

# Optimal anisotropic meshes and stabilized parameters for the stabilized finite element method

Yana Di\* Hehu Xie<sup>†</sup> Xiaobo Yin<sup>‡</sup>

## Abstract

We propose a numerical method to generate the anisotropic meshes and select the appropriate stabilized parameters simultaneously for convection diffusion equations by stabilized continuous linear finite elements. Since the discretized error in a suitable norm can be bounded by the sum of interpolation error and its variants in different norms, we replace them by some terms which contain the Hessian matrix of the true solution, convective fields, and the geometric properties such as directed edges and the area of the triangle. Based on this observation, the shape, size and equidistribution requirements are used to derive the corresponding metric tensor and the stabilized parameters. It is easily found from our derivation that the optimal stabilized parameter is coupled with the optimal metric tensor on each element. Some numerical results are also provided to validate the stability and efficiency of the proposed numerical method.

**Keywords.** Metric tensor; anisotropic; convection-diffusion equation, stabilized parameter, stabilized finite element method.

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\*LSEC, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China ([yandi@lsec.cc.ac.cn](mailto:yandi@lsec.cc.ac.cn)). This work is supported in part by National Science Foundations of China (NSFC 11271358) and the National Center for Mathematics and Interdisciplinary Science, CAS.

<sup>†</sup>LSEC, NCMIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China ([hxie@lsec.cc.ac.cn](mailto:hxie@lsec.cc.ac.cn)). This work is supported in part by National Science Foundations of China (NSFC 91330202, 11371026, 11001259, 2011CB309703) and the National Center for Mathematics and Interdisciplinary Science, CAS.

<sup>‡</sup>Corresponding author, School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, China, ([yinxb@mail.ccnu.edu.cn](mailto:yinxb@mail.ccnu.edu.cn)). This work was supported in part by the National Natural Science Foundation of China (11201167).

# 1 Introduction

This paper is concerned with the finite element solution of the following scalar convection-diffusion equation

$$\begin{cases} -\varepsilon\Delta u + \mathbf{b} \cdot \nabla u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathcal{R}^2$  is a bounded polygonal domain with boundary  $\partial\Omega$ ,  $\varepsilon > 0$  is the constant diffusivity,  $\mathbf{b} \in [W^{1,\infty}(\Omega)]^2$  is the given convective field satisfying the incompressibility condition  $\nabla \cdot \mathbf{b} = 0$  in  $\Omega$ ,  $f \in L^2(\Omega)$  is the source function, and  $g \in H^{1/2}(\partial\Omega)$  represents the Dirichlet boundary condition.

Despite the apparent simplicity of problem (1.1), its numerical solution become particularly challenging when convection dominates diffusion (i.e., when  $\varepsilon \ll \|\mathbf{b}\|$ ). In such cases, the solution usually exhibits very thin layers across which the derivatives of the solution are large. The widths of these layers are usually significantly smaller than the mesh size and hence the layers can be hardly resolved. As a result of this, on meshes which do not resolve the layers, standard Galerkin finite element methods have poor stability and accuracy properties.

To enhance the stability and accuracy of the Galerkin discretization of (1.1) in the convection-dominated regime, various stabilization strategies have been developed. Examples are upwind scheme [16], streamline diffusion finite element method (SDFEM), also known as streamline-upwind/Petrov-Galerkin formulation (SUPG) [9, 18], the Galerkin/Least-squares method (GLS) [19], residual free bubbles (RFB) functions [5, 6, 10], exponential fitting [3, 7, 33], discontinuous Galerkin methods [4, 17], and spurious oscillations at layers diminishing (SOLD) methods (also known as shock capturing methods) [21–23]. we refer to the monograph [30] for an extensive survey of the literature.

However, if uniform meshes are used for stabilized finite element method, oscillations still exist near the layers in some cases although very fine meshes are used [20]. Hence, it is more appropriate to generate adaptively anisotropic meshes to capture the layers. There are some recent efforts directed at constructing adaptive anisotropic meshes which combine a stabilized scheme and some mesh modification strategies. For example, the resolution of boundary layers occurring in the singularly perturbed case is achieved using anisotropic mesh refinement in boundary layer regions [2], where the actual choice of the element diameters in the refinement zone and the determination of the numerical damping parameters is addressed. In [29] an adaptive meshing algorithm is designed by combining SUPG method, an adapted metric tensor and an anisotropic centroidal Voronoi tessellation algorithm, which is shown to be robust in detecting layers and efficient in avoiding non-physical oscillations in the numerical approximation. Sun et al. [31] develop a multilevel-homotopic-adaptive finite element method (MHAFEM) by combining SDFEM, anisotropic mesh adaptation, and the homotopy of the diffusion coefficient. The authors use numerical experiments to demonstrate that MHAFEM can efficiently capture boundary or interior

layers and produce accurate solutions.

This list is by no means exhausted, and there are many efforts to construct adaptive anisotropic meshes by combining a stabilized scheme and some mesh modification strategies. However, so far as we know, there are still two key problems which haven't been solved in rigorous ways.

First, although there are many results on optimal anisotropic meshes for minimizing the interpolation error and also the discretized error of finite element method for solving the Laplace equation due to Céa's lemma, its extension to the discretized error of stabilized finite element method for the convection-dominated convection-diffusion equation is not clear. So far as we know most results on optimal anisotropic meshes for the stabilized finite element method applied to the convection-dominated convection-diffusion equation is just the same with that for the interpolation error. In fact, this strategy is not optimal, which will be illustrated experimentally later in this paper.

Second, there is a crucial factor to make the stabilized finite element method successful, that is, the proper selection of the stabilization parameter  $\alpha_K$  on element  $K$ . The standard choice for quasi-uniform triangulations is ([30, P. 305-306])

$$\alpha_K = \begin{cases} \alpha_0 h_K & Pe_K > 1, \text{ (convection-dominated case)} \\ \alpha_1 h_K^2 / \varepsilon & Pe_K \leq 1, \text{ (diffusion-dominated case)} \end{cases}$$

with appropriate positive constants  $\alpha_0$  and  $\alpha_1$ . Here  $Pe_K := \|\mathbf{b}\|_{0,\infty,K} h_K / (2\varepsilon)$  is the mesh Peclet number and  $h_K = \sup_{\mathbf{x}, \mathbf{x}' \in K} \|\mathbf{x} - \mathbf{x}'\|$  is the diameter of the mesh cell  $K$ . A more sophisticated choice is to replace the diameter  $h_K$  of the mesh cell in the above definition for  $\alpha_K$  by its streamline diameter  $h_{\mathbf{b},K}$  which is the maximal length of any characteristic running through  $K$ . How to extend this strategy to the case of anisotropic meshes? There are some attempts which basically use the analog of isotropic case to get the following form of stabilization parameters

$$\alpha_K = \frac{h_K}{2\|\mathbf{b}\|_K} \min \left\{ 1, \frac{Pe_K}{3} \right\}, \text{ with } Pe_K = \frac{\|\mathbf{b}\|_K h_K}{2\varepsilon}. \quad (1.2)$$

For example, Nguyen et al. [29] use the form of stabilization parameters (1.2) by setting  $h_K$  as the length of the longest edge of the element  $K$  projected onto the convective field  $\mathbf{b}$ , denoted by "LEP" in this paper. Another similar choice of  $h_K$  is the length of the projection of the longest edge of the element  $K$  onto the convective field  $\mathbf{b}$ , denoted by "PLE". As pointed in isotropic case, a more sophisticated choice is to replace the diameter  $h_K$  of the mesh cell by its streamline diameter  $h_{\mathbf{b},K}$  which is the maximal length of any characteristic running through  $K$ , i.e., the diameter of  $K$  in the direction of the convection  $\mathbf{b}$ , denoted by "DDC". There are also some other choices of stabilized parameters derived by relatively rigorous theory on anisotropic meshes. For example, in [2] an anisotropic a priori error analysis is provided for the advection-diffusion-reaction problem. It is shown that the height, say  $h_K$ , with respect to the diameter of each element  $K$  (denoted by "DEE"), should be used for the design of the stability parameters in the case of external

boundary layers. In [24] an alternative approach is proposed showing that the diameter of each element is again the correct choice. Micheletti et al. [27] consider the GLS methods for the scalar advection-diffusion and the Stokes problems with approximations based on continuous piecewise linear finite elements on anisotropic meshes, where new definitions of the stability parameters are proposed. Cangiani and Süli [10] use the stabilizing term derived from the RFB method to redefine the mesh Péclet number and propose a new choice of the streamline-diffusion parameters (This well-known fact that the RFB method and the SDFEM are equivalent under certain conditions was first observed by Brezzi and Russo [8]. The similar idea was also used in [25].) that is suitable for use on anisotropic partitions. Although there are so many strategies on selection of the stabilization parameters, it is still hard to show which is optimal. Besides, there is little result on the relationship between the strategy to generate the anisotropic meshes and the selection of stabilized parameters.

In this paper, we propose a strategy to generate the anisotropic meshes and select the appropriate stabilized parameters simultaneously for stabilized continuous linear finite elements. As in [13, 19, 27], the discretized error (the difference between the true solution and the stabilized finite element solution) in a suitable norm can be bounded by the sum of interpolation error and its variants in different norms. Based on this result, we use the idea in our recent work [32] to replace these norms of interpolation error by some terms which contain the Hessian matrix of the true solution, convective fields  $\mathbf{b}$ , and the geometric properties such as directed edges and the area of the triangle. After that, we use the shape, size and equidistribution requirements to derive the correspond metric tensor and the stabilized parameters. From our derivation it is easily found that the optimal stabilized parameter is coupled with the optimal metric tensor on each element. Specifically, the relationship between the optimal metric tensor and the optimal stabilized parameter on each element is given approximately by (4.7).

