

RECENT RESULTS ON TRUNCATED TOEPLITZ OPERATORS

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ABSTRACT. Truncated Toeplitz operators are compressions of Toeplitz operators on model spaces; they have received much attention in the last years. This survey article presents several recent results, which relate boundedness, compactness, and spectra of these operators to properties of their symbols. We also connect these facts with properties of the natural embedding measures associated to these operators.

1. INTRODUCTION

Truncated Toeplitz operators on model spaces have been formally introduced by Sarason in [34], although some special cases have long ago appeared in literature, most notably as model operators for completely nonunitary contractions with defect numbers one and for their commutant. This new area of study has been recently very active and many open questions posed by Sarason in [34] have now been solved. See [5, 8, 12, 21, 9, 20, 36, 19, 35, 6, 13]. Nevertheless, there are still basic and interesting questions which remain mysterious.

The truncated Toeplitz operators live on the model spaces K_Θ , which are the closed invariant subspaces for the backward shift operator S^* acting on the Hardy space H^2 (see Section 2 for precise definitions). Given a model space K_Θ and a function $\phi \in L^2 = L^2(\mathbb{T})$, the truncated Toeplitz operator A_ϕ^Θ (or simply A_ϕ if there is no ambiguity regarding the model space) is defined on a dense subspace of K_Θ as the compression to K_Θ of multiplication by ϕ . The function ϕ is then called a symbol of the operator. An alternate way of defining a truncated Toeplitz operator is by means of a measure; in case ϕ is bounded, then a possible choice of the defining measure for A_ϕ^Θ is ϕdm (with m Lebesgue measure).

Note that the symbol or the associated measure are never uniquely defined by the operator. From this and other points of view the truncated Toeplitz operators have much more in common with Hankel Operators than with Toeplitz operators. This point of view will be occasionally pursued throughout the paper.

We intend to survey several recent results that are mostly scattered in the literature. They focus on the relation between the operator and the symbol or the measure. Obviously the nonuniqueness is a main issue, and in some situations it may be avoided by considering the so-called standard symbol of the operator.

2010 *Mathematics Subject Classification.* 30J05, 30H10, 46E22.

Key words and phrases. Truncated Toeplitz operators, model spaces, compactness.

The properties under consideration are boundedness, compactness, and spectra. Most of the results presented are known, and our intention is only to put them in context and emphasize their connections, indicating the relevant references. Part of the embedding properties of measures have not appeared explicitly in the literature, so some proofs are provided only where references seemed to be lacking.

The structure of the paper is the following. After a preliminary section with generalities about Hardy spaces and model spaces, we discuss in section 3 Carleson measures, first for the whole H^2 and then for model spaces. Truncated Toeplitz operators are introduced in Section 4, where one also discusses some boundedness properties. Section 5 is dedicated to compactness of truncated Toeplitz operators, and Section 6 to its relation to embedding measures. The last two sections discuss Schatten–von Neumann and spectral properties, respectively.

2. PRELIMINARIES

For the content of this section, [17] is a classical reference for general facts about Hardy spaces, while [26] can be used for Toeplitz and Hankel operators as well as for model spaces.

2.1. Function spaces. Recall that the Hardy space H^p of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the space of analytic functions f on \mathbb{D} satisfying $\|f\|_p < +\infty$, where

$$\|f\|_p = \sup_{0 \leq r < 1} \left(\int_0^{2\pi} |f(re^{it})|^p \frac{dt}{2\pi} \right)^{1/p}, \quad (1 \leq p < +\infty).$$

The algebra of bounded analytic functions on \mathbb{D} is denoted by H^∞ . We denote also $H_0^p = zH^p$ and $H_-^p = \overline{zH^p}$. Alternatively, H^p can be identified (via radial limits) to the subspace of functions $f \in L^p = L^p(\mathbb{T})$ for which $\hat{f}(n) = 0$ for all $n < 0$. Here \mathbb{T} denotes the unit circle with normalized Lebesgue measure m .

In the case $p = 2$, H^2 becomes a Hilbert space with respect to the scalar product inherited from L^2 and given by

$$\langle f, g \rangle_2 = \int_{\mathbb{T}} f(\zeta) \overline{g(\zeta)} dm(\zeta), \quad f, g \in L^2.$$

The orthogonal projection from L^2 to H^2 will be denoted by P_+ . The space H_-^2 is precisely the orthogonal of H^2 , and the corresponding orthogonal projection is $P_- = I - P_+$.

The Poisson transform of a function $f \in L^1$ is

$$(2.1) \quad \hat{f}(z) = \int_{\mathbb{T}} f(\xi) \frac{1 - |z|^2}{|1 - \bar{\xi}z|^2} d\xi, \quad z \in \mathbb{D}.$$

Suppose now Θ is an inner function, that is a function in H^∞ whose radial limits are of modulus one almost everywhere on \mathbb{T} . Its spectrum is defined by

$$(2.2) \quad \mathfrak{s}(\Theta) := \{\zeta \in \mathbb{D} : \liminf_{\lambda \in \mathbb{D}, \lambda \rightarrow \zeta} |\Theta(\lambda)| = 0\}.$$

Equivalently, if $\Theta = BS$ is the decomposition of Θ into a Blaschke product and a singular inner function, then $\rho(\Theta)$ is the union between the closure of the limit points of the zeros of B and the support of the singular measure associated to S . We will also define

$$\rho(\Theta) = \mathfrak{s}(\Theta) \cap \mathbb{T}.$$

We define the corresponding *shift-coinvariant subspace* generated by Θ (also called *model space*) by the formula $K_{\Theta}^p = H^p \cap \Theta \overline{H_0^p}$, where $1 \leq p < +\infty$. We will be especially interested in the Hilbert case, that is when $p = 2$. In this case, we also denote by $K_{\Theta} = K_{\Theta}^2$ and it is easy to see that K_{Θ} is also given by the following

$$K_{\Theta} = H^2 \ominus \Theta H^2 = \left\{ f \in H^2 : \langle f, g \rangle = 0, \forall g \in H^2 \right\}.$$

The orthogonal projection of L^2 onto K_{Θ} is denoted by P_{Θ} . It is well known (see [26, page 34]) that $P_{\Theta} = P_+ - \Theta P_+ \overline{\Theta}$. Since P_+ acts boundedly on L^p , $1 < p < \infty$, this formula shows that P_{Θ} can also be regarded as a bounded operator from L^p into K_{Θ}^p , $1 < p < \infty$.

The spaces H^2 and K_{Θ} are reproducing kernel spaces over the unit disc \mathbb{D} . The respective reproducing kernels are, for $\lambda \in \mathbb{D}$,

$$k_{\lambda}(z) = \frac{1}{1 - \bar{\lambda}z},$$

$$k_{\lambda}^{\Theta}(z) = \frac{1 - \overline{\Theta(\lambda)}\Theta(z)}{1 - \bar{\lambda}z}.$$

Evaluations at certain points $\zeta \in \mathbb{T}$ may also be bounded sometimes; this happens precisely when Θ has an angular derivative in the sense of Caratheodory at ζ [1]. In this case the function $k_{\zeta}^{\Theta}(z) = \frac{1 - \overline{\Theta(\zeta)}\Theta(z)}{1 - \bar{\zeta}z}$ is in K_{Θ} , and it is the reproducing kernel for the point ζ .

