Analogs of Jacobian conditions for subrings

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Abstract

We present a generalization of the Jacobian Conjecture for m polynomials in n variables: $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, where k is a field of characteristic zero and $m \in \{1, \ldots, n\}$. We express the generalized Jacobian condition in terms of irreducible and square-free elements of the subalgebra $k[f_1, \ldots, f_m]$. We also discuss obtained properties in a more general setting – for subrings of unique factorization domains.

Introduction

The Jacobian Conjecture asserts that if k is a field of characteristic zero and polynomials $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$ satisfy the Jacobian condition

$$\operatorname{jac}(f_1, \ldots, f_n) \in k \setminus \{0\} \tag{1}$$

(where jac denotes the Jacobian determinant), then $k[f_1, \ldots, f_n] = k[x_1, \ldots, x_n]$. In terms of endomorphisms of the polynomial algebra $k[x_1, \ldots, x_n]$: if a k-endomorphism φ satisfies the Jacobian condition

$$\operatorname{jac}(\varphi(x_1), \ldots, \varphi(x_n)) \in k \setminus \{0\}, \tag{2}$$

then φ is an automorphism. For more information on the Jacobian Conjecture we refer the reader to van den Essen's book [7].

In Section 1 we present and discuss the following generalization of the Jacobian Conjecture, denoted by $\mathrm{JC}(m, n, k)$, where k is a field of characteristic zero, n is a positive integer, $m \in \{1, \ldots, n\}$ and jac denotes the Jacobian determinant with respect to given variables:

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"For arbitrary polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, if

$$\gcd\left(\operatorname{jac}^{f_1,\ldots,f_m}_{x_{i_1},\ldots,x_{i_m}},\ 1 \leqslant i_1 < \ldots < i_m \leqslant n\right) \in k \setminus \{0\}, \tag{3}$$

then $k[f_1, \ldots, f_m]$ is a ring of constants of some k-derivation of $k[x_1, \ldots, x_n]$."

This conjecture can be expressed in terms of polynomial homomorphisms (and algebraic closedness) in the following way:

"For every k-homomorphism $\varphi\colon k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_m]$, if

$$\gcd\left(\operatorname{jac}^{\varphi(x_1),\ldots,\varphi(x_m)}_{x_{i_1},\ldots,x_{i_m}},\ 1 \leqslant i_1 < \ldots < i_m \leqslant n\right) \in k \setminus \{0\},$$

then $\operatorname{Im}\varphi$ is algebraically closed in $k[x_1, \ldots, x_n]$."

One of the authors obtained in [12] a characterization of endomorphisms satisfying the Jacobian condition (2), where k is a field of characteristic zero, as mapping irreducible polynomials to square-free ones. De Bondt and Yan proved in [4] that mapping square-free polynomials to square-free ones is also equivalent to (2). We can express it in terms of polynomials $f_1 = \varphi(x_1), \ldots, f_n = \varphi(x_n)$: condition (1) holds if and only if all irreducible (resp. all square-free) elements of the ring $k[f_1, \ldots, f_n]$ are square-free in the ring $k[x_1, \ldots, x_n]$. In Theorem 2.4 we generalize this fact for m polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, where $m \in \{1, \ldots, n\}$. Namely, the generalized Jacobian condition (3) is equivalent to each of the following ones:

$$\begin{aligned} \operatorname{Irr} k[f_1, \ldots, f_m] &\subset \operatorname{Sqf} k[x_1, \ldots, x_n], \\ \operatorname{Sqf} k[f_1, \ldots, f_m] &\subset \operatorname{Sqf} k[x_1, \ldots, x_n], \end{aligned} \tag{4}$$

where Irr and Sqf denote the sets of irreducible and square-free elements of the respective ring. This fact is a consequence of a multidimensional generalization of Freudenburg's lemma ([8]) obtained in Theorem 2.3. A presentation of succeding generalizations of this lemma can be found in the Intoduction to [13].

The above conjecture motivates us in Section 2 to consider the following properties for a subring R of a unique factorization domain A:

$$\operatorname{Irr} R \subset \operatorname{Sqf} A, \quad \operatorname{Sqf} R \subset \operatorname{Sqf} A. \tag{5}$$

In Theorem 3.4, under some additional assumptions, we express the second condition in a kind of factoriality:

(6) *"For every* $x \in A,\ y \in \operatorname{Sqf} A,$ *if* $x^2 y \in R \setminus \{0\},$ *then* $x, y \in R$."