The rest of the paper is organized as follows. In Section 2, we state the GLS stabilized finite element method for the convection-diffusion equation (1.1). Section 3 is devoted to obtaining the estimate for the discretized error in a suitable norm via the anisotropic framework similar to that used in [32]. The optimal choice of the metric tensor and the stabilized parameters for the stabilized linear finite element method are then derived in Section 4. Some numerical examples are provided in Section 5 to demonstrate the stability and efficiency of the proposed numerical method. Some concluding remarks will be given in the last section.

## 2 Stabilized finite element discretization

We shall use the standard notations in for the Sobolev spaces  $H^s(\Omega)$  and their associated inner products  $(\cdot, \cdot)_s$ , norms  $\|\cdot\|_s$ , and seminorms  $|\cdot|_s$  for  $s \geq 0$ . The Sobolev space  $H^0(\Omega)$  coincides with  $L^2(\Omega)$ , in which case the norm and inner product are denoted by  $\|\cdot\|$  and

$(\cdot, \cdot)$ , respectively. Let  $H_g^1(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = g\}$  and  $H_0^1(\Omega) = \{v \in H^1(\Omega), v|_{\partial\Omega} = 0\}$ . The variational formulation of problem (1.1) reads as follows: find  $u \in H_g^1(\Omega)$  which satisfies

$$A(u, v) = F(v), \quad \forall v \in H_0^1(\Omega), \quad (2.1)$$

where  $A(\cdot, \cdot)$  and  $F(\cdot)$  define the bilinear and linear forms

$$A(u, v) = (\varepsilon \nabla u, \nabla v) + (\mathbf{b} \cdot \nabla u, v),$$

and

$$F(v) = (f, v),$$

respectively.

Given a triangulation  $\mathcal{T}_h$  of  $\Omega$ , we denote the piecewise linear and continuous finite element space by  $V^h$ , i.e.,

$$V^h = \{v \in H^1(\Omega), v|_K \in \mathcal{P}_1(K), \forall K \in \mathcal{T}_h\},$$

where  $\mathcal{P}_1(K)$  is linear polynomial space in one element  $K$ . We then define  $V_g^h := V^h \cap H_g^1(\Omega)$  and  $V_0^h := V^h \cap H_0^1(\Omega)$ . The standard finite element method discretization of (2.1) is to find  $u_h \in V_g^h$  such that

$$A(u_h, v_h) = f(v_h), \quad \forall v_h \in V_0^h. \quad (2.2)$$

For convection-dominated problems ( $\varepsilon \ll \|\mathbf{b}\|$ ), (2.2) using standard grid sizes are not able to capture steep layers without introducing non-physical oscillations. To enhance the stability and accuracy in the convection dominated regime, various stabilization strategies have been developed. Here we take the GLS stabilized finite element method as an example which reads as follows: find  $u_h \in V_g^h$  such that

$$A_h(u_h, v_h) = F(v_h), \quad \forall v_h \in V_0^h, \quad (2.3)$$

with

$$A_h(u_h, v_h) = A(u_h, v_h) + \sum_{K \in \mathcal{T}_h} \alpha_K (-\varepsilon \Delta u_h + \mathbf{b} \cdot \nabla u_h, -\varepsilon \Delta v_h + \mathbf{b} \cdot \nabla v_h)_K,$$

and

$$F_h(v_h) = F(v_h) + \sum_{K \in \mathcal{T}_h} \alpha_K (f, -\varepsilon \Delta v_h + \mathbf{b} \cdot \nabla v_h)_K.$$

In this paper we use linear finite element method, so the terms  $\Delta u_h|_K$  and  $\Delta v_h|_K$  in the two above equations are identically equal to zero. At this time, the GLS approach is the same as the SUPG method, which also enjoys the result in this paper if the linear finite element method is used. We endow the space  $H_0^1(\Omega)$  with the discrete norm  $\|\cdot\|_h$  defined, for any  $w \in H_0^1(\Omega)$ , by

$$\|w\|_h^2 := \varepsilon \|\nabla w\|_{L^2(\Omega)}^2 + \sum_{K \in \mathcal{T}_h} \alpha_K \|\mathbf{b} \cdot \nabla w\|_{L^2(K)}^2. \quad (2.4)$$

**Lemma 2.1.** *The stabilized finite element approximation  $u_h$  defined by (2.3) has the following error estimate*

$$\begin{aligned} \|u - u_h\|_h^2 \leq C \sum_{K \in \mathcal{T}_h} & \left( \alpha_K^{-1} \|u - \Pi_h u\|_{L^2(K)}^2 + \varepsilon \|\nabla(u - \Pi_h u)\|_{L^2(K)}^2 \right. \\ & \left. + \alpha_K \|\mathbf{b} \cdot \nabla(u - \Pi_h u)\|_{L^2(K)}^2 + \alpha_K \varepsilon^2 \|\Delta(u - \Pi_h u)\|_{L^2(K)}^2 \right), \end{aligned} \quad (2.5)$$

where  $\Pi_h$  denotes the standard continuous piecewise linear interpolation operator.

*Proof.* See, for example, [13, 19, 27]. □

### 3 Estimates for the interpolation error and its variants

As stated in Lemma 2.1, the discretized error in  $\|\cdot\|_h$  norm is bounded by four terms of interpolation error and its variants in different norms. In fact the interpolation error depends on the solution, the size and shape of the elements in the mesh. Understanding this relation is crucial for generating efficient meshes. In the mesh generation fields, this relation is often studied for the model problem of interpolating quadratic functions. For instance, Nadler [28] derived an exact expression for the  $L^2$ -norm of the linear interpolation error in terms of the three sides  $\ell_1$ ,  $\ell_2$ , and  $\ell_3$  of the triangle  $K$ :

$$\|u - \Pi_h u\|_{L^2(K)}^2 = \frac{|K|}{180} \left[ \left( d_{11} + d_{22} + d_{33} \right)^2 + d_{11}^2 + d_{22}^2 + d_{33}^2 \right], \quad (3.1)$$

where  $|K|$  is the area of the triangle,  $d_{ij} = \ell_i \cdot H(u) \ell_j$  with  $H(u)$  being the Hessian of  $u$ .

Three element-wise error estimates in different norms are derived by the following lemmas, which, together with (3.1), are fundamental for further discussion. Suppose  $u$  is a quadratic function on a triangle  $K$ . The function is given by its matrix representation:

$$\forall \mathbf{x} \in K, \quad u(\mathbf{x}) = \frac{1}{2} \mathbf{x}^t H(u) \mathbf{x}. \quad (3.2)$$

**Lemma 3.1.** *Let  $u$  be a quadratic function on a triangle  $K$ , and  $\Pi_h u$  be the Lagrangian linear interpolation of  $u$  on  $K$ . The following relationship holds:*

$$\|\nabla(u - \Pi_h u)\|_{L^2(K)}^2 = \frac{1}{48|K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} D_{ij} \ell_i^t \ell_j, \quad (3.3)$$

where

$$D_{11} = d_{12}^2 + d_{23}^2, D_{22} = d_{12}^2 + d_{13}^2, D_{12} = 2d_{12}^2. \quad (3.4)$$

*Proof.* The proof is similar to but easier than that of Lemma 3.2. □

**Remark 3.1.** *Lemma 3.1 is also used in [32].*

**Lemma 3.2.** *Let  $u$  be a quadratic function on a triangle  $K$ , and  $\Pi_h u$  be the Lagrangian linear interpolation of  $u$  on  $K$ . Assume  $\mathbf{b}$  is a constant vector on  $K$ , the following relationship holds:*

$$\|\mathbf{b} \cdot \nabla(u - \Pi_h u)\|_{L^2(K)}^2 = \frac{1}{48|K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} D_{ij} k_i k_j, \quad (3.5)$$

where

$$k_1 = (b_2, -b_1)\ell_1, \quad k_2 = (b_2, -b_1)\ell_2. \quad (3.6)$$

*Proof.* Following [26], we first derive an exact error estimate of the point-wise interpolation error in  $K$  :  $\nabla e(\mathbf{x}) = \nabla(u - \Pi_h u)(\mathbf{x})$  for  $\mathbf{x} \in K$ . This error is then integrated over  $K$ . Here the standard reference element technique is used. To do this we define the reference element  $K_r$  by its three vertices coordinates:

$$\hat{\mathbf{x}}_1 = (0, 0)^t, \hat{\mathbf{x}}_2 = (1, 0)^t, \text{ and } \hat{\mathbf{x}}_3 = (0, 1)^t.$$

All the terms are computed on  $K_r$  and then converted onto the element  $K$  at hand by using the following affine mapping:

$$\mathbf{x} = \mathbf{x}_1 + B_K \hat{\mathbf{x}} \text{ with } B_K = [\ell_1, -\ell_2], \mathbf{x} \in K, \hat{\mathbf{x}} \in K_r,$$

where

$$\ell_1 = \mathbf{x}_2 - \mathbf{x}_1, \text{ and } \ell_2 = \mathbf{x}_1 - \mathbf{x}_3.$$

In the frame of  $K_r$ , the quadratic function  $u$  turns into:

$$u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{2} \mathbf{x}_1^t H(u) \mathbf{x}_1 + \frac{1}{2} \mathbf{x}_1^t H(u) B_K \hat{\mathbf{x}} + \frac{1}{2} \hat{\mathbf{x}}^t B_K^t H(u) \mathbf{x}_1 + \frac{1}{2} \hat{\mathbf{x}}^t B_K^t H(u) B_K \hat{\mathbf{x}}.$$

Since the linear interpolation is concerned, linear and constant terms of  $u(\mathbf{x}(\hat{\mathbf{x}}))$  are exactly interpolated, these terms are neglected and only quadratic terms are kept. So we could set  $u(\mathbf{x}) = \frac{1}{2} \hat{\mathbf{x}}^t B_K^t H(u) B_K \hat{\mathbf{x}}$ , with a matrix form:

$$u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{2} \hat{\mathbf{x}}^t B_K^t H(u) B_K \hat{\mathbf{x}} = \frac{1}{2} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}^t \begin{bmatrix} d_{11} & -d_{12} \\ -d_{21} & d_{22} \end{bmatrix} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}.$$

