FAST DECAY OF COVARIANCES UNDER $\delta-$PINNING IN THE CRITICAL AND SUPERCRITICAL MEMBRANE MODEL

ERWIN BOLTHAUSEN, ALESSANDRA CIPRIANI, AND NOEMI KURT

Abstract. We consider the membrane model, that is the centered Gaussian field on $\mathbb{Z}^d$ whose covariance matrix is given by the inverse of the discrete Bilaplacian. We impose a $\delta-$pinning condition, giving a reward of strength ε for the field to be 0 at any site of the lattice. In this paper we prove that in dimensions $d \geq 4$ covariances of the pinned field decay at least stretched-exponentially, as opposed to the field without pinning, where the decay is polynomial in $d \geq 5$ and logarithmic in $d = 4$. The proof is based on estimates for certain discrete Sobolev norms, and on a Bernoulli domination result.

1. The model and main results

The membrane model, or Laplacian model, is an example of an effective random interface, see for example Sakagawa (2003), Velenik (2006) and Kurt (2008). We will work on the d-dimensional integer lattice $\mathbb{Z}^d$, and in the present paper our focus will be in $d \geq 4$, although the definition is well-posed in all dimensions. For $N \in \mathbb{N}$, let $V_N := [-N/2, N/2]^d \cap \mathbb{Z}^d$ and $V_N^c := \mathbb{Z}^d \setminus V_N$. The discrete Laplacian Δ on $\mathbb{Z}^d$ is defined as the operator acting on functions $f : \mathbb{Z}^d \to \mathbb{R}$ by

$$\Delta f(x) = \frac{1}{2d} \sum_{y:\, \|x-y\|=1} \big(f(y) - f(x)\big),$$

where $\|x\|$ denotes the ℓ^1-norm on the lattice. We sometimes write f_x for $f(x)$.

Definition 1.1. *The* membrane model *is the random field* $\{\varphi_x\}_{x\in\mathbb{Z}^d} \in \mathbb{R}^{\mathbb{Z}^d}$ *with zero boundary conditions outside* V_N*, whose distribution is given by*

$$P_N(\mathrm{d}\varphi) = \frac{1}{Z_N} \exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d} (\Delta\varphi_x)^2\right) \prod_{x\in V_N} \mathrm{d}\varphi_x \prod_{x\in V_N^c} \delta_0(\mathrm{d}\varphi_x), \tag{1.1}$$

where Z_N *is a normalizing constant.*

Note that by re-summation, the law P_N of the field is the law of the centered Gaussian field on V_N with covariance matrix

$$G_N(x,y) := \mathrm{cov}(\varphi_x, \varphi_y) = \left(\Delta_N^2\right)^{-1}(x,y), \quad x, y \in V_N.$$

Here, $\Delta_N^2 = \left(\Delta^2(x,y)\right)_{\{x,y\in V_N\}}$ is the Bilaplacian with 0-boundary conditions outside V_N. We extend both Δ_N^2 and G_N to $x, y \in \mathbb{Z}^d$ by setting the entries to 0 outside $V_N \times V_N$. For

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$x \in V_N$, the matrix G_N is determined by the boundary value problem [i.]

$$\begin{cases} \Delta^2 G_N(x,y) = \delta_x(y), & y \in V_N \\ G_N(x,y) = 0, & y \in \partial_2 V_N, \end{cases}$$

where we denote $\partial_2 V_N := \{y \in V_N^c : \exists z \in V_N : \|y - z\| \le 2\}$. It is known that in $d \ge 5$ there exists P on $\mathbb{R}^{\mathbb{Z}^d}$ such that $P_N \to P$ weakly (Sakagawa (2003)). Under P, the canonical coordinates $(\varphi_x)_{x \in \mathbb{Z}^d}$ form a centered Gaussian process with covariance given by

$$G(x,y) = \Delta^{-2}(x,y) = \sum_{z \in \mathbb{Z}^d} \Delta^{-1}(x,z)\Delta^{-1}(z,y) = \sum_{z \in \mathbb{Z}^d} \Gamma(x,z)\Gamma(z,y), \tag{1.2}$$

where Γ denotes the covariance of the discrete Gaussian Free Field (DGFF, see Sznitman (2012, Section 2) for an overview). The matrix Γ has an easy representation in terms of the simple random walk $(S_n)_{n \ge 0}$ on $\mathbb{Z}^d$ given by

$$\Gamma(x,\, y) = \sum_{m \ge 0} \mathrm{P}_x[S_m = y]$$

(P_x is the law of S starting at x). This entails that

$$G(x,\, y) = \sum_{m \ge 0} (m+1)\mathrm{P}_x[S_m = y] = \mathrm{E}_{x,y}\left[\sum_{\ell,\, m=0}^{+\infty} \mathbf{1}_{\{S_m = \tilde{S}_\ell\}}\right] \tag{1.3}$$

where S and $\tilde{S}$ are two independent simple random walks started at x and y respectively. One can note from this representation that $G(\cdot,\, \cdot)$ is translation invariant. The existence of the infinite volume measure in $d \ge 5$ gives that $G(0,0) < +\infty$. Using the above one can derive the following property of the covariance:

