)]\}) \\ &= \sum_{0\le i\le \#(\sigma)-|C|} S(\#(\sigma)-|C|,i)i!(-1)^i = (-1)^{\#(\sigma)-|C|}, \end{aligned}$$

where $S(n,k)$ is the *Stirling number of the second kind* and the last equality comes from the relation

$$\sum_{0\le k\le n} S(n,k)(x)_k = x^n$$

evaluated at $x=-1$ — here, $(x)_k := x(x-1)\cdots(x-k+1)$ denotes the falling factorial. This finishes the proof of Eq. (34).

This also completes the proof of the lemma by noticing that the right hand side of Eq. (34) vanishes for all set partitions σ such that $\#(\sigma)\ge 3$, which is ensured by Proposition 4.2. □

Lemma 4.4. *For any positive integer $r\ge 2$ and any partitions $\lambda^1,\dots,\lambda^r$, a following equality holds true*

(35)
$$A_2(\lambda^1,\dots,\lambda^r) = -\frac{1}{2}\sum_{1\le m\le N}\sum_{\emptyset\subsetneq I\subsetneq[r]} \left(x_m\frac{\partial}{\partial x_m}\kappa_I(\boldsymbol{u})\right)\left(x_m\frac{\partial}{\partial x_m}\kappa_{I^c}(\boldsymbol{u})\right).$$

Proof. Let us call RHS the right-hand side of Eq. (35). Using the definition of cumulants and Leibniz rule for the operator $x_m\frac{\partial}{\partial x_m}$ we get

$$-2\,\mathrm{RHS} = \sum_{\substack{\emptyset\subsetneq I\subsetneq[r]\\ \pi^1\in\mathcal{P}(I),\pi^2\in\mathcal{P}(I^c)}} \mu(\pi^1,\{I\})\mu(\pi^2,\{I^c\})\left(\sum_{\substack{B^1\in\pi^1\\ B^2\in\pi^2}} V_{B^1,B^2;C_1,\dots,C_s}\right),$$

where $C_1,\cdots,C_s$ are the blocks of π^1 and π^2 distinct from B^1 and B^2 and

$$V_{B^1,B^2;C_1,\dots,C_s} = \sum_{1\le m\le N}\left(x_m\frac{\partial}{\partial x_m}u_{B^1}\right)\left(x_m\frac{\partial}{\partial x_m}u_{B^2}\right)\left(\prod_{i=1}^{s}u_{C_i}\right).$$

Fix some partition $\{B^1,B^2,C_1,\cdots,C_s\}$ with two marked blocks B^1 and B^2 (the order of two marked blocks matters) and consider the coefficient of $V_{B^1,B^2;C_1,\dots,C_s}$ in $-2\,\mathrm{RHS}$. Pairs of set partitions (π^1,π^2) contributing to this coefficient are obtained as follows: take a subset J of $[s]$ and set

$$\pi^1 = B^1\cup\{C_j, j\in J\},\ \pi^2 = B^2\cup\{C_j, j\in[s]\setminus J\}.$$

Then $\mu(\pi^1, \{I\}) = (-1)^{|J|}(|J|)!$ and $\mu(\pi^2, \{I^c\}) = (-1)^{s-|J|}(s-|J|)!$. Thus the coefficient of $V_{B^1,B^2,C_1,\dots,C_s}$ in $-2\,\mathrm{RHS}$ is

$$\sum_{J\subset[s]} (-1)^{|J|}(|J|)!(-1)^{s-|J|}(s-|J|)! = \sum_{k=0}^{s} \binom{s}{k} (-1)^k k!(-1)^{s-k}(s-k)!$$
$$= \sum_{k=0}^{s} (-1)^s s! = (-1)^s (s+1)!.$$

Finally, we get

$$\mathrm{RHS} = \frac{1}{2} \sum_{\{B^1,B^2;C_1,\cdots,C_s\}} (-1)^{s+1}(s+1)! V_{B^1,B^2;C_1,\dots,C_s}.$$

where the sum runs over set partitions $\{B^1, B^2; C_1, \cdots, C_s\}$ with two *ordered* marked blocks. Note that $(-1)^{s+1}(s+1)!$ is simply the Möbius function of the underlying set partition (forgetting the marked blocks) and that one can remove the factor $1/2$ by summing over set partitions with two *unordered* marked blocks.

