

**NICE CONNECTING PATHS IN CONNECTED  
COMPONENTS OF SETS OF ALGEBRAIC  
ELEMENTS IN A BANACH ALGEBRA**

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*Dedicated to the 90th anniversary of Professor Miroslav Fiedler,  
from two grateful participants in mathematical olympiads*

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ABSTRACT. Generalizing earlier results about the set of idempotents in a Banach algebra, or of self-adjoint idempotents in a  $C^*$ -algebra, we announce constructions of nice connecting paths in the connected components of the set of elements in a Banach algebra, or of self-adjoint elements in a  $C^*$ -algebra, that satisfy a given polynomial equation, without multiple roots. In particular, we will prove that in the Banach algebra case every such non-central element lies on a complex line, all of whose points satisfy the given equation. We also formulate open questions.

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## 1. INTRODUCTION

Let  $A$  be a unital complex Banach algebra. Sometimes we will assume that moreover  $A$  is a  $C^*$ -algebra.

We let

$$E(A) := \{a \in A \mid a^2 = a\}$$

be the set of *idempotents* of  $A$ , and

$$S(A) := \{a \in A \mid a^2 = a = a^*\}$$

be the set of *self-adjoint idempotents* for the  $C^*$ -algebra case.

The *connected components* of  $E(A)$  and of  $S(A)$  have been investigated by many authors. To some of them we will refer later at the respective theorems. An ample literature is given in [AMMZ].

Let

$$p(\lambda) := \prod_{i=1}^n (\lambda - \lambda_i)$$

be a polynomial over  $\mathbb{C}$ , with all  $\lambda_i$ 's distinct. In the  $C^*$ -algebra case, when considering self-adjoint elements, we will assume that all  $\lambda_i$ 's are real. (In fact, if  $q(\lambda) := \prod\{(\lambda - \lambda_i) \mid 1 \leq i \leq n, \lambda_i \in \mathbb{R}\}$ , then  $p(a) = 0$  and  $a = a^*$  imply  $q(a) = 0$ . Thus below we could use  $q(\lambda)$  rather than  $p(\lambda)$ .) The  $\lambda_i$ 's are fixed throughout this paper.

We write

$$E_p(A) := \{a \in A \mid p(a) = 0\},$$

and

$$S_p(A) := \{a \in A \mid p(a) = 0, a = a^*\}$$

for the  $C^*$ -algebra case. Then  $E(A)$  and  $S(A)$  are special cases of  $E_p(A)$  and  $S_p(A)$ : namely, for  $p(\lambda) := \lambda(\lambda - 1)$ .

We say that  $\{e_1, \dots, e_n\} \subset A$  is a *partition of unity*, or in the  $C^*$ -algebra case that  $\{e_1, \dots, e_n\} \subset A$  is a *self-adjoint partition of unity*, if

$$\left\{ \begin{array}{l} \{e_1, \dots, e_n\} \subset E(A), \text{ or } \{e_1, \dots, e_n\} \subset S(A), \\ \text{and } e_i e_j = 0 \text{ for } 1 \leq i, j \leq n \text{ and } i \neq j, \\ \text{and } \sum_{i=1}^n e_i = 1. \end{array} \right.$$

The detailed proofs of the statements announced in Section 2 will be published in [MZ]. The idea of this development originates from personal conversations of the authors at the conference Operator Theory and Applications: Perspectives and Challenges, held in Jurata (Hel), Poland, March 18–28, 2010, and from the 2011 lecture by the first named author [Mak].

## 2. THEOREMS

The “only if” part of the following Proposition 1 comes from the Riesz decomposition theorem.