Lemma 1.2 (Sakagawa (2003, Lemma 5.1)). *Let $d \ge 5$. Then*

$$\lim_{\|x\| \to +\infty} \frac{G(0,x)}{\|x\|^{4-d}} = \eta \tag{1.4}$$

where

$$\eta = (2\pi)^{-d} \int_0^{+\infty} \int_{\mathbb{R}^d} \exp\left(\iota\langle \zeta,\, \theta\rangle - \frac{\|\theta\|^4 t}{4d^2}\right) \mathrm{d}\theta \mathrm{d}t$$

for any $\zeta \in \mathbb{S}^{d-1}$ and $\iota = \sqrt{-1}$.

In other words, as $\|x - y\| \to \infty$, the covariance between φ_x and φ_y decays like $\|x - y\|^{4-d}$ in the supercritical dimensions. For $d = 4$ it was shown that $G_N(x,y)$ behaves in first order as $\gamma_4(\log N - \log \|x - y\|)$ for some $\gamma_4 \in (0, \infty)$, if x and y are not too close to the boundary of V_N, see Cipriani (2013, Lemma 2.1).

The goal of this paper will be to show that this polynomial decay of covariances changes drastically if we introduce a so-called "δ-pinning" which gives a reward of size $\varepsilon > 0$ if the interface touches the 0-hyperplane at a site $x \in \mathbb{Z}^d$. More precisely, we introduce an atom of size ε in 0 to our model (1.1):

[i.] $\delta_x(y)$ is the Dirac delta mass at x, i. e., $\delta_x(y) = 1 \iff x = y$.

Definition 1.3. *Let $\varepsilon > 0$ and let P_N be defined as in (1.1). The* membrane model on V_N with pinning of strenght ε *is defined as*

$$P_N^\varepsilon(\mathrm{d}\varphi) = \frac{1}{Z_N^\varepsilon}\exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\varphi_x\Delta^2\varphi_x\right)\prod_{x\in V_N}(\mathrm{d}\varphi_x + \varepsilon\delta_0(\mathrm{d}\varphi_x))\prod_{x\in V_N^c}\delta_0(\mathrm{d}\varphi_x). \tag{1.5}$$

With this definition we have for any measurable function $f : \mathbb{R}^{\mathbb{Z}^d} \to \mathbb{R}$,

$$\begin{aligned} E_N^\varepsilon(f) &= \frac{1}{Z_N^\varepsilon}\int f(\varphi)\exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\varphi_x\Delta^2\varphi_x\right)\prod_{x\in V_N}(\mathrm{d}\varphi_x + \varepsilon\delta_0(\mathrm{d}\varphi_x))\prod_{x\in V_N^c}\delta_0(\mathrm{d}\varphi_x) = \\ &= \sum_{A\subseteq V_N}\varepsilon^{|A|}\frac{Z_{V_N\setminus A}}{Z_N^\varepsilon}E_{V_N\setminus A}(f) \end{aligned}$$

where $E_{V_N\setminus A}$ is the mean according to the measure $P_{V_N\setminus A}$ defined for $A \subseteq V_N$ by

$$P_{V_N\setminus A}(\mathrm{d}\varphi) = \frac{1}{Z_{V_N\setminus A}}\int\exp\left(-\frac{1}{2}\sum_{x\in\mathbb{Z}^d}\varphi_x\Delta^2\varphi_x\right)\prod_{x\in V_N\setminus A}\mathrm{d}\varphi_x\prod_{x\in A\cup V_N^c}\delta_0(\mathrm{d}\varphi_x).$$

Thus P_N^ε is a convex combination of probabilities $P_{V_N\setminus A}$ which are distributed according to a probability measure on $\mathcal{P}(V_N)$[ii.], namely

$$\zeta_N^\varepsilon(A) = \zeta_N^\varepsilon(\mathcal{A} = A) := \varepsilon^{|A|}\frac{Z_{V_N\setminus A}}{Z_N^\varepsilon}$$

(Velenik, 2006, Section 5). Here and in the following $\mathcal{A}$ denotes a $\mathcal{P}(V_N)$-valued random variable under some site percolation law (which will be specified in each occurrence). Using the above expansion, we obtain for the covariances with respect to P_N^ε

$$E_N^\varepsilon[\varphi_x\varphi_y] = \sum_{A\subseteq V_N}\zeta_N^\varepsilon(A)E_{V_N\setminus A}[\varphi_x\varphi_y]. \tag{1.6}$$