We call a subring R satisfying condition (6) square-factorially closed in A. In Theorem 3.6 we show that, under the same assumptions, square-factorially closed subrings are root closed.

1 A generalization of the Jacobian Conjecture for m polynomials in n variables

Let k be a field of characteristic zero. By $k[x_1, \ldots, x_n]$ we denote the k-algebra of polynomials in n variables.

Recall from [11] the following notion of a "differential gcd" for m polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$, $m \in \{1, \ldots, n\}$:

$$\operatorname{dgcd}(f_1, \ldots, f_m) = \gcd\left(\operatorname{jac}^{f_1,\ldots,f_m}_{x_{i_1},\ldots,x_{i_m}},\ 1 \leqslant i_1 < \ldots < i_m \leqslant n\right),$$

where $\operatorname{jac}^{f_1,\ldots,f_m}_{x_{i_1},\ldots,x_{i_m}}$ denotes the Jacobian determinant of $f_1, \ldots, f_m$ with respect to $x_{i_1}, \ldots, x_{i_m}$. For $m = n$ we have

$$\operatorname{dgcd}(f_1, \ldots, f_n) \sim \operatorname{jac}(f_1, \ldots, f_n),$$

for $m = 1$ we have

$$\operatorname{dgcd}(f) \sim \gcd\left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right),$$

where $g \sim h$ means that polynomials g and h are associated. We put $\operatorname{dgcd}(f_1, \ldots, f_m) = 0$ if $\operatorname{jac}^{f_1,\ldots,f_m}_{x_{i_1},\ldots,x_{i_m}} = 0$ for every $i_1, \ldots, i_m$, that is, $f_1, \ldots, f_m$ are algebraically dependent over k.

Let k be a field and let A be a k-algebra. A k-linear map $d\colon A \to A$ such that $d(fg) = d(f)g + fd(g)$ for $f, g \in A$, is called a *k-derivation* of A. The kernel of d is denoted by A^d and called the *ring of constants* of d. For more information on derivations and their rings of constants we refer the reader to Nowicki's book [15].

Consider the following conjecture for m polynomials in n variables.

Conjecture JC(m, n, k). *For arbitrary polynomials* $f_1, \ldots, f_m \in k[x_1, \ldots, x_n]$*, where* k *is a field of characteristic zero and* $m \in \{1, \ldots, n\}$*, if*

$$\operatorname{dgcd}(f_1, \ldots, f_m) \in k \setminus \{0\},$$

then $k[f_1, \ldots, f_m]$ is a ring of constants of some k-derivation of $k[x_1, \ldots, x_n]$.

Recall Nowicki's characterization of rings of constants ([16], Theorem 5.4, [15], Theorem 4.1.4, p. 47).

Theorem (Nowicki, 1994). *Let A be a finitely generated k-domain, where k is a field of characteristic zero. Let R be a k-subalgebra of A. The following conditions are equivalent:*

(i) *R is a ring of constants of some k-derivation of A,*

(ii) *R is integrally closed in A and $R_0 \cap A = R$.*

Let D be a family of k-derivations of a finitely generated k-domain A, where k is a field of characteristic zero. It follows from Nowicki's Theorem that the ring

$$A^D = \bigcap_{d \in D} A^d$$

is a ring of constants of some single k-derivation of A ([16], Theorem 5.5, [15], Theorem 4.1.5, p. 47).

Daigle observed ([6], 1.4) that condition (ii) of Nowicki's Theorem can be shortened to the following form:

(iii) *R is algebraically closed in A.*

Now we see for example that conjecture $\mathrm{JC}(2, 3, k)$ asserts that if polynomials $f, g \in k[x, y, z]$ satisfy the condition

$$\gcd\left(\mathrm{jac}^{f,g}_{x,y}, \mathrm{jac}^{f,g}_{x,z}, \mathrm{jac}^{f,g}_{y,z}\right) \in k \setminus \{0\},$$

then $k[f, g]$ is algebraically closed in $k[x, y, z]$.

Let us note some basic observations according to conjecture $\mathrm{JC}(m, n, k)$.