Then the function  $u$  in  $K_r$  reads:

$$u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{2} (d_{11} \hat{x}^2 + d_{22} \hat{y}^2 - 2d_{12} \hat{x} \hat{y}),$$

with its linear interpolation on  $K_r$ :

$$\Pi_h u(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{2} (d_{11} \hat{x} + d_{22} \hat{y}),$$

and the exact point-wise interpolation error:

$$e(\mathbf{x}(\hat{\mathbf{x}})) = \frac{1}{2}[d_{11}(\hat{x}^2 - \hat{x}) + d_{22}(\hat{y}^2 - \hat{y}) - 2d_{12}\hat{x}\hat{y}]. \quad (3.7)$$

It is obvious that the following formulas hold

$$B_K = [\ell_1, -\ell_2] = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}, \quad B_K^{-1} = \frac{1}{\det(B_K)} \begin{bmatrix} y_3 - y_1 & x_1 - x_3 \\ y_1 - y_2 & x_2 - x_1 \end{bmatrix}.$$

After that it is easily to obtain

$$\det(B_K) \cdot \mathbf{b}^t B_K^{-t} = \mathbf{b}^t \begin{bmatrix} y_3 - y_1 & y_1 - y_2 \\ x_1 - x_3 & x_2 - x_1 \end{bmatrix} = (b_2, -b_1)[\ell_2, \ell_1] = (k_2, k_1).$$

Then we have

$$\begin{aligned} \int_K (\mathbf{b} \cdot \nabla_{\mathbf{x}} e(\mathbf{x}))^2 dx dy &= \det(B_K) \int_{K_r} \left( \mathbf{b}^t B_K^{-t} \nabla_{\hat{\mathbf{x}}} e(\mathbf{x}(\hat{\mathbf{x}})) \right)^2 d\hat{x} d\hat{y} \\ &= \frac{1}{\det(B_K)} \left[ k_2^2 \int_{K_r} \left( \frac{\partial e(\mathbf{x}(\hat{\mathbf{x}}))}{\partial \hat{x}} \right)^2 d\hat{x} d\hat{y} + k_1^2 \int_{K_r} \left( \frac{\partial e(\mathbf{x}(\hat{\mathbf{x}}))}{\partial \hat{y}} \right)^2 d\hat{x} d\hat{y} \right. \\ &\quad \left. + 2k_1 k_2 \int_{K_r} \frac{\partial e(\mathbf{x}(\hat{\mathbf{x}}))}{\partial \hat{x}} \frac{\partial e(\mathbf{x}(\hat{\mathbf{x}}))}{\partial \hat{y}} d\hat{x} d\hat{y} \right]. \end{aligned} \quad (3.8)$$

Due to (3.7), we can easily obtain

$$\nabla_{\hat{\mathbf{x}}} e(\mathbf{x}(\hat{\mathbf{x}})) = \begin{pmatrix} \partial e(\mathbf{x}(\hat{\mathbf{x}}))/\partial \hat{x} \\ \partial e(\mathbf{x}(\hat{\mathbf{x}}))/\partial \hat{y} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} d_{11}(2\hat{x} - 1) - 2d_{12}\hat{y} \\ d_{22}(2\hat{y} - 1) - 2d_{12}\hat{x} \end{pmatrix}.$$

After simple calculation, the following results hold:

$$\begin{aligned} \int_{K_r} \hat{x}^2 d\hat{x} d\hat{y} &= \int_{K_r} \hat{y}^2 d\hat{x} d\hat{y} = \frac{1}{12}, & \int_{K_r} \hat{x}\hat{y} d\hat{x} d\hat{y} &= \frac{1}{24}, \\ \int_{K_r} \hat{x} d\hat{x} d\hat{y} &= \int_{K_r} \hat{y} d\hat{x} d\hat{y} = \frac{1}{6}, & \int_{K_r} 1 d\hat{x} d\hat{y} &= \frac{1}{2}. \end{aligned}$$

Then we have

$$24 \int_{K_r} \left( \frac{\partial e(\mathbf{x}(\hat{\mathbf{x}}))}{\partial \hat{x}} \right)^2 d\hat{x} d\hat{y} = d_{12}^2 + d_{13}^2 = D_{22}, \quad (3.9)$$

$$24 \int_{K_r} \left( \frac{\partial e(\mathbf{x}(\hat{\mathbf{x}}))}{\partial \hat{y}} \right)^2 d\hat{x} d\hat{y} = d_{12}^2 + d_{23}^2 = D_{11}, \quad (3.10)$$

$$48 \int_{K_r} \frac{\partial e(\mathbf{x}(\hat{\mathbf{x}}))}{\partial \hat{x}} \frac{\partial e(\mathbf{x}(\hat{\mathbf{x}}))}{\partial \hat{y}} d\hat{x} d\hat{y} = 2d_{12}^2 = D_{12}. \quad (3.11)$$

Substituting (3.9)-(3.11) into (3.8) we get the desired results (3.3) due to the fact  $\det(B_k) = 2|K|$ .  $\square$



**Lemma 3.3.** *Let  $u$  be a quadratic function on a triangle  $K$ . The following relationship holds:*

$$\|\Delta u\|_{L^2(K)}^2 = \frac{\left(|\ell_2|^2 d_{11} - 2\ell_1^t \ell_2 d_{12} + |\ell_1|^2 d_{22}\right)^2}{16|K|^3}. \quad (3.12)$$

*Proof.* Similar to that of Lemma 3.2 we can easily obtain

$$\begin{aligned} \int_K (\Delta u)^2 dx dy &= \det(B_K) \int_{K_r} \left( \frac{|\ell_2|^2}{\det(B_K)^2} d_{11} - 2 \frac{\ell_1^t \ell_2}{\det(B_K)^2} d_{12} + \frac{|\ell_1|^2}{\det(B_K)^2} d_{22} \right)^2 d\hat{x} d\hat{y} \\ &= \frac{\left(|\ell_2|^2 d_{11} - 2\ell_1^t \ell_2 d_{12} + |\ell_1|^2 d_{22}\right)^2}{2 \det(B_K)^3} = \frac{\left(|\ell_2|^2 d_{11} - 2\ell_1^t \ell_2 d_{12} + |\ell_1|^2 d_{22}\right)^2}{16|K|^3}. \end{aligned}$$

This is the desired result (3.12) and the proof is complete.  $\square$

**Remark 3.2.** *Since  $\Delta u$  is a constant under our assumption, there is a rather direct and easy way to prove Lemma 3.3. However, we still use the frame of proof for Lemma 3.2 to make the error expression be a consistent manner.*

**Theorem 3.1.** *Assume the exact solution  $u$  is quadratic on each element  $K$ , the error of the stabilized finite element approximation has the following estimate*

$$\|u - u_h\|_h^2 \leq C \sum_{K \in \mathcal{T}_h} E_K, \quad (3.13)$$

with

$$\begin{aligned} E_K &= \frac{|K|}{180\alpha_K} \left[ \left( \sum_{i=1}^3 d_{ii} \right)^2 + \sum_{i=1}^3 d_{ii}^2 \right] + \frac{\varepsilon}{48|K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} D_{ij} \ell_i^t \ell_j + \frac{\alpha_K}{48|K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} D_{ij} k_i k_j \\ &+ \frac{\alpha_K \varepsilon^2}{16|K|^3} \left( |\ell_2|^2 d_{11} - 2\ell_1^t \ell_2 d_{12} + |\ell_1|^2 d_{22} \right)^2, \end{aligned}$$

where  $D_{ij}$  ( $i, j = 1, 2, i \leq j$ ) and  $k_i$  ( $i = 1, 2$ ) are defined by (3.4) and (3.6), respectively.

*Proof.* Together with Lemma 2.1 and Lemma 3.1, 3.2, 3.3, the conclusion is obtained directly.  $\square$

Even the error estimate (3.13) is only valid for those piecewise quadratic functions, however, it could catch the main properties of the errors for general functions. In fact, the treatment to replace the general solution by its second order Taylor expansion yields a reliable and efficient estimator of the interpolation error for general functions provided a saturation assumption is valid [1, 12].

For simplicity of notation in the following discussion, for each element  $K \in \mathcal{T}_h$ , we denote by

$$Q_{1,K} = \frac{|K|}{180} \left[ \left( \sum_{i=1}^3 d_{ii} \right)^2 + \sum_{i=1}^3 d_{ii}^2 \right], \quad Q_{2,K} = \frac{1}{48|K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} D_{ij} \ell_i^t \ell_j,$$

$$\tilde{Q}_{2,K} = \frac{1}{48|K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} D_{ij} k_i k_j, \quad Q_{3,K} = \frac{1}{16|K|^3} \left( |\ell_2|^2 d_{11} - 2\ell_1^t \ell_2 d_{12} + |\ell_1|^2 d_{22} \right)^2.$$

And then  $E_K$  in (3.13) can be recast into

$$E_K = Q_{1,K} \cdot \alpha_K^{-1} + Q_{2,K} \cdot \varepsilon + \tilde{Q}_{2,K} \cdot \alpha_K + Q_{3,K} \cdot \alpha_K \varepsilon^2.$$

## 4 Metric tensors for anisotropic mesh adaptation

We now use the error estimates obtained in Section 3 to develop the metric tensor for the discretized error in  $\|\cdot\|_h$  norm and give a new definition of the stability parameters which are optimal in a certain sense. As a common practice in anisotropic mesh generation, the metric tensor,  $\mathcal{M}(\mathbf{x})$ , is used in a meshing strategy in such a way that an anisotropic mesh is generated as a quasi-uniform mesh in the metric space determined by  $\mathcal{M}(\mathbf{x})$ . Mathematically, this can be interpreted as the shape, size and equidistribution requirements as follows.

**The shape requirement.** The elements of the new mesh,  $\mathcal{T}_h$ , are (or are close to being) equilateral in the metric.