To write this even more concisely, let $A \subset \mathbb{Z}^d$ with $|A^c| < +\infty$, and denote by P_{A^c} the law of the membrane model with $0-$boundary conditions outside A^c. Let

$$G_A(x,y) := E_{A^c}[\varphi_x,\varphi_y], \quad x,\, y \in A^c,$$

which we again extend by setting it to 0 to all of $\mathbb{Z}^d$. Observe that in this notation $G_N = G_{V_N^c}$. Then we can rewrite (1.6) as

$$E_N^\varepsilon[\varphi_x\varphi_y] = E_{\zeta_N^\varepsilon}\left[G_{\mathcal{A}\cup V_N^c}[\varphi_x\varphi_y]\right]. \tag{1.7}$$

Our main result shows, in the following couple of theorems, that for any positive pinning strength ε the correlations between two points decay at least stretched-exponentially in the distance.

Theorem 1.4 (Decay of covariances, supercritical case)**.** *Let $d \geq 5$ and $\varepsilon > 0$. There exists $\alpha > 0$ independent of ε such that*

$$\limsup_{\|x-y\|\to+\infty}\ \limsup_{N\to+\infty} E_N^\varepsilon[\varphi_x\varphi_y]\mathrm{e}^{\|x-y\|^\alpha} = 0. \tag{1.8}$$

[ii.] $\mathcal{P}(A)$ is the powerset of $A \subset \mathbb{Z}^d$.

Theorem 1.5 (Decay of covariances, critical case)**.** *Let $d = 4$ and $\varepsilon > 0$. For every $0 < \lambda \leq 1$ there exists $\beta = \beta(\lambda) > 0$ independent of ε such that for $\delta \in (0, 1]$*

$$\limsup_{N \to +\infty} \sup_{x,y \in V_N : \|x-y\| \geq \delta N^\lambda} E_N^\varepsilon[\varphi_x \varphi_y] \mathrm{e}^{\|x-y\|^\beta} = 0. \tag{1.9}$$

This result complements the one of Sakagawa (2012), who proves, via a free-energy estimate, that in $d \geq 4$ the model is localized, in the sense that it exhibits a positive density of pinned sites.

The proof relies on two main steps: firstly, using certain equivalences of discrete Sobolev norms, we show in Theorem 3.5 that for "very good sets" A the decay is indeed exponential:

$$|G_A(x, y)| \leq c\mathrm{e}^{-c'\|x-y\|}.$$

Unfortunately these sets do not have probability high enough under ζ_N^ϵ, thus we need to make adjustments to the definition of "very good" to balance the effect of the random environment of pinned points and the exponential decay.

For the DGFF it was proved (see Bolthausen and Brydges (2001), Bolthausen and Velenik (2001), Deuschel and Velenik (2000), Ioffe and Velenik (2000)) that the decay of the covariances is in fact exponential in the critical and supercritical dimensions. We conjecture that this is also true for the membrane model, but due to the lack of the random walk representation (see Remark 2.5 below) we are not able to prove this at the moment. Results on the membrane model with pinning were shown in $(1+1)$ dimensions by Caravenna and Deuschel (2008).

The structure of the paper is as follows: we begin with general results, including Bernoulli domination, in Section 2. In Section 3 we prove our main theorems, starting with Theorems 3.5, 3.6 in Subsection 3.2, and then Theorems 1.4, 1.5 in Subsection 3.3.

2. General results on the membrane model

In this section we collect and prove some results on the membrane model that will be important for the proof of the main results. Just as the DGFF enjoys the spatial Markov property, the membrane model does too. In fact it holds that

Proposition 2.1 (Markov property, Cipriani (2013, Lemma 2.2))**.** *Let $(\varphi_x)_{x\in\mathbb{Z}^d}$ be the membrane model under the measure P_N. Let $B \subseteq V_N$. Let $\mathcal{F}_B := \sigma(\varphi_z, z \in V_N \setminus B)$. Then*

$$\{\varphi_x\}_{x\in B} \stackrel{d}{=} \left\{\mathbf{E}_N\left[\varphi_x | \mathcal{F}_B\right] + \varphi'_x\right\}_{x\in B} \tag{2.1}$$

where "$\stackrel{d}{=}$" indicates equality in distribution. In particular, under $\mathbf{P}_N(\cdot)$, φ'_x is independent of $\mathcal{F}_B$. Also $\{\varphi'_x\}_{x\in B}$ is distributed as the membrane model with 0-boundary conditions outside B.

A further important observation is that the variances of the membrane model are decreasing in the number of points in which the field is 0.

Lemma 2.2. *Let $A_1 \subset A_2 \subset V_N$. Then*

$$G_{A_2\cup V_N^c}(x,x) \leq G_{A_1\cup V_N^c}(x,x) \leq G_N(x,x)$$

for all $x \in V_N \setminus A_2$.