It is easy to check that, if $f, g \in K_{\Theta}$, then $fg \in H^1 \cap \bar{z}\Theta^2 H^1 \subset K_{\Theta^2}^1$. In particular, if f, g are also bounded, then $fg \in K_{\Theta^2}$. So $(k_{\lambda}^{\Theta})^2 \in K_{\Theta^2}$ for all $\lambda \in \mathbb{D}$.

The map C_{Θ} defined on L^2 by

$$(2.3) \quad C_{\Theta}f = \Theta \bar{z} \bar{f};$$

is a conjugation (i.e. C_{Θ} is anti-linear, isometric and involutive), which has the convenient supplementary property of mapping K_{Θ} precisely onto K_{Θ} .

2.2. One-component inner functions. In view of their main role in the study of operators on model spaces, we devote this subsection to a particular class of inner functions. Fix a number $0 < \epsilon < 1$, and define

$$(2.4) \quad \Omega(\Theta, \epsilon) = \{z \in \mathbb{D} : |\Theta(z)| < \epsilon\}.$$

The function Θ is called *one-component* if there exists a value of ϵ for which $\Omega(\Theta, \epsilon)$ is connected. (If this happens, then $\Omega(\Theta, \delta)$ is connected for every $\epsilon < \delta < 1$.) One-component functions have been introduced by Cohn [15]. An extensive study of these functions appears in [4, 3]; all results quoted below appear in [3].

The above definition is not very transparent. In fact, one-component functions are rather special: a first immediate reason is that they must satisfy $m(\rho(\Theta)) = 0$. This condition, of course, is not sufficient, but it suggests examining some simple cases.

The set $\rho(\Theta)$ is empty for finite Blaschke products, which are one-component. The next simplest case is when $\rho(\Theta)$ consists of just one point. One can prove easily that the elementary singular inner functions $\Theta(z) = e^{\frac{z+\zeta}{z-\zeta}}$ (for $\zeta \in \mathbb{T}$) are indeed one-component.

Suppose then that Θ is a Blaschke product whose zeros a_n tend nontangentially to a single point $\zeta \in \mathbb{T}$. If

$$(2.5) \quad \inf_{n \geq 1} \frac{|\zeta - a_{n+1}|}{|\zeta - a_n|} > 0,$$

then Θ is one-component. So, in particular, if $0 < r < 1$ and Θ is the Blaschke product with zeros $1 - r^n$, $n \geq 1$, then Θ is one-component. If condition (2.5) is not satisfied, then usually Θ is not one-component. A detailed discussion of such Blaschke products is given in [3], including the determination of the classes $\mathcal{C}_p(\Theta)$ (see Subsection 3.2).

One-component inner functions can be characterized by an estimate on the H^∞ norm of the reproducing kernels k_λ^Θ . While for a general inner function Θ we have $\|k_\lambda^\Theta\|_\infty = O(1 - |\lambda|^{-1})$, this estimate can be improved for one-component functions: Θ is one-component if and only if there exists a constant $C > 0$ such that for every $\lambda \in \mathbb{D}$, we have

$$\|k_\lambda^\Theta\|_\infty \leq C \frac{1 - |\Theta(\lambda)|}{1 - |\lambda|}.$$

2.3. Multiplication operators and their cognates. For $\phi \in L^\infty$, we denote by $M_\phi f = \phi f$ the multiplication operator on L^2 ; we have $\|M_\phi\| = \|\phi\|_\infty$. The Toeplitz operator $T_\phi : H^2 \rightarrow H^2$ and the Hankel operator $H_\phi : H^2 \rightarrow H_-^2 = L^2 \ominus H^2$ are given by the formulae

$$T_\phi = P_+ M_\phi, \quad H_\phi = P_- M_\phi.$$

In the case where ϕ is analytic, T_ϕ is just the restriction of M_ϕ to H^2 . We have $T_\phi^* = T_{\bar{\phi}}$ and $H_\phi^* = P_+ M_{\bar{\phi}} P_-$.

It should be noted that, while the symbols of M_ϕ and T_ϕ are uniquely defined by the operators, this is not the case with H_ϕ . Indeed, it is easy to check that $H_\phi = H_\psi$ if and only if $\phi - \psi \in H^\infty$. So statements about Hankel operators often imply only the existence of a symbol with corresponding properties.

The Hankel operators have the range and domain spaces different. It is sometimes preferable to work with an operator acting on a single space. For this, we introduce in L^2 the unitary symmetry \mathcal{J} defined by

$$\mathcal{J}(f)(z) = \bar{z}f(\bar{z}).$$

We have then $\mathcal{J}(H^2) = H_-^2$ and $\mathcal{J}(H_-^2) = H^2$. Define $\Gamma_\phi : H^2 \rightarrow H^2$ by

$$(2.6) \quad \Gamma_\phi = \mathcal{J}H_\phi.$$

Obviously properties of boundedness or compactness are the same for H_ϕ and Γ_ϕ .

The definition of M_ϕ , T_ϕ and H_ϕ can be extended to the case when the symbol ϕ is only in L^2 instead of L^∞ , obtaining (possibly unbounded) densely defined operators. Then M_ϕ and T_ϕ are bounded if and only if $\phi \in L^\infty$ (and $\|M_\phi\| = \|T_\phi\| = \|\phi\|_\infty$). The situation is more complicated for H_ϕ . Namely, H_ϕ is bounded if and only if there exists $\psi \in L^\infty$ with $H_\phi = H_\psi$, and

$$\|H_\phi\| = \inf\{\|\psi\|_\infty : H_\phi = H_\psi\}$$

This is known as Nehari's Theorem; see, for instance, [24, p. 182]. Moreover (but we will not pursue this in the sequel) an equivalent condition is $P_-\phi \in BMO$ (and $\|H_\phi\|$ is then a norm equivalent to $\|P_-\phi\|_{BMO}$).

Related results are known for compactness. The operators M_ϕ and T_ϕ are never compact except in the trivial case $\phi \equiv 0$. Hartman's Theorem states that H_ϕ is compact if and only if there exists $\psi \in C(\mathbb{T})$ with $H_\phi = H_\psi$; or, equivalently, $P_-\phi \in VMO$. If we know that ϕ is bounded, then H_ϕ is compact if and only if $\phi \in C(\mathbb{T}) + H^\infty$.

3. CARLESON MEASURES

3.1. Embedding of Hardy spaces. Let us discuss first some objects related to the Hardy space; we will afterwards see what analogous facts are true for the case of model spaces.

A positive measure μ on \mathbb{D} is called a Carleson measure if $H^2 \subset L^2(\mu)$ (such an inclusion is automatically continuous). It is known that this is equivalent to $H^p \subset L^p(\mu)$ for all $1 \leq p < \infty$. Carleson measures can also be characterized by a geometrical condition, as follows. For an arc $I \subset \mathbb{T}$ such that $|I| < 1$ we define

$$S(I) = \{z \in \mathbb{D} : 1 - |I| < |z| < 1 \text{ and } z/|z| \in I\}.$$

Then μ is a Carleson measure if and only if

$$(3.1) \quad \sup_I \frac{\mu(S(I))}{|I|} < \infty.$$

Condition (3.1) is called the *Carleson condition*.

The result can actually be extended (see [10]) to measures defined on $\overline{\mathbb{D}}$. Again the characterization does not depend on p , and it amounts to the fact that $\mu|_{\mathbb{T}}$ is absolutely continuous with respect to Lebesgue measure with essentially bounded density, while $\mu|_{\mathbb{D}}$ satisfies (3.1).