**The size requirement.** The elements of the new mesh  $\mathcal{T}_h$  have a unitary volume in the metric, i.e.,

$$\int_K \sqrt{\det(\mathcal{M}(\mathbf{x}))} dx = 1, \quad \forall K \in \mathcal{T}_h. \quad (4.1)$$

**The equidistribution requirement.** The anisotropic mesh is required to minimize the error for a given number of mesh points (or equidistribute the error on every element).

Notice that to derive the monitor function, we just need the shape and equidistribution requirements.

### 4.1 Optimal metric tensor and stabilized parameters

We derive the monitor function  $M(\mathbf{x})$  first. Assume  $H(u)$  is a symmetric positive definite matrix on every point  $\mathbf{x}$  and this restriction can be explained by Remark 2 in [26]. Set  $M(\mathbf{x}) = C(\mathbf{x})H(u)$ . Denoted by  $H_K$  and  $C_K$  the  $L^2$  projection of  $H(u)$  and  $C(\mathbf{x})$  to the constant space on  $K$ , and  $M_K = C_K H_K$ . Since  $H_K$  is a symmetric positive definite matrix,

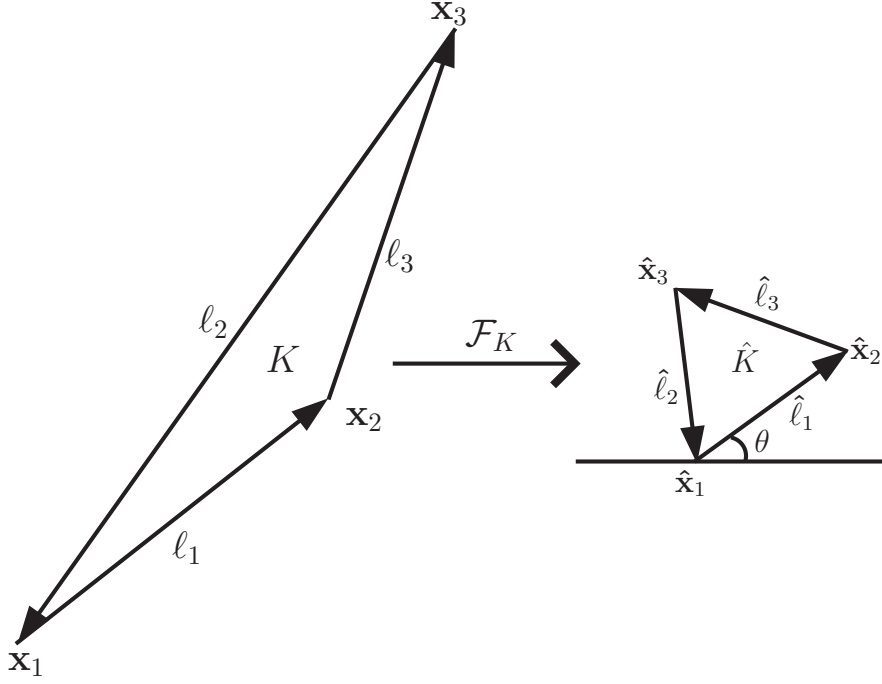


Figure 1: Affine map  $\hat{\mathbf{x}} = \mathcal{F}_K \mathbf{x}$  from triangle  $K$  to the reference triangle  $\hat{K}$ .

we do the singular value decomposition  $H_K = R^T \Lambda R$ , where  $\Lambda = \text{diag}(\lambda_{1,K}, \lambda_{2,K})$  is the diagonal matrix of the corresponding eigenvalues ( $\lambda_{1,K}, \lambda_{2,K} > 0$ ) and  $R$  is the orthogonal matrix with rows being the eigenvectors of  $H_K$ . Denote by  $F_K$  and  $\mathbf{t}_K$  the matrix and the vector defining the invertible affine map  $\hat{\mathbf{x}} = \mathcal{F}_K(\mathbf{x}) = F_K \mathbf{x} + \mathbf{t}_K$  from the generic element  $K$  to the reference triangle  $\hat{K}$  (see Figure 1). Here we take  $\hat{K}$  as an equilateral triangle with one edge which has angle  $\theta$  with the horizontal line. Let  $M_K = F_K^T F_K$ , then  $F_K = C_K^{\frac{1}{2}} \Lambda^{\frac{1}{2}} R$ . Mathematically, the shape requirement can be expressed as

$$|\hat{\ell}_1| = |\hat{\ell}_2| = |\hat{\ell}_3| = L, \quad (4.2)$$

and

$$\frac{\hat{\ell}_1 \cdot \hat{\ell}_3}{|\hat{\ell}_1| \cdot |\hat{\ell}_3|} = \frac{\hat{\ell}_2 \cdot \hat{\ell}_3}{|\hat{\ell}_2| \cdot |\hat{\ell}_3|} = \frac{\hat{\ell}_1 \cdot \hat{\ell}_2}{|\hat{\ell}_1| \cdot |\hat{\ell}_2|} = \cos(2\pi/3) = -\frac{1}{2}, \quad (4.3)$$

where  $L$  is a constant.

**Theorem 4.1.** *Under the shape requirement, the following results hold:*

$$Q_{1,K} = \frac{L^4 |K|}{15C_K^2}, \quad Q_{2,K} = \frac{L^4 \text{tr}(H_K)}{32\sqrt{3}C_K^2 \det(H_K)^{\frac{1}{2}}}, \quad Q_{3,K} = \frac{3L^4 \text{tr}(H_K)^2}{16C_K^2 |K| \det(H_K)},$$

and

$$\tilde{Q}_{2,K} = \frac{L^4 \det(H_K)^{\frac{1}{2}}}{24C_K^2} \left( \frac{A_1^2}{\lambda_{1,K}} + \frac{A_2^2}{\lambda_{2,K}} \right) = \frac{L^4}{32\sqrt{3}C_K^2} \cdot \frac{\mathbf{b}^t H_K \mathbf{b}}{\det(H_K)^{\frac{1}{2}}},$$

where

$$\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = R \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{b}.$$

*Proof.* From the definition for  $Q_{1,K}$  and the relation between  $\ell_i$  and  $\hat{\ell}_i$ , we have

$$Q_{1,K} = \frac{|K|}{180C_K^2} \left[ \left( \sum_{i=1}^3 |\hat{\ell}_i|^2 \right)^2 + \sum_{i=1}^3 |\hat{\ell}_i|^4 \right] = \frac{L^4 |K|}{15C_K^2}.$$

Similarly, together with (4.2) and (4.3), the following reduction is straight-forward:

$$\begin{aligned} Q_{2,K} &= \frac{1}{48|K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} D_{ij} \ell_i^t \ell_j = \frac{1}{48|K|} \left( |\ell_1|^2 (d_{12}^2 + d_{23}^2) + |\ell_2|^2 (d_{12}^2 + d_{13}^2) + 2(\ell_1 \cdot \ell_2) d_{12}^2 \right) \\ &= \frac{1}{48C_K^2 |K|} \left( |\ell_1|^2 ((\ell_1 \cdot M_K \ell_2)^2 + (\ell_2 \cdot M_K \ell_3)^2) + |\ell_2|^2 ((\ell_1 \cdot M_K \ell_2)^2 \right. \\ &\quad \left. + (\ell_1 \cdot M_K \ell_3)^2) + 2(\ell_1 \cdot \ell_2) (\ell_1 \cdot M_K \ell_2)^2 \right) \\ &= \frac{1}{48C_K^2 |K|} \left( |\ell_1|^2 ((\hat{\ell}_1 \cdot \hat{\ell}_2)^2 + (\hat{\ell}_2 \cdot \hat{\ell}_3)^2) + |\ell_2|^2 ((\hat{\ell}_1 \cdot \hat{\ell}_2)^2 + (\hat{\ell}_1 \cdot \hat{\ell}_3)^2) \right. \\ &\quad \left. + 2(\ell_1 \cdot \ell_2) (\hat{\ell}_1 \cdot \hat{\ell}_2)^2 \right) \\ &= \frac{L^4}{48C_K^2 |K|} \left( |\ell_1|^2 \left( \cos^2 \left( \frac{2\pi}{3} \right) + \cos^2 \left( \frac{2\pi}{3} \right) \right) + |\ell_2|^2 \left( \cos^2 \left( \frac{2\pi}{3} \right) + \cos^2 \left( \frac{2\pi}{3} \right) \right) \right. \\ &\quad \left. + 2(\ell_1 \cdot \ell_2) \cos^2 \left( \frac{2\pi}{3} \right) \right) \\ &= \frac{L^4}{96C_K^2 |K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} \ell_i^t \ell_j = \frac{L^4 \det(H_K)^{\frac{1}{2}}}{96C_K |\hat{K}|} \sum_{\substack{i,j=1,2 \\ i \leq j}} \left( C_K^{-\frac{1}{2}} R^{-1} \Lambda^{-\frac{1}{2}} \hat{\ell}_i \right) \cdot \left( C_K^{-\frac{1}{2}} R^{-1} \Lambda^{-\frac{1}{2}} \hat{\ell}_j \right) \\ &= \frac{L^4 \det(H_K)^{\frac{1}{2}}}{96C_K^2 |\hat{K}|} \sum_{\substack{i,j=1,2 \\ i \leq j}} \left( \Lambda^{-\frac{1}{2}} \hat{\ell}_i \right) \cdot \left( \Lambda^{-\frac{1}{2}} \hat{\ell}_j \right), \end{aligned}$$

where we use the equality  $|K| = |\hat{K}| / (C_K \sqrt{\det(H_K)})$ . Since  $\hat{\ell}_1 = L(\cos \theta, \sin \theta)^t$  and  $\hat{\ell}_2 = -L\left(\cos\left(\frac{\pi}{3} + \theta\right), \sin\left(\frac{\pi}{3} + \theta\right)\right)^t$ , using simple calculation, we obtain

$$\Lambda^{-\frac{1}{2}} \hat{\ell}_1 = L\left(\lambda_{1,K}^{-\frac{1}{2}} \cos \theta, \lambda_{2,K}^{-\frac{1}{2}} \sin \theta\right)^t, \quad \Lambda^{-\frac{1}{2}} \hat{\ell}_2 = -L\left(\lambda_{1,K}^{-\frac{1}{2}} \cos\left(\frac{\pi}{3} + \theta\right), \lambda_{2,K}^{-\frac{1}{2}} \sin\left(\frac{\pi}{3} + \theta\right)\right)^t,$$