On the other hand, from the definition of $D_{1,2}$ — Eq. (33) —, for any set partition π, one has:

$$D_{1,2}(u_B; B \in \pi) = \sum_{\cdots} V_{B^1,B^2;C_1,\dots,C_s},$$

where the sum runs over all ways to mark (in an unordered way) two blocks of π; the resulting marked partition is then denoted $\{B^1, B^2; C_1, \cdots, C_s\}$ as usual. Therefore, one has

$$\sum_{\pi\in\mathcal{P}([r])} \mu(\pi, \{[r]\}) D_{1,2}(u_B; B \in \pi) = \mathrm{RHS},$$

as claimed in the lemma. □

4.2. **Proof of Theorem 1.4.**

Proof of Theorem 1.4. The proof will by given by induction on r. For $r = 1$, we want to prove that Jack polynomials $J_\lambda^{(\alpha)}$ has no singularity in $\alpha = 0$. This follows, *e.g.*, from the specialization for $\alpha = 0$ given in Eq. (12). Moreover, we observed before stating Theorem 1.4 that the case $r = 2$ also follows from Eq. (12).

Let us assume that the statement holds true for all $m < r$. Notice first that, by Leibniz rule, for any $f_1, \dots, f_k \in \mathrm{Sym}$, one has the following expansions:

$$D_1 (f_1 \cdots f_k) = \sum_{1\le i\le k} f_1 \cdots (D_1 f_i) \cdots f_k;$$
$$D_2 (f_1 \cdots f_k) = \sum_{1\le i\le k} f_1 \cdots (D_2 f_i) \cdots f_k + D_{1,2}(f_1, \dots, f_k),$$

where $D_{1,2}$ is given by Eq. (33).

Fix some partitions $\lambda^1, \dots, \lambda^r$ and a set partition π of $[r]$. Then, one has

$$\begin{aligned}
D_\alpha \left(J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}}\right) &= \sum_{1\leq i\leq s} J_{\lambda^{\pi_1}} \cdots \left(D_\alpha J_{\lambda^{\pi_i}}\right) \cdots J_{\lambda^{\pi_s}} + \alpha D_{1,2}\left(J_{\lambda^{\pi_1}}, \dots, J_{\lambda^{\pi_s}}\right) \\
&= \left(\sum_{1\leq i\leq s} \left((N-1)|\lambda^{\pi_i}| - b\left((\lambda^{\pi_i})^t\right)\right)\right) J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} \\
&+ \alpha \left(\left(\sum_{1\leq i\leq s} b(\lambda^{\pi_i}) \right) J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} + D_{1,2}\left(J_{\lambda^{\pi_1}}, \dots, J_{\lambda^{\pi_s}}\right)\right) \\
&= \left((N-1)\left|\lambda^{[r]}\right| - b\left((\lambda^{[r]})^t\right)\right) J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} \\
&+ \alpha \left(\left(\sum_{1\leq i\leq s} b(\lambda^{\pi_i}) \right) J_{\lambda^{\pi_1}} \cdots J_{\lambda^{\pi_s}} + D_{1,2}\left(J_{\lambda^{\pi_1}}, \dots, J_{\lambda^{\pi_s}}\right)\right),
\end{aligned}$$

where the second equality comes from Proposition 2.2. Multiplying by the appropriate value of the Möbius function and summing over set partitions π, it gives us the following identity:

$$\begin{aligned}
D_\alpha \kappa_{[r]}(\boldsymbol{u}) = &\left((N-1)\left|\lambda^{[r]}\right| - b\left((\lambda^{[r]})^t\right)\right) \kappa_{[r]}(\boldsymbol{u}) \\
&+ \alpha \left(A_1(\lambda^1, \dots, \lambda^r) + A_2(\lambda^1, \dots, \lambda^r)\right),
\end{aligned} \tag{36}$$

where A_1 and A_2 are given by Eq. (31) and Eq. (32), respectively.

Consider the coefficient of α^j in the above expression. We have

$$\begin{aligned}
[\alpha^j] D_\alpha \kappa_{[r]}(\boldsymbol{u}) = &\left((N-1)\left|\lambda^{[r]}\right| - b\left((\lambda^{[r]})^t\right)\right) [\alpha^j]\kappa_{[r]}(\boldsymbol{u}) \\
&+ [\alpha^{j-1}] \left(A_1(\lambda^1, \dots, \lambda^r) + A_2(\lambda^1, \dots, \lambda^r)\right).
\end{aligned}$$

On the other hand, since $D_\alpha = D_1 + \alpha\, D_2$, one has

$$[\alpha^j] D_\alpha \kappa_{[r]}(\boldsymbol{u}) = D_1\left([\alpha^j]\kappa_{[r]}(\boldsymbol{u})\right) + D_2\left([\alpha^{j-1}]\kappa_{[r]}(\boldsymbol{u})\right). \tag{37}$$