Lemma 1.1. $\mathrm{JC}(m, n, k)$ *implies the Jacobian Conjecture for m variables over k.*

Proof. Assume that $\mathrm{JC}(m, n, k)$ holds and consider polynomials $f_1, \ldots, f_m \in k[x_1, \ldots, x_m]$ such that $\mathrm{jac}^{f_1,\ldots,f_m}_{x_1,\ldots,x_m} \in k \setminus \{0\}$.

In $k[x_1, \ldots, x_n]$ we have $\mathrm{dgcd}(f_1, \ldots, f_m) = \mathrm{jac}^{f_1,\ldots,f_m}_{x_1,\ldots,x_m}$, so $k[f_1, \ldots, f_m]$ is algebraically closed in $k[x_1, \ldots, x_n]$ by $\mathrm{JC}(m, n, k)$. Hence, $k[f_1, \ldots, f_m]$ is algebraically closed in $k[x_1, \ldots, x_m]$. And then $k[f_1, \ldots, f_m] = k[x_1, \ldots, x_m]$, because $f_1, \ldots, f_m$ are algebraically independent over k. □

Now, recall from [14] and [17] that a polynomial $f \in k[x_1, \ldots, x_n]$ over a field k is called *closed* if the ring $k[f]$ is integrally closed in $k[x_1, \ldots, x_n]$. When $\operatorname{char} k = 0$, a polynomial f is closed if and only if $k[f]$ is a ring of constants of some k-derivation of $k[x_1, \ldots, x_n]$ ([14], Theorem 2.1, [15], Theorem 7.1.4, p. 80). Necessary and sufficient conditions for a polynomial to be closed were collected and completed by Arzantsev and Petravchuk in [2], Theorem 1. Ayad proved ([3], Proposition 14) that a polynomial $f \in k[x, y]$, where $\operatorname{char} k = 0$, is closed if

$$\gcd\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \in k \setminus \{0\}.$$

His proof can be generalized to n variables (it was noted in [10], Proposition 4.2), so we obtain the following fact.

Theorem (Ayad, 2002). $\mathrm{JC}(1, n, k)$ *is true.*

Remark 1.2. The reverse implication in $\mathrm{JC}(m, n, k)$ need not to be true if $m < n$.

In this case $n \geqslant 2$. As an example we may consider a polynomial $f_1 = x_1^2 x_2 \in k[x_1, \ldots, x_n]$ and, if $m > 1$, polynomials $f_2 = x_3, \ldots, f_m = x_{m+1}$.

We have

$$\operatorname{dgcd}(x_1^2 x_2, x_3, \ldots, x_{m+1}) = \gcd(2x_1x_2, x_1^2) = x_1,$$

so $\operatorname{dgcd}(x_1^2 x_2, x_3, \ldots, x_{m+1}) \notin k \setminus \{0\}$.

On the other hand, $k[x_1^2 x_2, x_3, \ldots, x_{m+1}]$ is algebraically closed in $k[x_1, \ldots, x_n]$ as a ring of constants of a family of derivations

$$\left\{ x_1 \frac{\partial}{\partial x_1} - 2x_2 \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_{m+2}}, \ldots, \frac{\partial}{\partial x_n} \right\}.$$

2 Analogs of Jacobian conditions in terms of irreducible and square-free elements

If R is a commutative ring with unity, then R^* denotes the set of non-invertible elements of R. An element $a \in R$ is called *square-free* if it cannot be presented in the form $a = b^2 c$, where $b, c \in R$ and $b \notin R^*$. By $\operatorname{Irr} R$ we denote the set of irreducible elements of R and by $\operatorname{Sqf} R$ we denote the set of square-free elements of R.

Let k be a field of characteristic zero. Consider arbitrary polynomials $f_1, \dots, f_m \in k[x_1, \dots, x_n]$, where $m \in \{1, \dots, n\}$. Let $g \in k[x_1, \dots, x_n]$ be an irreducible polynomial.

The following lemma is a natural generalization of [12], Lemma 3.1. For the proof it is enough to add the argument from the beginning of the proof of [13], Proposition 3.4.a with $Q = (g)$.