**Proposition 1.** *Let  $A$  be a unital complex Banach algebra ( $C^*$ -algebra). Let  $a \in A$ . Then  $a \in E_p(A)$  ( $a \in S_p(A)$ ) if and only if there exists a (self-adjoint) partition of unity  $\{e_1, \dots, e_n\}$  such that*

$$a = \sum_{i=1}^n \lambda_i e_i.$$

*In the “only if” part, for  $a \in E_p(A)$  (for  $a \in S_p(A)$ ) one can choose the  $e_i$ ’s as polynomials of  $a$ , with complex (real) coefficients, which depend only on the  $\lambda_i$ ’s.*

This representation provides the tool for reducing questions about  $E_p(A)$  (about  $S_p(A)$ ) to those about  $E(A)$  (about  $S(A)$ ). Of course, for the respective proofs for  $E_p(A)$  (for  $S_p(A)$ ) one has to work still substantially. As an illustration, we include a sketch of proof of Theorem 7 in Section 3.

The distinctness of the  $\lambda_i$ ’s is essential in order that  $a$  should have such a simple form. For  $T \in A := B(l^2 \oplus l^2)$ , having a block matrix form  $(T_{ij})_{i,j=1}^2$ , which is subdiagonal (i.e., strictly lower triangular), we have  $T^2 = 0$ , but  $T_{21} \in B(l^2)$  can be as complicated as an element of  $B(l^2)$  can be.

A *path in a topological space  $X$*  is a continuous map  $f : [0, 1] \rightarrow X$ . We will say that  $f(0), f(1) \in X$  are *connected by this path  $f$* . By a small abuse of language we will also say that  $f([0, 1]) \subset X$  is a *path in  $X$*  (e.g., for polygonal paths). A topological space  $X$  is *pathwise connected* if any two of its points are connected by a path in  $X$ . A topological space  $X$  is *locally pathwise connected* if each point  $x \in X$  has a base of (not necessarily open) neighbourhoods consisting of pathwise connected sets.

**Theorem 2.** *Let  $A$  be a unital complex Banach algebra and  $C$  a connected component of  $E_p(A)$ . Then  $C$  is a relatively open subset of  $E_p(A)$ . Further,  $C$  is locally pathwise connected via each of the following types of paths:*

- 1) *similarity via an exponential function, i.e.,  $t \mapsto e^{-ct} a e^{ct}$ ;*
- 2) *a polynomial path of degree at most three;*
- 3) *a polygonal path of  $n$  segments.*

For  $E(A)$ , relative openness of  $C$  was proved by J. Zemánek [Ze], 1) was proved by J. Zemánek [Ze], 2) was proved by J. Esterle [Es] and M. Trémon [Tr85], 3) was proved by Z. V. Kovarik [Ko] (cf. also [Ze]).

**Theorem 3.** *Under the hypotheses of Theorem 2,  $C$  is pathwise connected via each of the following types of paths:*

1) *similarity via a finite product of exponential functions, i.e.,  $t \mapsto e^{-c_m t} \dots e^{-c_1 t} a e^{c_1 t} \dots e^{c_m t}$ ;*

2) *a polynomial path;*

3) *a polygonal path.*

*In fact, there is a path satisfying 1) and 2) simultaneously.*

For  $E(A)$ , 1) was proved by J. Zemánek [Ze], 2) was proved by J. Esterle [Es] and M. Trémon [Tr85], 3) was proved by Z. V. Kovarik [Ko] (cf. also [Ze]), and the last sentence was proved by [Es] and [Tr85].

**Problem.** Does there exist a uniform bound on the “minimum degree” of these polynomial connections, possibly depending on  $n$ , valid for all Banach algebras? Does such a bound exist, depending on  $n$  and on  $A$  (or even on  $C$ )? Even the case of a uniform bound for polynomial connections of idempotents is open, even if we allow dependence of the bound on  $A$  (or even on  $C$ ). For some particular cases, see [Tr85] and [MZ89]. ([Tr95] announced a further partial result, but his proof seems to be incorrect.)

Even the “simplest” case  $A := B(l^2)$  is open. (The case  $A =: B(\mathbb{C}^n)$  is solved positively by [Tr85], the uniform bound being 3, which is sharp. Here the connected components of  $E(A)$  consist of the projections of the same rank.) For  $A = B(l^2)$ , the connected components of  $E(A)$  are  $\{e \in A \mid \dim N(e) = \alpha, \dim R(e) = \beta\}$ , where  $0 \leq \alpha, \beta \leq \aleph_0$  are cardinalities with  $\alpha + \beta = \aleph_0$ , cf. [AMMZ] ( $N(\cdot)$  is the null-space and  $R(\cdot)$  is the range). By [MZ89], for  $\min\{\alpha, \beta\} < \aleph_0$ , in the respective connected component there exists an at most third degree polynomial path between any two elements of that component. But even the case  $\alpha = \beta = \aleph_0$  here is open.