and then the following three equalities hold:

$$\left( \Lambda^{-\frac{1}{2}} \hat{\ell}_1 \right) \cdot \left( \Lambda^{-\frac{1}{2}} \hat{\ell}_1 \right) = L^2 \left( \lambda_{1,K}^{-1} \cos^2 \theta + \lambda_{2,K}^{-1} \sin^2 \theta \right),$$

$$\left( \Lambda^{-\frac{1}{2}} \hat{\ell}_2 \right) \cdot \left( \Lambda^{-\frac{1}{2}} \hat{\ell}_2 \right) = L^2 \left( \lambda_{1,K}^{-1} \cos^2\left(\frac{\pi}{3} + \theta\right) + \lambda_{2,K}^{-1} \sin^2\left(\frac{\pi}{3} + \theta\right) \right),$$

$$\left(\Lambda^{-\frac{1}{2}}\hat{\ell}_1\right) \cdot \left(\Lambda^{-\frac{1}{2}}\hat{\ell}_2\right) = -L^2\left(\lambda_{1,K}^{-1}\cos\theta\cos\left(\frac{\pi}{3}+\theta\right) + \lambda_{2,K}^{-1}\sin\theta\sin\left(\frac{\pi}{3}+\theta\right)\right).$$

Thus, we get

$$\sum_{\substack{i,j=1,2 \\ i \leq j}} \left(\Lambda^{-\frac{1}{2}}\hat{\ell}_i\right) \cdot \left(\Lambda^{-\frac{1}{2}}\hat{\ell}_j\right) = \frac{3}{4}\left(\lambda_{1,K}^{-1} + \lambda_{2,K}^{-1}\right),$$

which gives to

$$\frac{L^4 \det(H_K)^{\frac{1}{2}}}{96C_K^2 |\hat{K}|} \sum_{\substack{i,j=1,2 \\ i \leq j}} \left(\Lambda^{-\frac{1}{2}}\hat{\ell}_i\right) \cdot \left(\Lambda^{-\frac{1}{2}}\hat{\ell}_j\right) = \frac{L^6 \det(H_K)^{\frac{1}{2}} \operatorname{tr}(H_K)}{128C_K^2 |\hat{K}| \det(H_K)} = \frac{L^4 \operatorname{tr}(H_K)}{32\sqrt{3}C_K^2 \det(H_K)^{\frac{1}{2}}}.$$

And then the formula for  $Q_{2,K}$  is proved.

Similar calculation can produce the corresponding formula for  $Q_{3,K}$ :

$$\begin{aligned} Q_{3,K} &= \frac{1}{16|K|^3} \left(|\ell_2|^2 d_{11} - 2\ell_1^t \ell_2 d_{12} + |\ell_1|^2 d_{22}\right)^2 = \frac{L^4}{16C_K^2 |K|^3} \left(\sum_{\substack{i,j=1,2 \\ i \leq j}} \ell_i^t \ell_j\right)^2 \\ &= \frac{L^4}{16C_K^4 |K|^3} \cdot \frac{9}{16} \left(\lambda_{1,K}^{-1} + \lambda_{2,K}^{-1}\right)^2 = \frac{9L^8}{256C_K^4 |K|^3 \det(H_K)^2} = \frac{3L^4 \operatorname{tr}(H_K)^2}{16C_K^2 |K| \det(H_K)}. \end{aligned}$$

To analyze the term  $\tilde{Q}_{2,K}$ , more patience should be paid. First,

$$\begin{aligned} \tilde{Q}_{2,K} &= \frac{1}{48|K|} \left(k_1^2(d_{12}^2 + d_{23}^2) + k_2^2(d_{12}^2 + d_{13}^2) + 2k_1 k_2 d_{12}^2\right) \\ &= \frac{1}{48C_K^2 |K|} \left(k_1^2((\ell_1 \cdot M_K \ell_2)^2 + (\ell_2 \cdot M_K \ell_3)^2) + k_2^2((\ell_1 \cdot M_K \ell_2)^2\right. \\ &\quad \left.+ (\ell_1 \cdot M_K \ell_3)^2) + 2k_1 k_2 (\ell_1 \cdot M_K \ell_2)^2\right) = \frac{L^4}{96C_K^2 |K|} \sum_{\substack{i,j=1,2 \\ i \leq j}} k_i k_j. \quad (4.4) \end{aligned}$$

Second, using  $\hat{\ell}_1 = L(\cos\theta, \sin\theta)^t$ , we have

$$\begin{aligned} k_1 &= (b_2, -b_1)\ell_1 = C_K^{-\frac{1}{2}}(b_2, -b_1)R^{-1}\Lambda^{-\frac{1}{2}}\hat{\ell}_1 = C_K^{-\frac{1}{2}}(A_1, A_2)\Lambda^{-\frac{1}{2}}\hat{\ell}_1 \\ &= C_K^{-\frac{1}{2}}L\left(A_1\lambda_{1,K}^{-\frac{1}{2}}\cos\theta + A_2\lambda_{2,K}^{-\frac{1}{2}}\sin\theta\right). \end{aligned}$$

Thus

$$k_1^2 = C_K^{-1}L^2\left(A_1^2\lambda_{1,K}^{-1}\cos^2\theta + A_2^2\lambda_{2,K}^{-1}\sin^2\theta + A_1A_2(\lambda_{1,K}\lambda_{2,K})^{-\frac{1}{2}}\sin(2\theta)\right).$$

Using  $\hat{\ell}_2 = -L\left(\cos\left(\frac{\pi}{3}+\theta\right), \sin\left(\frac{\pi}{3}+\theta\right)\right)^t$ , we have

$$k_2 = (b_2, -b_1)\ell_2 = C_K^{-\frac{1}{2}}(b_2, -b_1)R^{-1}\Lambda^{-\frac{1}{2}}\hat{\ell}_2 = C_K^{-\frac{1}{2}}(A_1, A_2)\Lambda^{-\frac{1}{2}}\hat{\ell}_2$$

$$= -C_K^{-\frac{1}{2}} L \left( A_1 \lambda_{1,K}^{-\frac{1}{2}} \cos \left( \frac{\pi}{3} + \theta \right) + A_2 \lambda_{2,K}^{-\frac{1}{2}} \sin \left( \frac{\pi}{3} + \theta \right) \right).$$

Thus

$$\begin{aligned} k_2^2 &= C_K^{-1} L^2 \left( A_1^2 \lambda_{1,K}^{-1} \cos^2 \left( \frac{\pi}{3} + \theta \right) + A_2^2 \lambda_{2,K}^{-1} \sin^2 \left( \frac{\pi}{3} + \theta \right) \right. \\ &\quad \left. + A_1 A_2 (\lambda_{1,K} \lambda_{2,K})^{-\frac{1}{2}} \sin \left( 2 \frac{\pi}{3} + 2\theta \right) \right), \end{aligned}$$

and

$$\begin{aligned} k_1 k_2 &= -C_K^{-1} L^2 \left( A_1^2 \lambda_{1,K}^{-1} \cos \theta \cos \left( \frac{\pi}{3} + \theta \right) + A_2^2 \lambda_{2,K}^{-1} \sin \theta \sin \left( \frac{\pi}{3} + \theta \right) \right. \\ &\quad \left. + A_1 A_2 (\lambda_{1,K} \lambda_{2,K})^{-\frac{1}{2}} \sin \left( \frac{\pi}{3} + 2\theta \right) \right). \end{aligned}$$

That is

$$\sum_{\substack{i,j=1,2 \\ i \leq j}} k_i k_j = \frac{3}{4} C_K^{-1} L^2 \left( A_1^2 \lambda_{1,K}^{-1} + A_2^2 \lambda_{2,K}^{-1} \right). \quad (4.5)$$

Finally, insert (4.5) into (4.4), it is easily got that

$$\begin{aligned} \tilde{Q}_{2,K} &= \frac{L^6}{128 C_K^3 |K|} \left( \frac{A_1^2}{\lambda_{1,K}} + \frac{A_2^2}{\lambda_{2,K}} \right) = \frac{L^6 \sqrt{\det(H_K)}}{128 C_K^2 |\hat{K}|} \left( \frac{A_1^2}{\lambda_{1,K}} + \frac{A_2^2}{\lambda_{2,K}} \right) \\ &= \frac{L^4 \sqrt{\det(H_K)}}{32 \sqrt{3} C_K^2} \left( \frac{A_1^2}{\lambda_{1,K}} + \frac{A_2^2}{\lambda_{2,K}} \right) = \frac{L^4}{32 \sqrt{3} C_K^2} \cdot \frac{\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} \end{aligned}$$

Now, we have obtained all the desired results in Theorem 4.1 and the proof is complete.  $\square$

To summarize,

$$\begin{aligned} E_K &= Q_{1,K} \cdot \alpha_K^{-1} + Q_{2,K} \cdot \varepsilon + \tilde{Q}_{2,K} \cdot \alpha_K + Q_{3,K} \cdot \alpha_K \varepsilon^2 \\ &= \frac{L^4}{C_K^2} \left( \frac{|K|}{15 \alpha_K} + \frac{\varepsilon \operatorname{tr}(H_K)}{32 \sqrt{3} \sqrt{\det(H_K)}} + \frac{\alpha_K}{32 \sqrt{3}} \frac{\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} + \frac{3 \varepsilon^2 \alpha_K \operatorname{tr}(H_K)^2}{16 |K| \det(H_K)} \right) \\ &=: \frac{L^4}{C_K^2} P(\alpha_K). \end{aligned}$$