Lemma 2.1. *For a given $i \in \{1, \dots, m\}$ consider the following condition:*

(∗) *there exist $s_1, \dots, s_m \in k[x_1, \dots, x_n]$, where $g \nmid s_i$, such that $g \mid s_1 d(f_1) + \ldots + s_m d(f_m)$ for every k-derivation d of $k[x_1, \dots, x_n]$.*

a) *Then $g \mid \operatorname{dgcd}(f_1, \dots, f_m)$ if and only if condition (∗) holds for some $i \in \{1, \dots, m\}$.*

b) *If, for a given $i \in \{1, \dots, m\}$, condition (∗) holds, then $\overline{f_i}$ is algebraic over the field $k(\overline{f_1}, \dots, \overline{f_{i-1}}, \overline{f_{i+1}}, \dots, \overline{f_m})$.*

Note the following consequence of Lemma 2.1 and [12], Lemma 3.2.b (where the polynomial w can be obtained as irreducible).

Corollary 2.2. *If $g \mid \operatorname{dgcd}(f_1, \dots, f_m)$, then $g \mid w(f_1, \dots, f_m)$ for some irreducible polynomial $w \in k[x_1, \dots, x_m]$.*

The following theorem is a multidimensional generalization of Freudenburg's lemma ([8]).

Theorem 2.3. *Let k be a field of characteristic zero, let $f_1, \dots, f_m \in k[x_1, \dots, x_n]$ be arbitrary polynomials, where $m \in \{1, \dots, n\}$, and let $g \in k[x_1, \dots, x_n]$ be an irreducible polynomial. The following conditions are equivalent:*

(i) $g \mid \operatorname{dgcd}(f_1, \dots, f_m)$,

(ii) $g^2 \mid w(f_1, \dots, f_m)$ *for some irreducible polynomial $w \in k[x_1, \dots, x_m]$,*

(iii) $g^2 \mid w(f_1, \dots, f_m)$ *for some square-free polynomial $w \in k[x_1, \dots, x_m]$.*

Proof. Implication (ii) ⇒ (i) was already proved in the proof of [12], Theorem 4.1, (ii) ⇒ (i). Implication (ii) ⇒ (iii) is trivial.

(i) ⇒ (ii) We combine the arguments from proofs of [12], Theorem 4.1, (i) ⇒ (ii) and [13], Theorem 3.6 (⇒). Assume that $g \mid \operatorname{dgcd}(f_1, \dots, f_m)$. By Corollary 2.2, $g \mid w(f_1, \dots, f_m)$ for some irreducible polynomial $w \in k[x_1, \dots, x_m]$. We proceed like in [12], using a derivation $d(f) = \operatorname{jac}^{f_1, \dots, f_{m-1}, f}_{x_{j_1}, \dots, x_{j_m}}$ for arbitrary $j_1 < \ldots < j_m$, instead of d_n, and applying Lemma 2.1 instead of [12], Lemma 3.1.

(iii) $\Rightarrow$ (ii) We apply the proof of [4], Theorem 2.1, 3) $\Rightarrow$ 2). Assume that $g^2 \mid w(f_1, \dots, f_m)$ for some square-free polynomial $w \in k[x_1, \dots, x_m]$, so $w(f_1, \dots, f_m) = g^2 h$, where $h \in k[x_1, \dots, x_n]$. Then there exist polynomials $w_1, w_2 \in k[x_1, \dots, x_m]$ such that $w = w_1 w_2$, w_1 is irreducible and $g \mid w_1(f_1, \dots, f_m)$. Then we proceed like in [4]. □

As a consequence of Theorem 2.3 we obtain the following generalization of [12], Theorem 5.1 and [4], Corollary 2.2.

Theorem 2.4. *Let $A = k[x_1, \dots, x_n]$, where k is a field of characteristic zero. Assume that $f_1, \dots, f_m \in A$ are algebraically independent over k, where $m \in \{1, \dots, n\}$. Put $R = k[f_1, \dots, f_m]$. Then the following conditions are equivalent:*

(i) $\operatorname{dgcd}(f_1, \dots, f_m) \in k \setminus \{0\}$,

(ii) $\operatorname{Irr} R \subset \operatorname{Sqf} A$,

(iii) $\operatorname{Sqf} R \subset \operatorname{Sqf} A$.

The above theorem allows us call conditions (ii) and (iii) analogs of the Jacobian condition (i) for a subring R.