**Theorem 4.** *Let  $A$  be a unital complex  $C^*$ -algebra, and  $C$  a connected component of  $S_p(A)$ . Then  $C$  is a relatively open subset of  $S_p(A)$ . Further,  $C$  is locally pathwise connected by similarities via exponential functions, i.e.,  $t \mapsto e^{-ict} a e^{ict}$ , where additionally  $c = c^*$ .*

For  $S(A)$ , Theorem 4 was proved by S. Maeda [Mae] (cf. also [Ze]).

**Theorem 5.** *Under the hypotheses of Theorem 4,  $C$  is pathwise connected by similarities via finite products of exponential functions, i.e.,  $t \mapsto e^{-ic_m t} \dots e^{-ic_1 t} a e^{ic_1 t} \dots e^{ic_m t}$ , where additionally  $c_1 = c_1^*, \dots, c_m = c_m^*$ .*

For  $S(A)$ , Theorem 5 was proved by S. Maeda [Mae] (cf. also [Ze]).

For the  $C^*$ -algebra case, the analogues of 2) and 3) from Theorems 2 and 3 are false for  $S_p(A)$ . In fact, already the connected component of  $S(B(\mathbb{C}^2))$  consisting of all rank-one orthogonal projections does not contain any non-constant polynomial path. (The connected components of  $S(B(\mathbb{C}^n))$  consist of the orthogonal projections of the same rank.)

**Theorem 6.** *Let  $A$  be a unital complex Banach algebra ( $C^*$ -algebra). Let  $a \in E_p(A)$  (let  $a \in S_p(A)$ ). Then  $a$  belongs to the centre of  $A$  if and only if its connected component in  $E_p(A)$  (in  $S_p(A)$ ) is  $\{a\}$ .*

Theorem 6 for  $E(A)$  was proved by J. Zemánek [Ze], for  $S(A)$  by S. Maeda [Mae]. In Theorem 6, of course, the “only if” part for  $S_p(A)$  follows from the “only if” part for  $E_p(A)$ .

**Theorem 7.** *Let  $A$  be a unital complex Banach algebra, and  $C$  a connected component of  $E_p(A)$ . If  $C$  is disjoint from the centre of  $A$ , then any element of  $C$  belongs to a complex line entirely contained in  $C$ . In particular,  $C$  is unbounded.*

For  $E(A)$ , Theorem 7 was proved by J. Zemánek [Ze].

In the  $C^*$ -algebra case even the entire  $S_p(A)$  has a distance  $\max\{|\lambda_i| \mid 1 \leq i \leq n\}$  from 0, so the analogue of Theorem 7 for  $S_p(A)$  is false.

Theorem 6 and Theorem 7 yield the next Corollary 8.

**Corollary 8.** *Let  $A$  be a unital complex Banach algebra. Then  $E_p(A)$  is a union of its isolated points and of complex lines. ■*

**Theorem 9.** *There exists an explicit constant  $c(\lambda_1, \dots, \lambda_n) > 0$  (depending on  $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ , and invariant under any map  $(\lambda_1, \dots, \lambda_n) \mapsto (a + b\lambda_1, \dots, a + b\lambda_n)$  with  $a, b \in \mathbb{R}$  and  $b \neq 0$ ) such that the following holds. If  $A$  is a unital complex  $C^*$ -algebra, and  $C_1, C_2$  are distinct connected components of  $S_p(A)$ , then the distance of  $C_1$  and  $C_2$  is at least  $c(\lambda_1, \dots, \lambda_n) \cdot \min\{|\lambda_i - \lambda_j| \mid 1 \leq i, j \leq n, i \neq j\}$ .*

**Conjecture.** Let  $A$  be a unital complex Banach algebra ( $C^*$ -algebra) and  $C_1, C_2$  distinct connected components of  $E_p(A)$  (of  $S_p(A)$ ). Then the distance of  $C_1$  and  $C_2$  is at least  $\min\{|\lambda_i - \lambda_j| \mid 1 \leq i, j \leq n, i \neq j\}$ .