To minimize the term

$$P(\alpha_K) = \frac{|K|}{15 \alpha_K} + \frac{\alpha_K}{32 \sqrt{3}} \left( \frac{\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} + \frac{6 \sqrt{3} \varepsilon^2 \operatorname{tr}(H_K)^2}{|K| \det(H_K)} \right) + \frac{\varepsilon \operatorname{tr}(H_K)}{32 \sqrt{3} \sqrt{\det(H_K)}},$$

the following condition should be satisfied

$$\frac{|K|}{15 \alpha_K} = \frac{\alpha_K}{32 \sqrt{3}} \left( \frac{\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} + \frac{6 \sqrt{3} \varepsilon^2 \operatorname{tr}(H_K)^2}{|K| \det(H_K)} \right).$$

This equation gives the following choice for the optimal stabilized parameters

$$\alpha_K^* = \sqrt{\frac{32\sqrt{3}}{15}} \cdot |K| \cdot \left( \frac{|K|\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} + \frac{6\sqrt{3}\varepsilon^2 \text{tr}(H_K)^2}{\det(H_K)} \right)^{-\frac{1}{2}}. \quad (4.6)$$

At this time,

$$\begin{aligned} P(\alpha_K^*) &= \sqrt{\frac{1}{120\sqrt{3}} \left( \frac{|K|\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} + \frac{6\sqrt{3}\varepsilon^2 \text{tr}(H_K)^2}{\det(H_K)} \right)} + \frac{\varepsilon \text{tr}(H_K)}{32\sqrt{3}\sqrt{\det(H_K)}} \\ &\approx \sqrt{\frac{1}{120\sqrt{3}} \left( \frac{|K|\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} + \frac{6\sqrt{3}\varepsilon^2 \text{tr}(H_K)^2}{\det(H_K)} \right)} = \frac{2|K|}{15\alpha_K^*}. \end{aligned}$$

Here we omit the small term  $\frac{\varepsilon \text{tr}(H_K)}{32\sqrt{3}\sqrt{\det(H_K)}}$  to simplify the formula of the metric tensor. Of course if we do not omit this term the derivation can be carried out similarly, however, in this case the expression of the metric tensor will seem rather complicated. In fact the numerical efficiency is almost the same no matter this term is omitted or not (we have done numerical experiments to verify this point, however, due to the length reason we do not list the comparison in this paper). To proceed the derivation, at this time

$$E_K = \frac{L^4}{C_K^2} P(\alpha_K^*) \approx \frac{2L^4|K|}{15\alpha_K^* C_K^2}.$$

To satisfy the equidistribution requirement, we require that

$$\frac{L^4}{C_K^2} P(\alpha_K^*) \approx \frac{2L^4|K|}{15\alpha_K^* C_K^2} = \frac{e}{N},$$

where  $N$  is the number of elements of  $\mathcal{T}_h$ . Then  $C_K$  could be the form

$$C_K \sim \sqrt{\frac{|K|}{\alpha_K^*}}, \quad (4.7)$$

and  $M(\mathbf{x})$  could be the form

$$M(\mathbf{x}) = \sqrt[4]{\frac{|K|\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} + \frac{6\sqrt{3}\varepsilon^2 \text{tr}(H_K)^2}{\det(H_K)}} H(u). \quad (4.8)$$

To establish the metric tensor  $\mathcal{M}(\mathbf{x})$ , set  $\mathcal{M}(\mathbf{x}) = \theta M(\mathbf{x})$ . At this time, the size requirement (4.1) should be used, which leads to

$$\theta \int_K \rho(\mathbf{x}) d\mathbf{x} = 1,$$

where

$$\rho(\mathbf{x}) = \sqrt{\det(M(\mathbf{x}))}.$$

Summing the above equation over all the elements of  $\mathcal{T}_h$ , one gets

$$\theta\sigma = N,$$

where

$$\sigma = \int_{\Omega} \rho(\mathbf{x}) d\mathbf{x}.$$

Thus, we have

$$\mathcal{M}(\mathbf{x}) = \frac{N}{\sigma} C_K H(u). \quad (4.9)$$

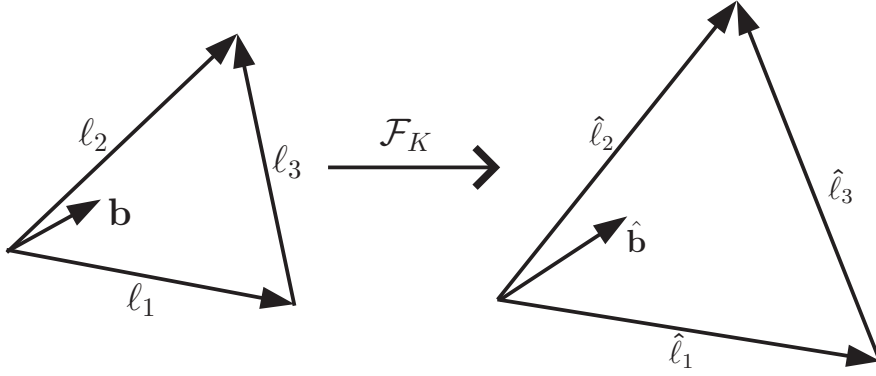


Figure 2: Affine map  $\hat{\mathbf{x}} = \mathcal{F}_K \mathbf{x}$  from triangle  $K$  to the reference triangle  $\hat{K}$ : isotropic case.

## 4.2 Practical use of stabilized parameters

Since there exists a constant  $C$  in equation (2.5), the exact ratios between the terms  $\alpha_K^{-1} \|u - \Pi_h u\|_{L^2(K)}^2$ ,  $\varepsilon \|\nabla(u - \Pi_h u)\|_{L^2(K)}^2$ ,  $\alpha_K \|\mathbf{b} \cdot \nabla(u - \Pi_h u)\|_{L^2(K)}^2$ , and  $\alpha_K \varepsilon^2 \|\Delta(u - \Pi_h u)\|_{L^2(K)}^2$  can hardly be precisely estimated. So the stabilized parameters  $\alpha_K$  (4.6) and the monitor function (4.8) can be regarded just as quasi-optimal. To establish the practical form of the stabilized parameters with the same scale with other stabilized strategies, consider the special case: the true solution is isotropic. Due to (4.6), when convection dominates diffusion, that is,

$$\frac{|K| \mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} \gg \frac{\varepsilon^2 \text{tr}(H_K)^2}{\det(H_K)},$$

the following estimate holds:

$$\frac{15}{32\sqrt{3}} \cdot (\alpha_K^*)^2 = \frac{|K| \sqrt{\det(H_K)}}{\mathbf{b}^t H_K \mathbf{b}} = \frac{C_K |K| \sqrt{\det(H_K)}}{\mathbf{b}^t C_K H_K \mathbf{b}} = \frac{|\hat{K}|}{\mathbf{b}^t M_K \mathbf{b}} = \frac{|\hat{K}|}{\hat{\mathbf{b}}^t \hat{\mathbf{b}}} = \frac{|K|}{\mathbf{b}^t \mathbf{b}}.$$



And then

$$\sqrt{\frac{15}{32\sqrt{3}}} \cdot \alpha_K^* = \frac{\sqrt{|K|}}{\|\mathbf{b}\|} = \frac{\sqrt[4]{3}\|\ell_1\|}{2\|\mathbf{b}\|}.$$

On the contrary, when diffusion dominates convection,

$$\frac{|K|\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} \ll \frac{\varepsilon^2 \text{tr}(H_K)^2}{\det(H_K)},$$

we have

$$\sqrt{\frac{15}{32\sqrt{3}}} \cdot \alpha_K^* = \frac{1}{\sqrt{6\sqrt{3}}} \cdot \frac{|K|\sqrt{\det(H_K)}}{\varepsilon \text{tr}(H_K)} = \frac{1}{\sqrt{6\sqrt{3}}} \cdot \frac{\sqrt{3}\|\ell_1\|^2}{8\varepsilon}.$$

To compare with the other stabilized parameters in the same scale we suggest

$$\alpha_K^* = |K| \cdot \left( \frac{\sqrt{3} \cdot |K|\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} + \frac{27\varepsilon^2 \text{tr}(H_K)^2}{4\det(H_K)} \right)^{-\frac{1}{2}}. \quad (4.10)$$

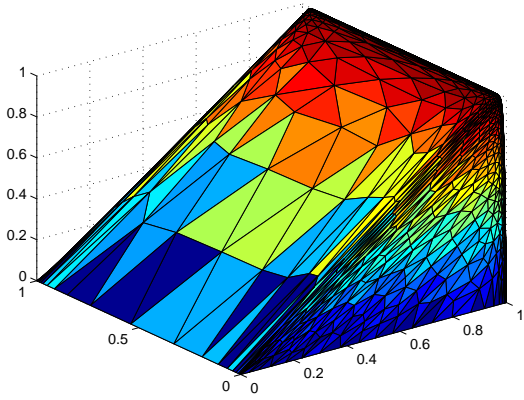
## 5 Numerical examples

In this section, we will demonstrate several numerical examples to see the superiority of our strategy to others. All the presented experiments are performed using the BAMG project [15] via software FreeFem++ [14]. Given a background mesh, the nodal values of the solution are obtained by solving a PDE through the GLS stabilized finite element method. Then the second order derivatives of the solution are obtained by using some recovery techniques (we use the gradient recovery technique [34] twice to obtain the approximated Hessian in our computation). After that, the metric tensor is computed according to the formulas derived in the previous section. Finally, a new mesh according to the computed metric tensor is generated by BAMG. The process is repeated several times in the computation until the approximate solution satisfies the prescribed tolerance.

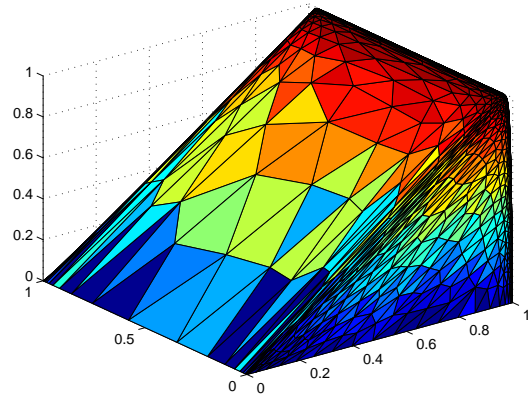
We compare our new stabilized parameters (4.10) denoted by “new stabilized parameter” with the form of stabilization parameters (1.2) using different definition of  $h_K$ , namely LEP, PLE, DEE and DDC. Also we will demonstrate that metric tensors suitable for diffusion-dominated equations are not always optimal choices for convection-dominated cases.