Comparing both expressions, we have the following identity, which will be a key tool in the proof:

$$\begin{aligned}
&D_1\left([\alpha^j]\kappa_{[r]}(\boldsymbol{u})\right) + D_2\left([\alpha^{j-1}]\kappa_{[r]}(\boldsymbol{u})\right) \\
&\quad = \left((N-1)\left|\lambda^{[r]}\right| - b\left((\lambda^{[r]})^t\right)\right) [\alpha^j]\kappa_{[r]}(\boldsymbol{u}) \\
&\qquad + [\alpha^{j-1}] \left(A_1(\lambda^1, \dots, \lambda^r) + A_2(\lambda^1, \dots, \lambda^r)\right).
\end{aligned} \tag{38}$$

We recall that our goal is to prove that

$$[\alpha^j]\kappa_{[r]}(\boldsymbol{u}) = 0 \tag{39}$$

for any $0 \leq j \leq r-2$. We proceed by induction on j.

Consider the case $j = 0$. Since $\kappa_{[r]}(\boldsymbol{u})$, A_1 and A_2 are polynomials in α, Eq. (38) simplifies in this case to

$$D_1 f = \left((N-1)\left|\lambda^{[r]}\right| - b\left((\lambda^{[r]})^t\right)\right) f,$$

where $f = [a^0]\kappa_{[r]}(\boldsymbol{u})$. Thanks to Proposition 4.1 we know that f satisfies the assumptions of Corollary 2.3 and hence it is equal to zero.

Now, we fix $j \leq r-2$, and we assume that $[\alpha^i]\kappa_{[r]}(\boldsymbol{u}) = 0$ holds true for all $0 \leq i < j$. We are going to show that it holds true for $i = j$ as well.

Since $[\alpha^{j-1}]\kappa_{[r]}(\boldsymbol{u}) = 0$ by the inductive hypothesis, Eq. (38) reads

$$D_1\left([\alpha^j]\kappa_{[r]}(\boldsymbol{u})\right) = \left((N-1)\left|\lambda^{[r]}\right| - b\left((\lambda^{[r]})^t\right)\right)[\alpha^j]\kappa_{[r]}(\boldsymbol{u}) + [\alpha^{j-1}]\left(A_1(\lambda^1,\dots,\lambda^r) + A_2(\lambda^1,\dots,\lambda^r)\right).$$

First, we claim that $[\alpha^{j-1}]A_1(\lambda^1,\dots,\lambda^r) = 0$. Indeed, from the induction hypothesis, for each subset I with $\emptyset \subsetneq I \subsetneq [r]$, one has $\kappa_I(\boldsymbol{u}) = O(\alpha^{|I|-1})$ and $\kappa_{I^c}(\boldsymbol{u}) = O(\alpha^{|I^c|-1}) = O(\alpha^{r-|I|-1})$. We then use Lemma 4.3 and write:

$$[\alpha^{j-1}]A_1(\lambda^1,\dots,\lambda^r) = b\left(\lambda^{[r]}\right)[\alpha^{j-1}]\kappa_{[r]}(\boldsymbol{u}) + \frac{1}{2}\sum_{\emptyset\subsetneq I\subsetneq[r]}[\alpha^{j-1}]\,\mathrm{InEx}\left(\lambda^I,\lambda^{I^c}\right)\kappa_I(\boldsymbol{u})\kappa_{I^c}(\boldsymbol{u}) = 0,$$

since $j-1 < r-2$.

Similarly, one can prove that $[\alpha^{j-1}]A_2(\lambda^1,\dots,\lambda^r) = 0$. Indeed, using a similar argument as before, we have

$$\frac{1}{2}\sum_{1\leq m\leq N}\sum_{\emptyset\subsetneq I\subsetneq[r]}\left(x_m\frac{\partial}{\partial x_m}\kappa_I(\boldsymbol{u})\right)\left(x_m\frac{\partial}{\partial x_m}\kappa_{I^c}(\boldsymbol{u})\right) = O(\alpha^{r-2}).$$

But, from Lemma 4.4, the left-hand side is $A_2(\lambda^1,\dots,\lambda^r)$. Since $j-1 < r-2$, we know that $[\alpha^{j-1}]A_2(\lambda^1,\dots,\lambda^r) = 0$, as wanted.