There is a link between Hankel operators and Carleson measures that has first appeared in [29, 39]; a comprehensive presentation can be found in [28, 1.7]. Let μ be

a finite complex measure on $\overline{\mathbb{D}}$. Define the operator $\Gamma[\mu]$ on analytic polynomials by the formula

$$\langle \Gamma[\mu]f, g \rangle = \int_{\overline{\mathbb{D}}} zf(z)\overline{g(\bar{z})} d\mu(z).$$

Note that if μ is supported on \mathbb{T} , then the matrix of $\Gamma[\mu]$ in the standard basis of H^2 is $(\hat{\mu}(i+j))_{i,j \geq 0}$, where $\hat{\mu}(i)$ are the Fourier coefficients of μ .

Then the operator $\Gamma[\mu]$ is bounded whenever μ is a Carleson measure. Conversely, if $\Gamma[\mu]$ is bounded, then there exists a Carleson measure ν on \mathbb{D} such that $\Gamma[\mu] = \Gamma[\nu]$.

It is easy to see that if $d\mu = \phi dm$ for some $\phi \in L^\infty$, then $\Gamma[\mu] = \Gamma_\phi$, where Γ_ϕ has been defined by (2.6) and is the version of a Hankel operator acting on a single space.

Analogous results may be proved concerning compactness. In this case the relevant notion is that of *vanishing Carleson measure*, which is defined by the property

$$(3.2) \quad \lim_{|I| \rightarrow 0} \frac{\mu(S(I))}{|I|} = 0.$$

Note that vanishing Carleson measures cannot have mass on the unit circle (intervals containing a Lebesgue point of the corresponding density would contradict the vanishing condition). Then the embedding $H^p \subset L^p(\mu)$ is compact if and only if μ is a vanishing Carleson measure.

A similar connection exists to compactness of Hankel operators. If μ is a vanishing Carleson measure on $\overline{\mathbb{D}}$, then $\Gamma[\mu]$ is compact. Conversely, if $\Gamma[\mu]$ is compact, then there exists a vanishing Carleson measure ν on $\overline{\mathbb{D}}$ such that $\Gamma[\mu] = \Gamma[\nu]$.

3.2. Embedding of model spaces. Similar questions for model spaces have been developed starting with the papers [15, 16] and [38]; however, the results in this case are less complete. Let us introduce first some notations. For $1 \leq p < \infty$, define

$$\begin{aligned} \mathcal{C}_p(\Theta) &= \{\mu \text{ finite measure on } \mathbb{T} : K_\Theta^p \hookrightarrow L^p(|\mu|) \text{ is bounded}\}, \\ \mathcal{C}_p^+(\Theta) &= \{\mu \text{ positive measure on } \mathbb{T} : K_\Theta^p \hookrightarrow L^p(\mu) \text{ is bounded}\}, \\ \mathcal{V}_p(\Theta) &= \{\mu \text{ finite measure on } \mathbb{T} : K_\Theta^p \hookrightarrow L^p(|\mu|) \text{ is compact}\}, \\ \mathcal{V}_p^+(\Theta) &= \{\mu \text{ positive measure on } \mathbb{T} : K_\Theta^p \hookrightarrow L^p(\mu) \text{ is compact}\}. \end{aligned}$$

It is clear that $\mathcal{C}_p(\Theta)$ and $\mathcal{V}_p(\Theta)$ are complex vectorial subspaces of the complex measures on the unit circle. Using the relations $K_{\Theta^2} = K_\Theta \oplus \Theta K_\Theta$ and $K_\Theta \cdot K_\Theta \subset K_{\Theta^2}^1$, it is easy to see that $\mathcal{C}_2(\Theta^2) = \mathcal{C}_2(\Theta)$, $\mathcal{C}_1(\Theta^2) \subset \mathcal{C}_2(\Theta)$, and $\mathcal{V}_1(\Theta^2) \subset \mathcal{V}_2(\Theta)$.

It is natural to look for geometric conditions to characterize these classes. Things are, however, more complicated, and the results are only partial. We start by fixing a number $0 < \epsilon < 1$; then the (Θ, ϵ) -Carleson condition asserts that

$$(3.3) \quad \sup_I \frac{\mu(S(I))}{|I|} < \infty,$$

where the supremum is taken only over the intervals $|I|$ such that $S(I) \cap \Omega(\Theta, \epsilon) \neq \emptyset$. (Remember that $\Omega(\Theta, \epsilon)$ is given by (2.4).)

It is then proved in [38] that if μ satisfies the (Θ, ϵ) -Carleson condition, then the embedding $K_{\Theta}^p \subset L^p(\mu)$ is continuous. The converse is true if Θ is *one-component*; in which case the embedding condition does not depend on p , while fulfilling of the (Θ, ϵ) -Carleson condition does not depend on $0 < \epsilon < 1$ (see Theorem 3.1 below).

As concerns the general case, it is shown by Aleksandrov [3] that if the converse is true for some $1 \leq p < \infty$, then Θ is one-component. Also, Θ is one-component if and only if the embedding condition does not depend on p . More precisely, the next theorem is proved in [3] (note that a version of this result for $p \in (1, \infty)$ already appears in [38]).

Theorem 3.1. *The following are equivalent for an inner function Θ :*

- (1) Θ is one-component.
- (2) For some $0 < p < \infty$ and $0 < \epsilon < 1$, $\mathcal{C}_p(\Theta)$ coincides with the class of measures that satisfy the (Θ, ϵ) -Carleson condition.
- (3) For all $0 < p < \infty$ and $0 < \epsilon < 1$, $\mathcal{C}_p(\Theta)$ coincides with the class of measures that satisfy the (Θ, ϵ) -Carleson condition.
- (4) The class $\mathcal{C}_p(\Theta)$ does not depend on $p \in (0, \infty)$.

In particular, if Θ is one component, then so is Θ^2 , whence $\mathcal{C}_1(\Theta^2) = \mathcal{C}_2(\Theta^2) = \mathcal{C}_2(\Theta)$.

Note that a general characterization of $\mathcal{C}_2(\Theta)$ has recently been obtained in [22]; however, the geometric content of this result is not easy to see.

The question of compactness of the embedding $K_{\Theta}^p \subset L^p(\mu)$ in this case should be related to a vanishing Carleson condition. In fact, there are two vanishing conditions, introduced in [14]. What is called therein the *second vanishing condition* is easier to state. We say that μ satisfies the *second (Θ, ϵ) -vanishing condition* [7, 14] if for each $\eta > 0$ there exists $\delta > 0$ such that $\mu(S(I))/|I| < \eta$ whenever $|I| < \delta$ and $S(I) \cap \Omega(\Theta, \epsilon) \neq \emptyset$. The following result is then proved in [7].

Theorem 3.2. *If the positive measure μ satisfies the second (Θ, ϵ) -vanishing condition, then the embedding $K_{\Theta}^p \subset L^p(\mu)$ is compact for $1 < p < \infty$.*

The converse is true in case Θ is one-component.

In other words, the theorem thus states that positive measures that satisfy the second vanishing condition are in $\mathcal{V}_p^+(\Theta)$ for all $1 < p < \infty$, and the converse is true for Θ one-component.