### 5.1 Stability vs parameters

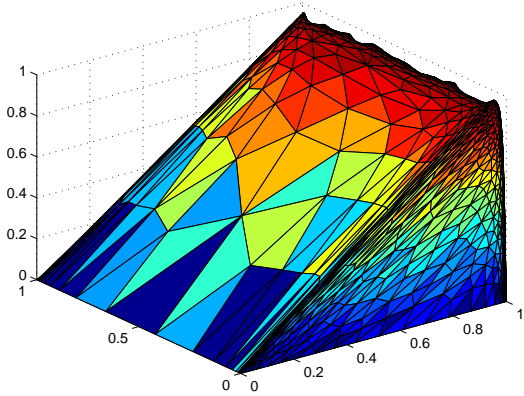
**Example 5.1.** *We consider the problem (1.1) with  $\Omega = (0, 1)^2$ ,  $\varepsilon = 10^{-4}, 10^{-6}, 10^{-8}$ ,  $\mathbf{b} = (1, 0)^T$ ,  $f = 1$  and the homogeneous boundary condition. The solution for this problem possess an exponential boundary layer at  $x_1 = 1$  and parabolic boundary layers at  $x_2 = 0$  and  $x_2 = 1$ .*



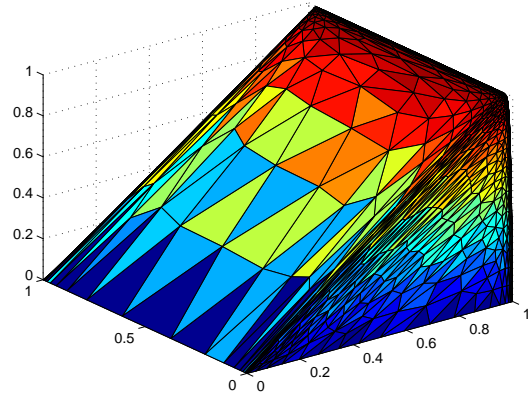
(a)



(b)

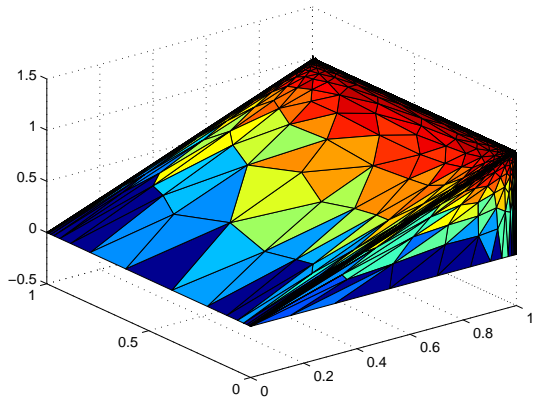


(c)

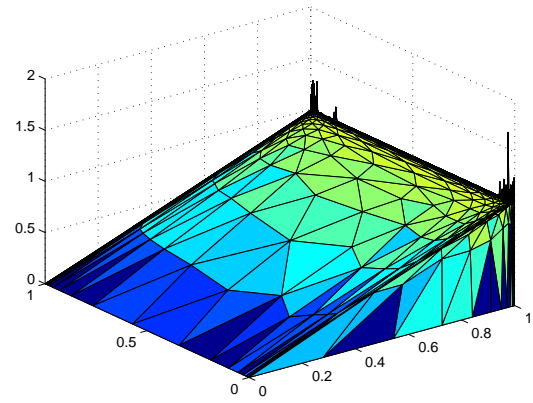


(d)

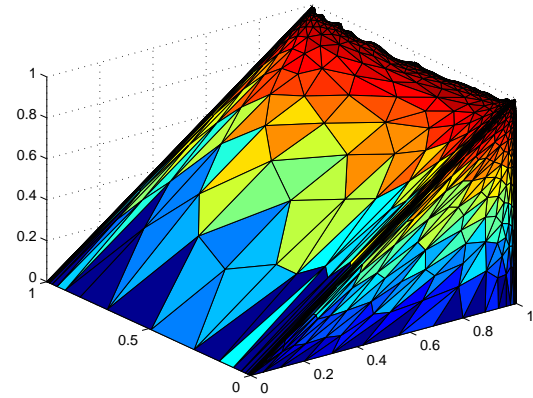
Figure 3: Example 5.1:  $\varepsilon = 10^{-4}$ , solution generated by the stabilized strategy (1.2) with  $h_K$  being (a) PLE, (b) LPE, (c) DDC, and (d) new stabilized parameter (4.10), respectively, using monitor function (4.8).



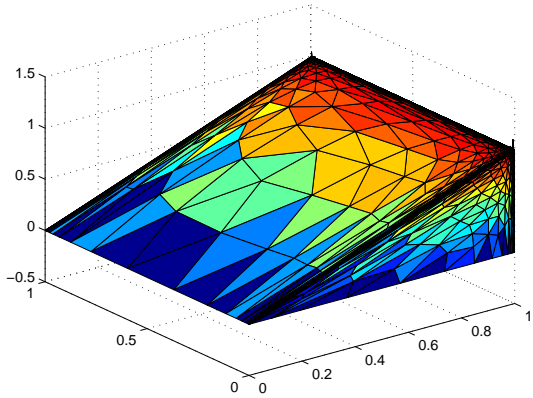
(a)



(b)

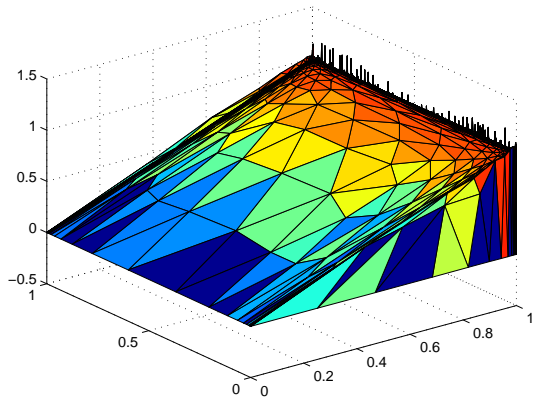


(c)

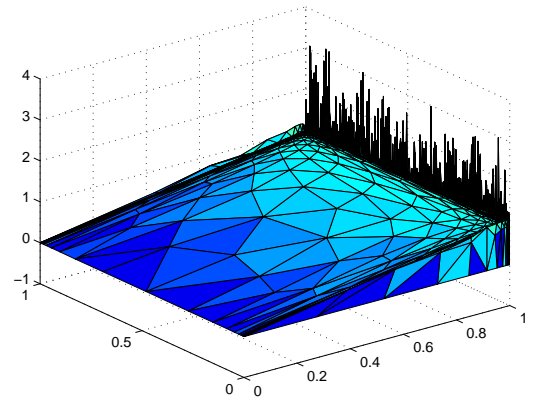


(d)

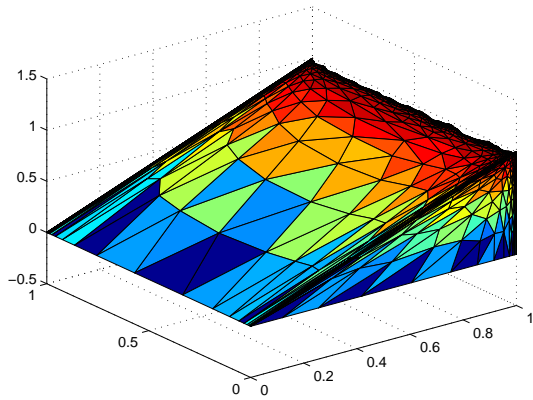
Figure 4: Example 5.1:  $\varepsilon = 10^{-6}$ , solution generated by the stabilized strategy (1.2) with  $h_K$  being (a) PLE, (b) LPE, (c) DDC, and (d) new stabilized parameter (4.10), respectively, using monitor function (4.8).



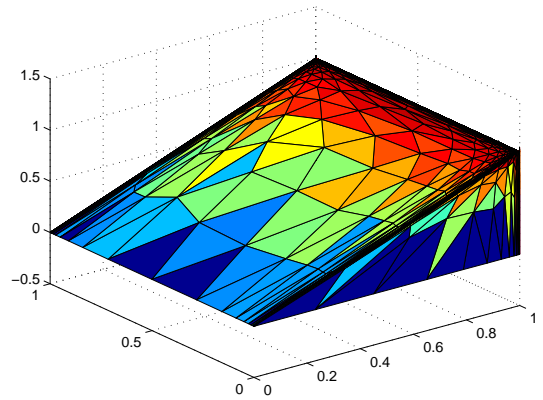
(a)



(b)



(c)



(d)

Figure 5: Example 5.1:  $\varepsilon = 10^{-8}$ , solution generated by the stabilized strategy (1.2) with  $h_K$  being (a) PLE, (b) LPE, (c) DDC, and (d) new stabilized parameter (4.10), respectively, using monitor function (4.8).

Four stabilized strategies using monitor function (4.8) are compared in stability for  $\varepsilon = 10^{-4}$  (Figure 3),  $\varepsilon = 10^{-6}$  (Figure 4),  $\varepsilon = 10^{-8}$  (Figure 5). We could see from Figures 3-5 that the stability for the form of stabilization parameters (1.2) using definition of  $h_K$  as “the longest edge of projection” and “projection of the longest edge” are good for relatively large  $\varepsilon$ , e.g.,  $\varepsilon = 10^{-4}$ . However, with the decreasing of  $\varepsilon$  the stability becomes worse for these two strategies. For the form of stabilization parameter (1.2) using definition of  $h_K$  as the diameter in the convection direction, the stability seems to be the similar pattern for different  $\varepsilon$ , however, there still exists obvious oscillation. Our new stabilized parameter (4.10) has better stability for wide range of  $\varepsilon$  from  $10^{-4}$  to  $10^{-8}$ .