Above computations show that Eq. (38) simplifies to

$$D_1 f = \left((N-1)\left|\lambda^{[r]}\right| - b\left((\lambda^{[r]})^t\right)\right) f,$$

where $f = [a^j]\kappa_{[r]}(\boldsymbol{u})$. Again, thanks to Proposition 4.1 we know that f satisfies assumptions from Corollary 2.3 and thus it is equal to zero, which finishes the proof. □

5. POLYNOMIALITY IN b-CONJECTURE

5.1. **Cumulants and Young diagrams.** Consider a function F on Young diagrams and some diagrams $\lambda^1,\cdots,\lambda^r$. Then we consider the family defined by (recall that we use $\oplus$ for entry-wise sum of partitions):

$$u_I = F\left(\bigoplus_{i\in I}\lambda^i\right). \tag{40}$$

Definition 5.1. We say that a function G on Young diagrams has the small cumulant property if, for any $r \geq 1$ and *for any partitions* $\lambda^1, \cdots, \lambda^r$, the above-defined family has the small cumulant property.

With this notation, the results of the previous sections can be reformulate as:

Theorem 1.4: For a fixed alphabet $\boldsymbol{x}$, the function $\lambda \mapsto J_\lambda^\alpha(\boldsymbol{x})$ has the small cumulant property.

Proposition 3.6: The function h_α has the small cumulant property.

Proposition 3.7: The function h''_α has the small cumulant property.

Corollary 3.4: If G_1 and G_2 have the small cumulant properties and take non-zero values, then so have $G_1 \cdot G_2$ and G_1/G_2.

As a consequence, the function

$$\lambda \mapsto \frac{1}{h_\alpha(\lambda) h''_\alpha(\alpha)} J_\lambda^\alpha(\boldsymbol{x}) J_\lambda^\alpha(\boldsymbol{y}) J_\lambda^\alpha(\boldsymbol{z})$$

has the small cumulant property. We will use that later in this section.

Another consequence is that the function $\lambda \mapsto \frac{J_\lambda^\alpha(\boldsymbol{x})}{h_\alpha(\lambda)}$ also has the small cumulant property. We will not use this result here, but since this function is the P-normalization of Jack polynomials, we have considered relevant to mention it here.

5.2. **Cumulants and logarithm.** Let $\boldsymbol{t} = (t_1, t_2, \cdots)$ be an infinite alphabet of formal variables. We use the notation $\boldsymbol{t}^\lambda = t_{\lambda_1} \cdots t_{\lambda_r}$

Lemma 5.2. *Let F be a function on Young diagrams. Denote $\kappa^F(\lambda^1, \cdots, \lambda^r)$ the cumulant $\kappa_{[r]}(\boldsymbol{u})$, where $\boldsymbol{u}$ is defined by Eq.* (40). *Then we have the following equality of formal power series in $\boldsymbol{t}$:*

$$\log \sum_\lambda \frac{F(\lambda)}{\alpha^{\lambda_1} \prod_i m_i(\lambda^t)!} \boldsymbol{t}^{\lambda^t} = \sum_{r \geq 1} \frac{1}{r! \alpha^r} \sum_{(j_1, \cdots, j_r)} \kappa^F(1^{j_1}, \cdots, 1^{j_r}) t_{j_1} \cdots t_{j_r}.$$

Proof. Both sides expand as linear combinations of products

$$F_{\lambda^1, \dots, \lambda^s} := F(\lambda^1) \cdots F(\lambda^s) \boldsymbol{t}^{(\lambda^1)^t} \cdots \boldsymbol{t}^{(\lambda^s)^t},$$

where $\lambda^1, \dots, \lambda^s$ are partitions. Fix some partitions $\lambda^1, \dots, \lambda^s$. The coefficient of $F_{\lambda^1, \dots, \lambda^s}$ on the left-hand side is given by

$$\frac{(-1)^{s-1}}{s} \frac{s!}{|\operatorname{Aut}(\lambda^1, \cdots, \lambda^s)|} \prod_{h=1}^s \frac{1}{\alpha^{\lambda_1^h} \prod_i m_i\left((\lambda^h)^t\right)!}. \tag{41}$$

Here, $|\operatorname{Aut}(\lambda^1, \cdots, \lambda^s)|$ denotes the number of permutations σ of size s such that $\lambda^j = \lambda^{\sigma(j)}$ for all $j \leq s$.