Remark 2.5. Conditions (ii) and (iii) of Theorem 2.4 can be expressed in the following way, respetively:

(ii) for every irreducible polynomial $w \in k[x_1, \dots, x_m]$ the polynomial $w(f_1, \dots, f_m)$ is square-free,

(iii) for every square-free polynomial $w \in k[x_1, \dots, x_m]$ the polynomial $w(f_1, \dots, f_m)$ is square-free.

3 Square-factorially closed subrings

We will present some basic observations concerning conditions (ii) and (iii) from Theorem 2.4 for a subring R of an arbitrary unique factorization domain A. Conjecture $\mathrm{JC}(m, n, k)$ motivates us to state the following open question.

A general question. *Let R be a subring of a domain A such that*

$$\operatorname{Irr} R \subset \operatorname{Sqf} A \quad \text{or} \quad \operatorname{Sqf} R \subset \operatorname{Sqf} A.$$

When R is algebraically closed in A?

Lemma 3.1. *Let A be a unique factorization domain. Let R be a subring of A such that $R^* = A^*$. Consider the following conditions:*

(i) $\operatorname{Irr} R \subset \operatorname{Irr} A$,

(ii) $\operatorname{Sqf} R \subset \operatorname{Sqf} A$,

(iii) $\operatorname{Irr} R \subset \operatorname{Sqf} A$.

Then the following implications hold:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}.$$

Proof. (i) $\Rightarrow$ (ii)

Assume that $\operatorname{Irr} R \subset \operatorname{Irr} A$ and consider an element $a \in \operatorname{Sqf} R$. Let $a = q_1 \dots q_r$, where $q_1, \dots, q_r \in \operatorname{Irr} R$ are pairwise non-associated in R. Then, by the assumption, $q_1, \dots, q_r \in \operatorname{Irr} A$. Moreover, since $A^* = R^*$, $q_1, \dots, q_r$ are pairwise non-associated in A.

(ii) $\Rightarrow$ (iii)

Assume that $\operatorname{Sqf} R \subset \operatorname{Sqf} A$ and consider an element $a \in \operatorname{Irr} R$. Suppose that $a = b^2 c$, where $b \in R \setminus R^*$ and $c \in R$. Then $a = b \cdot (bc)$, where $b, bc \in R \setminus R^*$, a contradiction. Hence, $a \in \operatorname{Sqf} R$. □

Recall that a subring R of a ring A is called *factorially closed* in A if the following implication:

$$xy \in R \setminus \{0\} \Rightarrow x, y \in R$$

holds for every $x, y \in A$. The ring of constants of any locally nilpotent derivation of a domain of characteristic zero is factorially closed. We refer the reader to [6] and [9] for more information about locally nilpotent derivations.

Lemma 3.2. *Let A be a unique factorization domain. Let R be a subring of A such that $R^* = A^*$. The following conditions are equivalent:*

(i) $\operatorname{Irr} R \subset \operatorname{Irr} A$,

(ii) *R is factorially closed in A.*

Proof. (i) $\Rightarrow$ (ii)

Assume that $\operatorname{Irr} R \subset \operatorname{Irr} A$ and consider elements $x, y \in A$ such that $xy \in R \setminus \{0\}$. Let $xy = q_1 \dots q_r$, where $q_1, \dots, q_r \in \operatorname{Irr} R$. Then $q_1, \dots, q_r \in \operatorname{Irr} A$, so without loss of generality we may assume that $x = a q_1 \dots q_s$ and $y = b q_{s+1} \dots q_r$ for some $a, b \in A^*$. Since $A^* = R^*$, we infer that $x, y \in R$.

(ii) $\Rightarrow$ (i)

Assume that condition (ii) holds and consider an element $a \in \operatorname{Irr} R$. Let $a = bc$, where $b, c \in A$. By the assumption, $b, c \in R$, so $b \in R^*$ or $c \in R^*$. Hence, $b \in A^*$ or $c \in A^*$. □

Note the following easy fact.