To discuss the case $p = 1$, we have to introduce what is called in [14] the *first vanishing condition*. Let us call the supremum in (3.3) the (Θ, ϵ) -Carleson constant of μ . Define

$$(3.4) \quad H_{\delta} = \{z \in \overline{\mathbb{D}} : \text{dist}(z, \rho(\Theta)) < \delta\},$$

and $\mu_\delta(A) = \mu(A \cap H_\delta)$. Then μ_δ are also Θ -Carleson measures, with (Θ, ϵ) -Carleson constants decreasing when δ decreases. We say that μ satisfies the first (Θ, ϵ) -vanishing condition if these Carleson constants tend to 0 when $\delta \rightarrow 0$.

It is shown in [7] that the first vanishing condition implies the second, and that the converse is not true: there exist measures which satisfy the second vanishing condition but not the first.

The next theorem is proved in [14].

Theorem 3.3. *If a positive measure μ satisfies the first (Θ, ϵ) -vanishing condition, then $\mu \in \mathcal{V}_p^+(\Theta)$ for $1 \leq p < \infty$.*

In case $\mu \in \mathcal{C}_p(\Theta)$, we will denote by $\iota_{\mu,p} : K_\Theta^p \rightarrow L^p(|\mu|)$ the embedding (which is then known to be a bounded operator). Then $\mu \in \mathcal{V}_p(\Theta)$ means that $\iota_{\mu,p}$ is compact. We will also write ι_μ instead of $\iota_{\mu,2}$.

4. TRUNCATED TOEPLITZ OPERATORS

Let Θ be an inner function and $\phi \in L^2$. The truncated Toeplitz operator $A_\phi = A_\phi^\Theta$, introduced by Sarason in [34], will be a densely defined, possibly unbounded operator on K_Θ . Its domain is $K_\Theta \cap H^\infty$, on which it acts by the formula

$$A_\phi f = P_\Theta(\phi f), \quad f \in K_\Theta \cap H^\infty.$$

If A_ϕ thus defined extends to a bounded operator, that operator is called a TTO. The class of all TTOs on K_Θ is denoted by $\mathcal{T}(\Theta)$, and the class of all nonnegative TTO's on K_Θ is denoted by $\mathcal{T}(\Theta)^+$.

Although these operators are called truncated *Toeplitz*, they have more in common with *Hankel* operators H_ϕ , or rather with their cognates Γ_ϕ , which act on a single space. As a first example of this behavior, we note that the symbol of a truncated Toeplitz operators is not unique. It is proved in [34] that

$$(4.1) \quad A_{\phi_1} = A_{\phi_2} \iff \phi_1 - \phi_2 \in \Theta H^2 + \overline{\Theta H^2}.$$

Let us denote $\mathfrak{S}_\Theta = L^2 \ominus (\Theta H^2 + \overline{\Theta H^2})$; it is called the space of *standard* symbols. It follows from (4.1) that every TTO has a unique standard symbol. One proves in [34, Section 3] that \mathfrak{S} is contained in $K_\Theta + \overline{K_\Theta}$ as a subspace of codimension at most one; this last space is sometimes easier to work with.

It is often the case that the assumption $\Theta(0) = 0$ simplifies certain calculations. For instance, in that case we have precisely $\mathfrak{S} = K_\Theta + \overline{K_\Theta}$; we will see another example in Section 7. Fortunately, there is a procedure to pass from a general inner Θ to one that has this property: it is called the *Crofoot transform*. For $a \in \mathbb{D}$ let Θ_a be given by the formula

$$\Theta_a(z) = \frac{\Theta(z) - a}{1 - \bar{a}\Theta(z)}.$$

If we define the Crofoot transform by

$$J(f) := \frac{\sqrt{1-|a|^2}}{1-\bar{a}\Theta} f,$$

then J is a unitary operator from K_Θ to K_{Θ_a} , and

$$(4.2) \quad J\mathcal{T}(\Theta)J^* = \mathcal{T}(\Theta_a).$$

In particular, if $a = \Theta(0)$, then $\Theta_a(0) = 0$, and (4.2) allows the transfer of properties from TTOs on K_{Θ_a} to TTOs on K_Θ .

Especially nice properties are exhibited by TTOs which have an analytic symbol $\phi \in H^2$ (of course, this is never a standard symbol). It is a consequence of interpolation theory [33] that

$$\{A_\phi^\Theta \in \mathcal{T}(\Theta) : \phi \in H^2\} = \{A_z^\Theta\}'$$

(A_z^Θ is called a *compressed shift*, or a *model operator*).

One should also mentioned that other two classes of TTOs have already been studied in different contexts. First, the classical finite Toeplitz matrices are TTOs with $\Theta(z) = z^n$ written in the basis of monomials. Secondly, TTOs with $\Theta(z) = e^{\frac{z+1}{z-1}}$ correspond, after some standard transformations, to a class of operators alternately called Toeplitz operators on Paley–Wiener spaces [31], or truncated Wiener–Hopf operators [11].

There is an alternate manner to introduce TTOs, related to the Carleson measures in the previous section. For every $\mu \in \mathcal{C}_2(\Theta)$ the sesquilinear form

$$(f, g) \mapsto \int f\bar{g} d\mu$$

is bounded, and therefore there exists a bounded operator A_μ^Θ on K_Θ such that

$$(4.3) \quad \langle A_\mu^\Theta f, g \rangle = \int f\bar{g} d\mu.$$

It is shown in [34, Theorem 9.1] that A_μ^Θ thus defined is actually a TTO. In fact, the converse is also true, as stated in Theorem 4.2 below. An interesting open question is the characterization of the measures μ for which $A_\mu = 0$.

The definition of TTOs does not make precise the class of symbols $\phi \in L^2$ that produce bounded TTOs. A first remark is that the standard symbol of a bounded truncated Toeplitz operator is not necessarily bounded. To give an example, consider an inner function Θ with $\Theta(0) = 0$, for which there exists a singular point $\zeta \in \mathbb{T}$ where Θ has an angular derivative in the sense of Caratheodory. It is shown then in [34, Section 5] that $k_\zeta^\Theta \otimes k_\zeta^\Theta$ is a bounded rank one TTO with standard symbol $k_\zeta^\Theta + \overline{k_\zeta^\Theta} - 1$, and that this last function is unbounded.

A natural question is therefore whether every bounded TTO has a bounded symbol (such as is the case with Hankel operators). In the case of T_ϕ with ϕ analytic, the answer is readily seen to be positive, being proved again in [33]; moreover,

$$\inf\{\|\psi\|_\infty : \psi \in H^\infty, A_\psi^\Theta = A_\phi^\Theta\} = \|A_\phi^\Theta\|.$$

The first negative answer for the general situation has been provided in [6], and the counterexample is again given by the rank one TTO $k_\zeta^\Theta \otimes k_\zeta^\Theta$. The following result is proved in [6].

Theorem 4.1. *Suppose Θ has an angular derivative in the sense of Caratheodory in $\zeta \in \mathbb{T}$ (equivalently, $k_\zeta^\Theta \in L^2$), but $k_\zeta^\Theta \notin L^p$ for some $p \in (2, \infty)$. Then $k_\zeta^\Theta \otimes k_\zeta^\Theta$ has no bounded symbol.*

A more general result has been obtain in [5], where one also makes clear the relation between measures and TTO. In particular, one characterizes the inner functions Θ which have the property that every bounded TTO on K_Θ has a bounded symbol.