For  $n = 2$  this conjecture is equivalent to the statement that this distance for  $E_p(A) := E(A)$  (for  $S_p(A) := S(A)$ ) is at least 1, which is due to J. Zemánek [Ze] (due to S. Maeda [Mae]). For  $n \geq 3$  we do not even know whether this distance for the Banach algebra case is positive.

If true, this conjecture would be sharp, for any Banach algebra: consider  $\lambda_i \cdot 1$  and  $\lambda_j \cdot 1$ .

The Conjecture for the case of  $S_p(A)$  would follow from the Conjecture in the case of  $E_p(A)$ . In fact, different connected components of  $S_p(A)$  are subsets of different connected components of  $E_p(A)$ , by [BFML], Section 1, Applications, 2), also taking into consideration our Proposition 1 and Theorem 3.

### 3. A PROOF

*Proof of Theorem 7 from Theorem 3 and Theorem 6.* If  $C$  is disjoint from the centre, then by Theorem 6 it has more than one elements. Let  $a_0 \in C$  be an arbitrary element of  $C$ , and let  $a_1 \in C$ , with  $a_1 \neq a_0$ . Then, by Theorem 3, 3), there exists a non-constant polygonal path connecting  $a_0$  to  $a_1$  in  $C$ . Its first non-constant segment (counted from  $a_0$ ) is the graph of a non-constant polynomial of degree 1, say of

$$\lambda \mapsto a_0 + b\lambda, \text{ from } [0, 1] \text{ to } C (\subset E_p(A) \subset A).$$

Hence

$$(1) \quad b \neq 0 \text{ and we have for all } \lambda \in [0, 1] \text{ identically } p(a_0 + b\lambda) = 0.$$

Then the equation in (1) is a polynomial equation, with coefficients from  $A$  and of degree at most  $n$ , for  $\lambda \in \mathbb{C}$ . (Attention: here the coefficient of  $\lambda^n$  is  $b^n$ , which may be 0 even for  $b \neq 0$ .)

*We make an indirect assumption. If the polynomial*

$$(2) \quad \mathbb{C} \ni \lambda \mapsto p(a_0 + b\lambda) \in A$$

*were not identically 0 for all  $\lambda \in \mathbb{C}$ , then for some  $\lambda_0 \in \mathbb{C}$  we would have*

$$p(a_0 + b\lambda_0) \neq 0.$$

Then for some continuous linear functional  $a'$  on  $A$  we would have

$$\langle p(a_0 + b\lambda_0), a' \rangle \neq 0.$$

The polynomial

$$(3) \quad \mathbb{C} \ni \lambda \mapsto \langle p(a_0 + b\lambda), a' \rangle \in \mathbb{C}$$

is a  $\mathbb{C}$ -valued polynomial on  $\mathbb{C}$  of degree at most  $n$ , which would not vanish at  $\lambda_0 \in \mathbb{C}$ . Hence *the polynomial in (3) would have at most  $n$  distinct roots.*

However, by (1) we have that *the polynomial in (3) vanishes for all  $\lambda \in [0, 1]$  identically.* This is a contradiction, showing that our indirect assumption is false.

That is, the polynomial in (2) is identically 0 for all  $\lambda \in \mathbb{C}$ . In other words, for all  $\lambda \in \mathbb{C}$  we have

$$p(a_0 + b\lambda) = 0, \quad \text{i.e., } a_0 + b\lambda \in E_p(A),$$

which implies by connectedness of  $\mathbb{C}$  that for all  $\lambda \in \mathbb{C}$  we have even

$$a_0 + b\lambda \in C.$$

Since by (1)  $b \neq 0$ , we see that

$C$  contains a complex line passing through its arbitrary point  $a_0$ .

■

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