Lemma 3.3. *Let A be a domain, let R be a subring of A. The following conditions are equivalent:*

(i) $R_0 \cap A = R$,

(ii) *for every $x \in R$, $y \in A$, if $xy \in R \setminus \{0\}$, then $y \in R$.*

Theorem 3.4. *Let A be a unique factorization domain. Let R be a subring of A such that $R^* = A^*$ and $R_0 \cap A = R$. The following conditions are equivalent:*

(i) $\operatorname{Sqf} R \subset \operatorname{Sqf} A$,

(ii) *for every $x \in A$, $y \in \operatorname{Sqf} A$, if $x^2 y \in R \setminus \{0\}$, then $x, y \in R$.*

Proof. (i) $\Rightarrow$ (ii)

Assume that $\operatorname{Sqf} R \subset \operatorname{Sqf} A$ and consider $x, y \in A$ such that $y \in \operatorname{Sqf} A$ and $x^2 y \in R \setminus \{0\}$. If $x \in R$, then $x^2 \in R$, and hence $y \in R$ by Lemma 3.3.

Now suppose that $x \notin R$ and x is a minimal element (with respect to a number of irreducible factors in A) with this property. In this case $x \notin A^*$, so $x^2 y \notin \operatorname{Sqf} A$, and then $x^2 y \notin \operatorname{Sqf} R$. Hence, $x^2 y = z^2 t$ for some $z, t \in R$, $z \notin R^*$. We can present t in the form $t = u^2 v$, where $u, v \in A$, $v \in \operatorname{Sqf} A$. We have $u^2 v \in R \setminus \{0\}$, so $u \in R$ by the minimality of x. We obtain $x^2 y = z^2 u^2 v$, where $y, v \in \operatorname{Sqf} A$, hence $x = czu$ for some $c \in A^*$. By the assumptions, $c \in R$, so $x \in R$, a contradiction.

(ii) $\Rightarrow$ (i)

Assume that condition (ii) holds. Consider an element $r \in \operatorname{Sqf} R$. Suppose that $r \notin \operatorname{Sqf} A$, so $r = x^2 y$ for some $x, y \in A$ such that $x \notin A^*$ and $y \in \operatorname{Sqf} A$. Since $x^2 y \in R \setminus \{0\}$, we obtain that $x, y \in R$. We have $x \notin R^*$, so $x^2 y \notin \operatorname{Sqf} R$, a contradiction. □

Definition 3.5. Let A be a UFD. A subring R of A such that the implication

$$x^2 y \in R \setminus \{0\} \;\Rightarrow\; x, y \in R$$

holds for every $x \in A$, $y \in \operatorname{Sqf} A$, will be called *square-factorially closed* in A.

Recall that a subring R of a ring A is called *root closed* in A if the following implication:

$$x^n \in R \;\Rightarrow\; x \in R$$

holds for every $x \in A$ and $n \geqslant 1$. The properties of root closed subrings were investigated in many papers, see [1] and [5] for example.

Theorem 3.6. *Let A be a unique factorization domain. Let R be a subring of A such that $R^* = A^*$ and $R_0 \cap A = R$. If R is square-free closed in A, then R is root closed in A.*

Proof. Assume that R is square-free closed in A. Consider an element $x \in A$, $x \neq 0$, such that $x^n \in R$ for some $n \geqslant 1$. Let $n = 2^r(2l+1)$, where $r, l \geqslant 0$. Observe first that since $(x^{2l+1})^{2^r} \in R$, then applying Theorem 3.4 we obtain $x^{2l+1} \in R$.

Now, note that x can be presented in the form $x = s_0^{2^m} s_1^{2^{m-1}} \dots s_{m-1}^2 s_m$, where $s_0, \dots, s_m \in \operatorname{Sqf} A$ and $m \geqslant 0$. We will show by induction on m that the following implication:

$$x^{2l+1} \in R \;\Rightarrow\; x \in R$$

holds for every $x \in A$, $x \neq 0$. Let $m > 0$ and assume that the above implication holds for $m-1$. Put $t = s_0^{2^{m-1}} s_1^{2^{m-2}} \dots s_{m-1}$, so $x = t^2 s_m$ and $x^{2l+1} = (t^{2l+1} s_m^l)^2 s_m$. If $x^{2l+1} \in R$, then $t^{2l+1} s_m^l, s_m \in R$ by Theorem 3.4, so $t^{2l+1} \in R$ by Lemma 3.3. Hence, $t \in R$ by the induction assumption and, finally, $x \in R$. □

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