Theorem 4.2. *Suppose Θ is an inner function.*

- (1) *For every bounded TTO $A \geq 0$ there exists a positive measure $\mu \in \mathcal{C}_2^+(\Theta)$ such that $A = A_\mu^\Theta$.*
- (2) *For every bounded $A \in \mathcal{T}(\Theta)$ there exists a complex measure $\mu \in \mathcal{C}_2(\Theta)$ such that $A = A_\mu^\Theta$.*
- (3) *A bounded TTO $A \in \mathcal{T}(\Theta)$ admits a bounded symbol if and only if $A = A_\mu^\Theta$ for some $\mu \in \mathcal{C}_1(\Theta^2)$.*
- (4) *Every bounded TTO on K_Θ admits a bounded symbol if and only if $\mathcal{C}_1(\Theta^2) = \mathcal{C}_2(\Theta^2)$.*

In particular, as shown by Theorem 3.1, the second condition is satisfied if Θ is one-component (since then all classes $\mathcal{C}_p(\Theta)$ coincide). It is still an open question whether Θ one-component is actually equivalent to $\mathcal{C}_1(\Theta^2) = \mathcal{C}_2(\Theta^2)$. (As mentioned previously, Θ is one-component if and only if Θ^2 is one-component.) Such a result would be a significant strengthening of Theorem 3.1.

As a general observation, one may say that, if Θ is one-component, then TTOs on K_Θ have many properties analogous to those of Hankel operators. This is the class of inner functions for which the current theory is more developed.

5. COMPACT OPERATORS

Surprisingly enough, the first result about compactness of TTOs dates from 1970. In [1, Section 5] one introduces what are, in our terminology, TTOs with continuous symbol, and one proves the following theorem.

Theorem 5.1. *If Θ is inner and ϕ is continuous on \mathbb{T} , then A_ϕ^Θ is compact if and only if $\phi|_{\rho(\Theta)} = 0$.*

This result has been rediscovered more recently in [21]; see also [20].

Thinking of Hartmann's theorem, it seems plausible to believe that continuous symbols play for compact TTOs the role played by bounded symbols for general TTOs. However, as shown by Theorem 4.1, there exist inner functions Θ for which even rank-one operators might not have bounded symbols (not to speak about continuous). So we have to consider only certain classes of inner functions, suggested by the boundedness results in the previous section. In this sense one has the following result proved by Bessonov [8].

Theorem 5.2. *Suppose that $\mathcal{C}_1(\Theta^2) = \mathcal{C}_2(\Theta)$ and A is a truncated truncated Toeplitz operator. Then the following are equivalent:*

- (1) A is compact.
- (2) $A = A_{\phi\Theta}$ for some $\phi \in \mathbf{C}(\mathbb{T})$.

In particular, this is true if Θ is one component.

One can see that instead of $\mathbf{C}(\mathbb{T})$ the main role is played by $\Theta\mathbf{C}(\mathbb{T})$. We give below some ideas about the connection between these two classes.

Theorem 5.3. *Suppose $\mathcal{C}_2(\Theta) = \mathcal{C}_1(\Theta^2)$ and $m(\rho(\Theta)) = 0$. Then the following are equivalent for a truncated Toeplitz operator A .*

- (i) A is compact.
- (ii) $A = A_{\phi}^{\Theta}$ for some $\phi \in \mathbf{C}(\mathbb{T})$ with $\phi|_{\rho(\Theta)} = 0$.

Proof. (ii) \implies (i) is proved in Theorem 5.1.

Suppose now (i) is true. By Theorem 5.2 $A = A_{\Theta\psi}$ for some $\psi \in \mathbf{C}(\mathbb{T})$. By the Rudin–Carleson interpolation theorem (see, for instance, [18, Theorem II.12.6]), there exists a function $\psi_1 \in \mathbf{C}(\mathbb{T}) \cap H^{\infty}$ (that is, in the disk algebra) such that $\psi|_{\rho(\Theta)} = \psi_1|_{\rho(\Theta)}$. Then one checks easily that $\phi = \Theta(\psi - \psi_1)$ is continuous on \mathbb{T} , and $A_{\phi} = A_{\Theta\psi}$ (since $A_{\Theta\psi_1} = 0$). \square

In particular, Theorem 5.3 applies to the case Θ one-component, since for such functions we have $\mathcal{C}_2(\Theta) = \mathcal{C}_1(\Theta^2)$ and $m(\rho(\Theta)) = 0$ [2, Theorem 6.4].

We also have the following result which is contained in [8, Proposition 2.1]; here is a simpler proof.

Proposition 5.4. (i) *If $\phi \in \Theta\mathbf{C}(\mathbb{T}) + \Theta H^{\infty}$, then A_{ϕ} is compact.*
(ii) *If $\phi \in \mathbf{C}(\mathbb{T}) + H^{\infty}$, then the converse is also true.*

Proof. First note that

$$(5.1) \quad A_{\phi} = (\Theta H_{\bar{\Theta}\phi} - H_{\phi})|_{K_{\Theta}}.$$

By Hartmann's Theorem we know that a Hankel operator with bounded symbol is compact if and only if its symbol is in $\mathbf{C}(\mathbb{T}) + H^{\infty}$. Since $\mathbf{C}(\mathbb{T}) + H^{\infty}$ is an algebra, $\phi \in \Theta\mathbf{C}(\mathbb{T}) + \Theta H^{\infty}$, that is, $\bar{\Theta}\phi \in \mathbf{C}(\mathbb{T}) + H^{\infty}$, implies $\phi \in \mathbf{C}(\mathbb{T}) + H^{\infty}$. Then applying (5.1) proves (i).

On the other hand, if $\phi \in \mathbf{C}(\mathbb{T}) + H^{\infty}$, again (5.1) proves (ii). \square

It is interesting to compare Theorem 5.2 to Proposition 5.4. Suppose that a TTO A_ϕ is compact. Proposition 5.4 says that, if we know that $\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$, then it has actually to be in $\Theta\mathbf{C}(\mathbb{T}) + \Theta H^\infty$. So there exists $\psi \in \mathbf{C}(\mathbb{T}) + H^\infty$ such that $\phi = \Theta\psi$. This is true with no special assumption on Θ , but the symbol ϕ is assumed to be in a particular class.

On the other hand, suppose that Θ satisfies the assumption $\mathcal{C}_2(\Theta) = \mathcal{C}_1(\Theta^2)$, and again A_ϕ is compact. Without any a priori assumption on the symbol, applying Theorem 5.2 yields the existence of $\psi \in \mathbf{C}(\mathbb{T}) + H^\infty$ such that $A_\phi = A_{\Theta\psi}$. However, in this case we will not necessarily have $\phi = \Theta\psi$, but, according to (4.1), $\phi - \Theta\psi \in \Theta H^2 + \overline{\Theta H^2}$.

It would be interesting to give an example of a compact operator, with a symbol $\psi \in \Theta\mathbf{C}(\mathbb{T}) + \Theta H^\infty$, that has no continuous symbol.

Since A_ϕ is compact if and only if $A_\phi^* = A_{\bar{\phi}}$ is, any condition on the symbol produces another one by conjugation. So one expects a definitive result to be invariant by conjugation. This is not the case, for instance, with Proposition 5.4: by conjugation we obtain that if $\phi \in \Theta\mathbf{C}(\mathbb{T}) + \overline{\Theta H^\infty}$, then A_ϕ^Θ is compact. Also, in Theorem 5.1 one could add a third equivalent condition, namely that $A = A_{\phi\bar{\Theta}}$ for some $\phi \in \mathbf{C}(\mathbb{T})$. From this point of view, Theorem 5.3 is more satisfactory. Naturally, if Θ is one-component would actually be equivalent to $\mathcal{C}_1(\Theta^2) = \mathcal{C}_2(\Theta)$ (the open question stated above), then Theorems 5.1 and 5.3 would turn out to be equivalent to a simple and symmetric statement for this class of functions.