The situation on the right-hand side is more intricate. First, rewrite it as

$$\sum_{r\geq 1} \frac{1}{r!\alpha^r} \sum_{(j_1,\cdots,j_r)} \kappa^F(1^{j_1},\cdots,1^{j_r}) t_{j_1}\cdots t_{j_r} \\ = \sum_{r\geq 1} \sum_{(j_1,\cdots,j_r)} \sum_{\pi\in\mathcal{P}([r])} \frac{\mu(\pi,\{[r]\})}{r!\alpha^r} \prod_{B\in\pi} F\left(\bigoplus_{h\in B} 1^{j_h}\right) t_{j_1}\cdots t_{j_r}. \tag{42}$$

We are interested in which summation indices contribute to the coefficient of $F_{\lambda^1,\ldots,\lambda^s}$, that is indices such that one has the following equality of the multisets

$$\left\{\bigoplus_{h\in B} 1^{j_h}, B\in\pi\right\} = \{\lambda^h, 1\leq h\leq s\}.$$

First, $(j_1,\cdots,j_r)$ should be a reordering of list of column lengths in $\lambda^1,\ldots,\lambda^s$. If m'_i denotes the number of i in this list of column lengths, there are $r!/(\prod_i m'_i!)$ such reordering and each gives the same contribution to the coefficient of $F_{\lambda^1,\ldots,\lambda^s}$. We now suppose that we have fixed such a reordering $(j_1,\cdots,j_r)$.

Let $m'_i(\lambda^j)$ denotes the number of columns of length i in λ^j. Then the number of *ordered* set partitions $(B_1,\cdots,B_s)$ of $[r]$ such that

$$\bigoplus_{b\in B_h} 1^{j_b} = \lambda^h \text{ for } 1\leq h\leq s$$

is $(\prod_i m_i!)/\left(\prod_{i,j} m'_i\left((\lambda^j)\right)\right)$. Indeed, for each value i, one has to choose $m'_i\left((\lambda^1)\right)$ entries equal to i in the list $(j_1,\cdots,j_r)$ that go in B_1, $m'_i\left((\lambda^2)\right)$ entries equal to i that go in B_2, and so on. This gives for each i a multinomial $m'_i!/\left(\prod_j m'_i\left((\lambda^j)\right)\right)$, as claimed. But we want to count (unordered) set partitions and not ordered set partitions as above, so that we should divide by $|\operatorname{Aut}(\lambda^1,\cdots,\lambda^s)|$.

All these set partitions have s blocks so that the corresponding value of the Möbius function is $\mu(\pi,\{[r]\}) = (-1)^{s-1}(s-1)!$.

Finally, the coefficient of $F_{\lambda^1,\ldots,\lambda^s}$ in Eq. (42) is

$$\frac{r!}{\prod_i m_i!} \frac{\prod_i m_i!}{\prod_{i,j} m'_i(\lambda^j)} \frac{1}{|\operatorname{Aut}(\lambda^1,\cdots,\lambda^s)|} \frac{(-1)^{s-1}(s-1)!}{r!\alpha^r}, \tag{43}$$

where r is the total number of columns in the $\lambda^1,\ldots,\lambda^s$, that is $r=\sum_h \lambda^h_1$.

Comparing Eq. (41) and Eq. (43), we get our result. □

Remark. The statement and proof of this lemma are similar to the fact that cumulants can be alternatively defined as a sum over set partitions or as coefficients in the generating series of the logarithm of the moment generating series; see, *e.g.* Eqs (3) and (II.c) in [LS59].

5.3. **Conclusion.** We have now all the tools needed to prove the polynomiality in b-conjecture.

Proof of Theorem 1.2. Thanks to Corollary 2.5, it is enough to prove that $h^\tau_{\mu,\nu}(\beta)$ has no pole in $\alpha = 0$, *i.e.* that $h^\tau_{\mu,\nu}(\beta) = O(1)$. From Eq. (1), this amounts to establish that

$$\log\left(\sum_{\tau\in\mathcal{P}} \frac{J^{(\alpha)}_\tau(\boldsymbol{x})\, J^{(\alpha)}_\tau(\boldsymbol{y})\, J^{(\alpha)}_\tau(\boldsymbol{z})\, t^{|\tau|}}{h_\alpha(\lambda) h'_\alpha(\lambda)}\right) = O(\alpha^{-1}).$$

But, using Eq. (29), we see that this quantity is the left-hand side of Lemma 5.2 for

$$F(\lambda) = \frac{1}{h_\alpha(\lambda) h''_\alpha(\alpha)} J^\alpha_\lambda(\boldsymbol{x}) J^\alpha_\lambda(\boldsymbol{y}) J^\alpha_\lambda(\boldsymbol{z}),$$

and $t_1 = t_2 = \cdots = t$. It was observed at the end of Section 5.1 that this function F has the small cumulant property. Therefore, for any $j_1, \cdots, j_r$, the cumulant $\kappa^F(1^{j_1}, \cdots, 1^{j_r})$ is $O(\alpha^{r-1})$ and, thus, the right-hand side of Lemma 5.2 is $O(\alpha^{-1})$. This finishes the proof of the polynomiality.

The bound on the degree follows from the polynomiality and work of La Croix, see [LaC09, Lemma 5.7 and Theorem 5.18]. □

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