## 5.2 Accuracy vs metric tensors

**Example 5.2.** Let  $\Omega = (0, 1)^2$ ,  $\varepsilon = 10^{-8}$ ,  $\mathbf{b} = (2, 3)^T$  and  $c = 0$  be in (1.1). The right-hand side  $f$  and the Dirichlet data  $g$  are chosen in such a way that

$$\begin{aligned} u(x_1, x_2) = & x_1 x_2^2 - x_2^2 \exp\left(\frac{2(x_1 - 1)}{\varepsilon}\right) - x_1 \exp\left(\frac{3(x_2 - 1)}{\varepsilon}\right) \\ & + \exp\left(\frac{2(x_1 - 1) + 3(x_2 - 1)}{\varepsilon}\right), \end{aligned}$$

which exhibits layers at the outflow boundary part.

Consider Example 5.2, since the exact solution is given, we demonstrate in this subsection that the metric tensor proposed in this paper is more suitable for stabilized finite element method approximating the convection-diffusion equation than those optimal ones for the interpolation error in some norms, e.g.  $L^2$  norm ([11]) which is defined in term of monitor function by

$$M(\mathbf{x}) = \frac{1}{\sqrt[6]{\det(H_K)}} H(u). \quad (5.1)$$

From Figures 6 we conclude as follows:

(1) For every type of stabilized parameter, the error in  $L^2$  norm by using monitor function (4.8) is smaller than that of (5.1). Since the differences are not so obvious just from Figure 6, we list a part of comparison in Table 1. However, we don't list rest details due to the length reason.

(2) Our new stabilized parameter (4.10) behaves better than other strategies no matter which monitor function is used.

(3) We could divide these stabilized parameters into two types by the errors of  $L^2$  norm. The first type contains the strategy (1.2) with  $h_K$  being the diameter of each element, projection of the longest edge, and the longest edge of projection. The second type contains the strategy (1.2) with  $h_K$  being the diameter in the convection direction, and our new stabilized parameter (4.10). This type of stabilized parameters behaves better than the first one no matter which metric tensor is used.

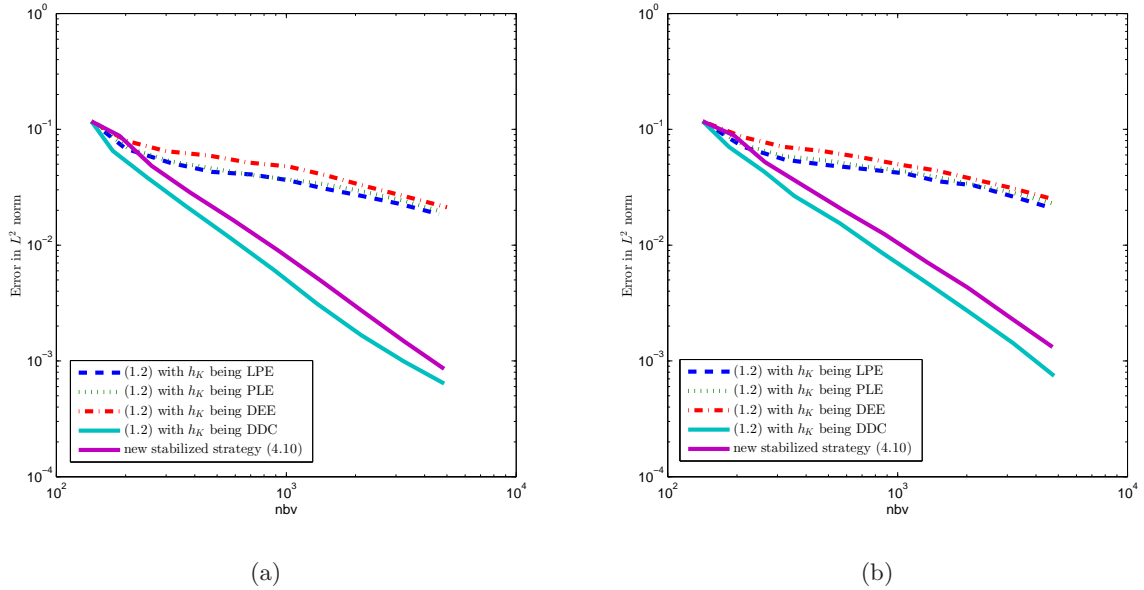


Figure 6: Example 5.2:  $\varepsilon = 10^{-8}$ , error of  $L^2$  norm by different stabilized strategies using monitor functions (a) (4.8), and (b) (5.1).

(4) For the two stabilized parameters of the second type, our new stabilized parameter (4.10) behaves better than the strategy (1.2) with  $h_K$  being the diameter in the convection direction in the sense of stability. However, the contrary conclusion will be found when the error of  $L^2$  norm is concerned. To be specific we list the errors of  $L^2$  norm for two parameters in Table 1, where  $nbv^{1,L}$  and  $nbv^{1,N}$  stand for the number of vertices using our new parameter strategy (4.10) using monitor functions (5.1) and (4.8), respectively. While  $nbv^{2,L}$  and  $nbv^{2,N}$  stand for the number of vertices using the strategy (1.2) with  $h_K$  being the diameter in the convection direction using monitor functions (5.1) and (4.8), respectively.  $u_h^1$  and  $u_h^2$  are defined similarly. We could find that the error in  $L^2$  norm by our parameter strategy is slightly larger than that by the parameter strategy (1.2) with  $h_K$  being the diameter in the convection direction no matter which monitor function is used.

### 5.3 Relationship between our parameter and the best one among others

In view of the above considerations, besides our new stability parameter (4.10) it seems to be the best choice to select the stabilized parameter as (1.2) with  $h_K$  as the diameter of  $K$  in the direction of the convection  $\mathbf{b}$ . It is interesting to study the relationship between (1.2) with  $h_K$  as the diameter of  $K$  in the direction of the convection  $\mathbf{b}$  and (4.10), so we give the following analysis. First, we set  $\mathbf{b}_h$  as the unit vector in the direction of  $\mathbf{b}$ , that

$nbv^{1,N}$	$\ u - u_h^{1,N}\ $	$nbv^{2,N}$	$\ u - u_h^{2,N}\ $	$nbv^{1,L}$	$\ u - u_h^{1,L}\ $	$nbv^{2,L}$	$\ u - u_h^{2,L}\ $
142	1.171e-01	142	1.171e-01	142	1.171e-01	142	1.171e-01
188	8.796e-02	176	6.500e-02	195	8.886e-02	185	7.038e-02
260	4.820e-02	249	3.817e-02	266	5.234e-02	260	4.297e-02
381	2.852e-02	363	2.193e-02	400	3.182e-02	360	2.594e-02
582	1.664e-02	560	1.186e-02	595	1.941e-02	579	1.566e-02
925	8.942e-03	885	6.101e-03	915	1.149e-02	895	9.054e-03
1413	4.940e-03	1363	3.114e-03	1381	7.123e-03	1397	5.092e-03
2142	2.692e-03	2121	1.660e-03	2129	3.964e-03	2158	2.745e-03
3257	1.477e-03	3247	9.909e-04	3207	2.267e-03	3289	1.484e-03
4858	8.536e-04	4862	6.374e-04	4944	1.274e-03	4953	7.889e-04

Table 1: Error tendency in  $L^2$  norm

is  $\mathbf{b} = |\mathbf{b}|\mathbf{b}_h$ , then

$$\begin{aligned} \frac{|K|\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} &= \frac{|K||\mathbf{b}|^2 \mathbf{b}_h^t H_K \mathbf{b}_h}{|\mathbf{b}_h|^2 \sqrt{\det(H_K)}} = \frac{|K||\mathbf{b}|^2 \mathbf{b}_h^t C_K H_K \mathbf{b}_h}{|\mathbf{b}_h|^2 C_K \sqrt{\det(H_K)}} = \frac{|K|^2 |\mathbf{b}|^2 \hat{\mathbf{b}}_h^t \hat{\mathbf{b}}_h}{h_K^2 |\hat{K}|} \\ &\in (4/\sqrt{3}, \sqrt{3}) \frac{|K|^2 |\mathbf{b}|^2}{h_K^2} \end{aligned}$$

On the triangle where convection dominates diffusion,

$$\alpha_K^* \approx \sqrt[4]{\frac{1}{3}} |K| \cdot \left( \frac{|K|\mathbf{b}^t H_K \mathbf{b}}{\sqrt{\det(H_K)}} \right)^{-\frac{1}{2}} \in \left( \frac{1}{2}, 1/\sqrt{3} \right) \cdot \frac{h_K}{|\mathbf{b}|},$$

the lower bound is the same as the choice of (1.2) with  $h_K$  as the diameter of  $K$  in the direction of the convection  $\mathbf{b}$ . This result shows that the two parameters are very similar on those triangles where convection dominates diffusion. So the difference between the two strategies comes from those triangles where diffusion dominates convection, which indicates that the selection of stabilized parameter in the diffusion-dominated area play an important role in the stabilized finite element element.

## 6 Conclusion

In this paper, we propose a strategy which generate optimal anisotropic meshes and select the optimal stabilized parameters for the GLS or SUPG linear finite element method to solve the convection-dominated convection-diffusion equation. This strategy basically solve the two key problems mentioned at the beginning of this paper in a relatively rigorous way, i.e., (1) how to generate optimal anisotropic meshes for minimizing the discretized error of stabilized finite element method for the convection-dominated convection-diffusion equation, and (2) based on (1) how to select optimal stabilized parameters. Numerical

examples also indicate that the new strategy proposed in this article is superior than any existed one in term of stability and competitive with the best existed one in term of accuracy (“the best one” here is in fact not the complete existing one but combining the best existed way to select stabilized parameters and our strategy to generate optimal anisotropic meshes).

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