6. COMPACT TTOs AND EMBEDDING MEASURES

In the present section we discuss some relations between compactness of TTOs and embedding measures. Let us first remember that a (finite) complex measure on the unit circle can be decomposed by means of nonnegative finite measures, as stated more precisely in the following lemma [32, chap. 6].

Lemma 6.1. *If μ is a complex measure, one can write $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ with $0 \leq \mu_j \leq |\mu|$ for $1 \leq j \leq 4$.*

We will also use the following simple result.

Lemma 6.2. *If $0 \leq \nu_1 \leq \nu_2$, then $\nu_2 \in \mathcal{C}_p^+(\Theta)$ implies $\nu_1 \in \mathcal{C}_p^+(\Theta)$, and $\nu_2 \in \mathcal{V}_p^+(\Theta)$ implies $\nu_1 \in \mathcal{V}_p^+(\Theta)$.*

Proof. If $0 \leq \nu_1 \leq \nu_2$, then we have a contractive embedding $J : L^p(\nu_2) \rightarrow L^p(\nu_1)$, and the lemma follows from the equality $\iota_{\nu_1,p} = J\iota_{\nu_2,p}$. \square

The ultimate goal would be to obtain for compact TTOs statements similar to those for boundedness appearing in Theorem 4.2. But one can only obtain partial results: measures in $\mathcal{V}_2(\Theta)$ produce compact TTOs, but the converse can be obtained only for positive operators.

Theorem 6.3. *Suppose $A \in \mathcal{T}(\Theta)$.*

- (1) If there exists $\mu \in \mathcal{V}_2(\Theta)$ such that $A = A_\mu$, then A is compact.
(2) If A is compact and positive, then there exists $\mu \in \mathcal{V}_2^+(\Theta)$ such that $A = A_\mu$.

Proof. (1) Take A_μ with $\mu \in \mathcal{V}_2(\Theta)$. Writing $\mu = \mu_1 - \mu_2 + i\mu_3 - i\mu_4$ as in Lemma 6.1, one has $A_\mu = A_{\mu_1} - A_{\mu_2} + iA_{\mu_3} - iA_{\mu_4}$. Since $0 \leq \mu_j \leq |\mu|$, it follows that $\mu_j \in \mathcal{V}_2^+(\Theta)$ by Lemma 6.2. So we may suppose from the beginning that $\mu \in \mathcal{V}_2^+(\Theta)$.

To show that A_μ is compact, take a sequence (f_n) tending weakly to 0 in K_Θ , and $g \in K_\Theta$ with $\|g\|_2 = 1$. Formula (4.3) can be written

$$\langle A_\mu f_n, g \rangle_2 = \int \iota_\mu(f_n) \overline{\iota_\mu(g)} d\mu,$$

and thus

$$|\langle A_\mu f_n, g \rangle| \leq \|\iota_\mu(f_n)\|_{L^2(\mu)} \|\iota_\mu(g)\|_{L^2(\mu)} \leq \|\iota_\mu(f_n)\|_{L^2(\mu)} \|\iota_\mu\|.$$

Taking the supremum with respect to g , we obtain

$$\|A_\mu f_n\|_2 \leq \|\iota_\mu(f_n)\|_{L^2(\mu)} \|\iota_\mu\|.$$

But $f_n \rightarrow 0$ weakly and ι_μ compact imply that $\|\iota_\mu(f_n)\|_{L^2(\mu)} \rightarrow 0$. So $\|A_\mu f_n\| \rightarrow 0$ and therefore A_μ is compact.

(2) If $A \geq 0$, by Theorem 4.2, there exists $\mu \in \mathcal{C}_2^+(\Theta)$ such that $A = A_\mu$. We must then show that $\mu \in \mathcal{V}_2^+(\Theta)$; that is, ι_μ is compact.

Take then a sequence f_n tending weakly to 0 in K_Θ ; in particular, (f_n) is bounded, so we may assume $\|f_n\| \leq M$ for all n . Applying again formula (4.3), we have

$$\langle A_\mu f_n, f_n \rangle = \int \iota_\mu(f_n) \overline{\iota_\mu(f_n)} d\mu = \|\iota_\mu(f_n)\|_{L^2(\mu)}^2.$$

Therefore

$$\|\iota_\mu(f_n)\|_{L^2(\mu)}^2 \leq \|A_\mu f_n\| \|f_n\| \leq M \|A_\mu f_n\|.$$

Since A_μ is compact, $\|A_\mu f_n\| \rightarrow 0$. The same is then true about $\|\iota_\mu(f_n)\|_{L^2(\mu)}^2$; thus ι_μ is compact, that is, $\mu \in \mathcal{V}_2^+(\Theta)$. \square

This approach leads to an alternate proof of Theorem 5.1.

Proposition 6.4. *If $\phi \in \mathbf{C}(\mathbb{T})$ and $\phi|_\rho(\Theta) = 0$, then the measure $\mu = \phi dm$ is in $\mathcal{V}_p(\Theta)$ for every $1 \leq p < \infty$. In particular, A_ϕ is compact.*

Proof. Since $\phi \in L^\infty$, the measure $|\mu|$ is an obvious Θ -Carleson measure. Now fix $\epsilon > 0$. Since ϕ is uniformly continuous on \mathbb{T} , there exists $\eta > 0$ such that, if $\zeta \in \mathbb{T}$, $\text{dist}(\zeta, \rho(\Theta)) < \eta$, then $|\phi(\zeta)| < \epsilon$. In other words, if $\zeta \in H_\eta$, then $|\phi(\zeta)| < \epsilon$ (where H_η is defined by (3.4)).

Let $\delta < \eta$ and I be any arc of \mathbb{T} . Then we have

$$\begin{aligned} |\mu|_\delta(T(I)) &= |\mu|(T(I) \cap H_\delta) \\ &= \sup \left\{ \sum_{i \geq 1} |\mu(E_i)| : \bigcup_{i \geq 1} E_i = T(I) \cap H_\delta, E_i \cap E_j = \emptyset \text{ for } i \neq j \right\} \end{aligned}$$

Since $E_i \subset H_\delta \subset H_\eta$, note that

$$\begin{aligned} |\mu(E_i)| &= \left| \int_{E_i} \phi \, dm \right| \\ &\leq \int_{E_i} |\phi| \, dm \leq \epsilon m(E_i). \end{aligned}$$

Hence

$$|\mu|_\delta(T(I)) \leq \epsilon |I|,$$

which shows that the Θ -Carleson constant of $|\mu|_\delta$ is smaller than ϵ . We conclude the proof applying Theorem 3.3 and Theorem 6.3. \square

The next theorem is a partial analogue of Theorem 4.2 (3).

Theorem 6.5. *Suppose $\mu \in \mathcal{V}_1(\Theta^2)$. Then $A_\mu = A_{\Theta\phi}$ for some $\phi \in \mathbf{C}(\mathbb{T})$.*

Proof. By [8, Lemma 3.1] we know that $K_{z\Theta^2}^1 \cap zH^1$ is w^* -closed when we consider it embedded in $H^1 = \mathbf{C}(\mathbb{T})/H_0^1$. We define on $K_{z\Theta^2}^1 \cap zH^1$ the linear functional ℓ by

$$\ell(f) = \int \bar{\Theta}f \, d\mu.$$

It is clear that ℓ is continuous, but we assert that it is also w^* -continuous. Indeed, the w^* topology is metrizable (since $\mathbf{C}(\mathbb{T})$ is separable), and therefore we can check w^* -continuity on sequences.

If $f_n \rightarrow 0$ w^* , then, in particular, the sequence (f_n) is bounded. Then, since ι_μ is compact, the sequence $(\iota_\mu(f_n))$ is compact in $L^1(\mu)$, and a standard argument says that, in fact, $\iota_\mu f_n \rightarrow 0$ in $L^1(\mu)$. Then

$$\ell(f_n) = \int \bar{\Theta}(\iota_\mu f_n) \, d\mu \rightarrow 0.$$

It follows that there exists $\phi \in \mathbf{C}(\mathbb{T})$, such that

$$\ell(f) = \int \bar{\Theta}f \, d\mu = \int \phi f \, dm$$

for every $f \in K_{z\Theta^2}^1 \cap zH^1$, or, equivalently,

$$\int f \, d\mu = \int \Theta\phi f \, dm$$

for every $f \in \bar{\Theta}(K_{z\Theta^2}^1 \cap zH^1)$. If $g, h \in K_\Theta^2$, then $g\bar{h} \in \bar{\Theta}(K_{z\Theta^2}^1 \cap zH^1)$, so

$$\langle A_\mu g, h \rangle = \int g\bar{h} \, d\mu = \int \Theta\phi g\bar{h} \, dm = \langle A_{\Theta\phi} f, g \rangle,$$

which proves the theorem. \square

In particular, it follows from Proposition 5.4 that if $\mu \in \mathcal{V}_1(\Theta^2)$ then A_μ^Θ is compact.

7. TTOS IN OTHER IDEALS

The problem of deciding when certain TTOs are in Schatten–von Neumann classes S_p has no clear solution yet, even in the usually simple case of the Hilbert–Schmidt ideal. In [23] one gives criteria for particular cases; to convey their flavour, below is an example (Theorem 3 of [23]). Remember that Θ is called an interpolating Blaschke product if its zeros (z_i) form an interpolation sequence, or, equivalently, if they satisfy the Carleson condition

$$\inf_{i \in \mathbb{N}} \prod_{j \neq i} \left| \frac{z_i - z_j}{1 - \bar{z}_i z_j} \right| > 0.$$

Theorem 7.1. *Suppose Θ is an interpolating Blaschke product and ϕ is an analytic function. Then:*

- (1) A_ϕ is compact if and only if $\phi(z_i) \rightarrow 0$.
- (2) For $1 \leq p < \infty$, $A_\phi \in S_p$ if and only if $(\phi(z_i)) \in \ell^p$.

More satisfactory results are obtained in [23] in the case of Hilbert–Schmidt operators, but even in this case an explicit equivalent condition on the symbol is hard to formulate. Let us start by assuming that $\Theta(0) = 0$ (see the discussion of the Crofoot transform in Section 4); in this case the space of standard symbols \mathfrak{S} is precisely $K_\Theta + \overline{K_\Theta}$. We define then $\Phi = \Theta^2/z$; Φ is also an inner function with $\Phi(0) = 0$, and $C_\Theta(K_\Theta + \overline{K_\Theta}) = K_\Phi$ (remember that C_Θ is given by formula (2.3)).

Let then K_Φ^0 be the linear span (nonclosed) of the reproducing kernels k_λ^Φ , $\lambda \in \mathbb{D}$. It can be checked that for every $\lambda \in \mathbb{D}$ we have $(k_\lambda^\Phi)^2 \in K_\Phi$, and therefore the formula

$$\mathcal{D}_0 k_\lambda^\Phi = (k_\lambda^\Phi)^2$$

defines an (unbounded) linear operator $\mathcal{D}_0 : K_\Phi^0 \rightarrow K_\Phi$.

The result that is proved in [23] is the following.

Theorem 7.2. *With the above notations, the following assertions are true:*

- (1) \mathcal{D}_0 is a positive symmetric operator. Its Friedrichs selfadjoint extension (see [30, Theorem X.23]) will be denoted by \mathcal{D} ; it has a positive root $\mathcal{D}^{1/2}$.
- (2) A TTO A_ϕ^Θ , with $\phi \in K_\Theta + \overline{K_\Theta}$, is a Hilbert–Schmidt operator if and only if $C_\Theta \phi$ is in the domain of $\mathcal{D}^{1/2}$, and the Hilbert–Schmidt norm is $\|\mathcal{D}^{1/2}(C_\Theta \phi)\|$.

Since the square of a reproducing kernel is also a reproducing kernel, let us denote by \mathcal{H}_Θ^2 the reproducing kernel Hilbert space that has as kernels $(k_\lambda^\Theta)^2$ ($\lambda \in \mathbb{D}$). It is a space of analytic functions defined on \mathbb{D} , and it provides another characterization of Hilbert–Schmidt TTOs obtained in [23].

Theorem 7.3. *Define, for $\phi \in K_\Theta + \overline{K_\Theta}$,*

$$(\Delta\phi)(\lambda) = \langle C_\Theta \phi, (k_\lambda^\Theta)^2 \rangle.$$

Then:

- (1) $\Delta\phi$ is a function analytic in \mathbb{D} , which coincides on K_ϕ^0 with $\mathcal{D}(C_\Theta\phi)$.
(2) An alternate formula for $\Delta\phi$ is

$$(\Delta\phi)(\lambda) = (z\alpha)'(\lambda) - 2\Theta(\lambda)(z\alpha_2)'(\lambda),$$

where $C_\Theta\phi = \alpha = \alpha_1 + \Theta\alpha_2$, with $\alpha_1, \alpha_2 \in K_\Theta$.

- (3) A_ϕ^Θ is a Hilbert–Schmidt operator if and only if $\Delta\phi \in \mathcal{H}_\Theta^2$, and the Hilbert–Schmidt norm is $\|\Delta\phi\|_{\mathcal{H}_\Theta^2}$.

The proof of these two theorems uses the theory of Hankel forms as developed in [27]. Admittedly, none of the characterizations is very explicit.

For the case of one-component functions, a conjecture is proposed in [8, 4.3] for the characterization of Schatten–von Neumann TTOs. It states essentially that a truncated Toeplitz operator is in S_p if and only if it has at least one symbol ϕ in the Besov space $B_{pp}^{1/p}$ (note that this would not necessarily be the standard symbol). This last space admits several equivalent characterizations; for instance, if we define, for $\tau \in \mathbb{T}$, $\Delta_\tau f(z) = f(\tau z) - f(z)$, then

$$B_{pp}^{1/p} = \left\{ f \in L^p : \int_{\mathbb{T}} \frac{\|\Delta_\tau f\|_p^p}{|1-\tau|^2} dm(\tau) < \infty \right\}.$$

The conjecture is suggested by the similar result in the case of Hankel operators [28, Chapter 6]. It is true if $\Theta(z) = e^{\frac{z+1}{z-1}}$, as shown in [31].

Bessonov also proposes some alternate characterizations in terms of Clark measures; we will not pursue this approach here.

8. INVERTIBILITY AND FREDHOLMNESS

Invertibility and, more generally, spectrum of a TTO has been known since several decades in the case of analytic symbols. The main result here is stated in the next theorem (see, for instance, [25, 2.5.7]). It essentially says that $\sigma(A_\phi^\Theta) = \phi(\mathfrak{s}(\Theta))$, but we have to give a precise meaning to the quantity on the right, since $\mathfrak{s}(\Theta)$ (as defined by (2.2)) intersects the set \mathbb{T} , where $\phi \in H^\infty$ is defined only almost everywhere.

Theorem 8.1. *If $\phi \in H^\infty$, then*

$$\begin{aligned} \sigma(A_\phi^\Theta) &= \{\zeta \in \mathbb{C} : \inf_{z \in \mathbb{D}} (|\Theta(z)| + |\phi(z) - \zeta|) = 0\}, \\ \{\lambda \in \mathbb{C} : \lambda = \phi(z), \text{ where } z \in \mathbb{D}, \Theta(z) = 0\} &\subset \sigma_p(A_\phi^\Theta), \\ \sigma_e(A_\phi^\Theta) &= \{\zeta \in \mathbb{C} : \liminf_{z \in \mathbb{D}, |z| \rightarrow 1} (|\Theta(z)| + |\phi(z) - \zeta|) = 0\}. \end{aligned}$$

As noted above, the class of TTOs is invariant by conjugation, and therefore we may obtain corresponding characterizations for coanalytic symbols. But what happens for more general TTOs? Again a result in [1] seems to be historically the

first one. It deals with the essential spectrum of a TTO with continuous symbol. More precisely, it states that

$$\sigma_e(A_\phi^\Theta) = \phi(\rho(\Theta)).$$

There is a more extensive development of these ideas in [21], which, in particular, studies the C^* -algebra generated by TTOs with continuous symbols.

The above characterization of the essential spectrum is extended in [8] to symbols in $\mathbf{C}(\mathbb{T}) + H^\infty$. Since functions $\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$ are defined only almost everywhere on \mathbb{T} , one should explain the meaning of the right hand side. The following is the precise statement of Bessonov's result; its form is similar to that of Theorem 8.1.

Theorem 8.2. *Suppose $\phi \in \mathbf{C}(\mathbb{T}) + H^\infty$. Then*

$$\sigma_e(A_\phi^\Theta) = \{\zeta \in \mathbf{C} : \liminf_{z \in \mathbb{D}, |z| \rightarrow 1} (|\Theta(z)| + |\hat{\phi}(z) - \zeta|) = 0\}$$

(for the definition of $\hat{\phi}$, see (2.1)).

It is harder to find criteria for invertibility of TTOs with nonanalytic symbols. The next part of the section uses embedding measures to obtain some partial results. We start with a statement which is essentially about bounded below TTOs.

Theorem 8.3. *Let A be a (bounded) TTO, and let μ a complex measure such that $A = A_\mu$.*

- (1) *If A is bounded below, then ι_μ is also bounded below, i.e. there exists $C > 0$ (depending only on μ and Θ) such that*

$$\int_{\mathbb{T}} |f|^2 dm \leq C \int_{\mathbb{T}} |\iota_\mu(f)|^2 d|\mu|$$

for all $f \in K_\Theta$.

- (2) *Suppose $A \in \mathcal{T}(\Theta)^+$ and let $\mu \in \mathcal{C}_2(\Theta)^+$ such that $A = A_\mu$. The following assertions are equivalent:*

- (a) *the operator A is invertible;*
(b) *there exists $C > 0$ (depending only on μ and Θ) such that*

$$\int_{\mathbb{T}} |f|^2 dm \leq C \int_{\mathbb{T}} |\iota_\mu(f)|^2 d\mu$$

for all $f \in K_\Theta$.

Proof. (1) By definition of A_μ , for all $f, g \in K_\Theta$, we have

$$|\langle A_\mu f, g \rangle| = \left| \int_{\mathbb{T}} \iota_\mu(f) \overline{\iota_\mu(g)} d\mu \right| \leq \int_{\mathbb{T}} |\iota_\mu(f)| |\iota_\mu(g)| d|\mu|.$$

The Cauchy–Schwarz inequality implies that

$$|\langle A_\mu f, g \rangle| \leq \|\iota_\mu(f)\|_{L^2(|\mu|)} \|\iota_\mu(g)\|_{L^2(|\mu|)} \leq \|\iota_\mu(f)\|_{L^2(|\mu|)} \|\iota_\mu\| \|g\|_2.$$

Then, taking the supremum over all $g \in K_\Theta$ of unit norm, we get:

$$\|A_\mu f\|_2 \leq \|\iota_\mu\| \|\iota_\mu(f)\|_{L^2(|\mu|)}.$$

Now, if A_μ is bounded below, there exists $C > 0$ such that

$$\|\iota_\mu(f)\|_{L^2(|\mu|)} \geq \frac{C}{\|\iota_\mu\|} \|f\|_2,$$

and thus ι_μ is bounded below.

(2) First, recall that since $A_\mu = A_\mu^*$, A_μ is invertible if and only if A_μ is bounded below. Thus (a) \implies (b) follows from part (1).

Conversely, assume that ι_μ is bounded below and let C be the constant defined in (b). It remains to check that A_μ is bounded below.

For a nonzero $f \in K_\Theta$, we have:

$$\|A_\mu f\|_2 \geq |\langle A_\mu f, f / \|f\|_2 \rangle| = \frac{1}{\|f\|_2} \|\iota_\mu(f)\|_{L^2(\mu)}^2 \geq \frac{1}{\|f\|_2} \frac{1}{C} \|f\|_2^2 = \frac{\|f\|_2}{C},$$

as expected. \square

Volberg [37] proved that given $\varphi \in L^\infty(\mathbb{T})$, and an inner function Θ , the following are equivalent:

- there exist $C_1, C_2 > 0$ such that

$$C_1 \|f\|_2 \leq \|f\|_{L^2(|\varphi|dm)} \leq C_2 \|f\|_2,$$

for all $f \in K_\Theta$;

- there exists $\delta > 0$ such that

$$|\widehat{\varphi}(\lambda) + |\Theta(\lambda)|| \geq \delta,$$

for all $\lambda \in \mathbb{D}$.

Volberg's result allows the translation of the embedding conditions in Theorem 8.3 into concrete functional inequalities, leading to the following statement.

Theorem 8.4. *Let $\varphi \in L^\infty(\mathbb{T})$ and let Θ be an inner function.*

- (1) *If A_φ^Θ is bounded below, then there exists $\delta > 0$ such that*

$$|\widehat{\varphi}(\lambda) + |\Theta(\lambda)|| \geq \delta,$$

for all $\lambda \in \mathbb{D}$.

- (2) *If $\varphi \geq 0$, the following assertions are equivalent:*

- (a) *The operator A_φ^Θ is invertible;*
- (b) *there exists $\delta > 0$ such that*

$$|\widehat{\varphi}(\lambda) + |\Theta(\lambda)|| \geq \delta,$$

for all $\lambda \in \mathbb{D}$.

Denote by $\sigma_{ap}(T)$ the approximate point spectrum of a bounded operator T .

Corollary 8.5. *Let $\varphi \in L^\infty(\mathbb{T})$ and let Θ be an inner function. Then*

$$\{\mu \in \mathbb{C} : \inf_{\lambda \in \mathbb{D}} (|\widehat{\varphi - \mu}(\lambda) + |\Theta(\lambda)||) = 0\} \subset \sigma_{ap}(A_\varphi^\Theta).$$

ACKNOWLEDGEMENTS

The authors thank Richard Rochberg for some useful discussions. The authors were partially supported by French-Romanian project LEA-Mathmode.

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