Proof. Let $B := V_N \setminus A_1$. By Proposition 2.1, for a membrane model φ under P_N

$$\{\varphi_x\}_{x\in B} \stackrel{d}{=} \{E_N [\varphi_x|\mathcal{F}_B] + \varphi'_x\}_{x\in V_N\setminus B}$$

where φ' has the law of a membrane model on B with zero boundary conditions on $A_1 \cup V_N^c$. Therefore

$$G_N(x, x) - G_{A_1\cup V_N^c}(x, x) = E_N \left[(E_N [\varphi_x|\mathcal{F}_B])^2\right] \geq 0.$$

For $A_1 \subset A_2$, the proof follows exactly the same lines replacing V_N with $V_N \setminus A_1$ above. □

Next we prove that G_A satisfies a similar boundary value problem as G_N.

Lemma 2.3. *Let $d \geq 4$, $A \subset \mathbb{Z}^d$ such that $|A^c| < +\infty$. Let N be large enough such that $A^c \subset V_N$, and fix $x \in A^c$. Then $G_A(x, y)$ solves the discrete boundary value problem*

$$\begin{cases} \Delta^2 G_A(x,y) = \delta_x(y) & y \in A^c, \\ G_A(x,y) = 0 & y \in A \cup V_N^c. \end{cases} \tag{2.2}$$

Moreover, there exists a constant $\gamma = \gamma(d)$ such that for all $x \in \mathbb{Z}^d$,

$$G_A(x, x) \leq \begin{cases} \gamma & \text{if } d \geq 5, \\ \gamma \log N & \text{if } d = 4. \end{cases} \tag{2.3}$$

Proof. By Proposition 2.1, G_A is the covariance matrix of the membrane model on V_N conditioned to be 0 in $A \cup V_N^c$. A well-known fact about Gaussian random vectors is that conditioning on the values of some of the entries yields again a Gaussian vector, whose covariance matrix can be calculated by a simple formula. In our case, this formula looks as follows: let

$$\Sigma_{A,N} := (G_N(x, y))_{x,y\in A\cup V_N^c}.$$

Then (Zhang, 2006, Chapter 6)

$$G_A(x, y) = G_N(x, y) - \sum_{z,w\in A\cup V_N^c} G_N(x, z)\Sigma_{A,N}^{-1}(z, w)G_N(w, y). \tag{2.4}$$

From (1.2) we immediately obtain (2.2), and using the fact that G_A is positive semi-definite (since it is a covariance matrix) and Kurt (2008, Proposition 2.1.1 resp. Proposition 2.1.2) we get (2.3). □

For $d \geq 5$ we obtain the same result for any $A \subseteq \mathbb{Z}^d$.

Lemma 2.4. *Let $d \geq 5$, $A \subset \mathbb{Z}^d$, and $x \in A^c$ (thus A^c is possibly infinite). The membrane model on A^c is well-defined, and its covariance matrix $G_A(x, y)$ solves the discrete boundary value problem*

$$\begin{cases} \Delta^2 G_A(x,y) = \delta_x(y) & y \in A^c, \\ G_A(x,y) = 0 & y \in A. \end{cases} \tag{2.5}$$

Moreover, there exists a constant $\gamma = \gamma(d)$ such that

$$G_A(x, x) \leq \gamma$$

for all $x \in \mathbb{Z}^d$.

Proof. By Lemma 2.2, $G_A(x, x) := \lim_{N\to+\infty} G_{A\cup V_N^c}(x, x)$ exists for $x \in \mathbb{Z}^d$, and from (2.3) we know that the sequence of measures $P_{A^c\cap V_N}$ is tight. Since we are dealing with Gaussian measures, it is enough to prove the existence of the weak limit P_{A^c} of $P_{A^c\cap V_N}$ to show the statement. Then (2.5) follows by taking limits in (2.2). □

Remark 2.5. At this point it is important to note that G_A is *not* the convolution of the covariance matrix of the DGFF with 0-boundary conditions outside A^c, which is only the case for the infinite volume situation, c. f. (1.2). Therefore the random walk representation (1.3) doesn't carry over to G_A. This is an important difference between the membrane model and the DGFF. To study properties of the pinned DGFF one can rely on the random walk representation, as for example Bolthausen and Brydges (2001), Bolthausen and Velenik (2001), Coquille and Miłoś (2013), Ioffe and Velenik (2000), Velenik (2006) do. In the membrane model one can, as in Cipriani (2013) and Kurt (2009), approximate G_N by a random walk representation and thus derive useful estimates. However, this approximation is only valid for convex connected A^c, and thus cannot be applied to the pinning case. We therefore need to apply very different methods in order to find estimates for $G_A(x,y)$ for general $A \subset V_N$. Our approach is based on equivalences of certain discrete Sobolev norms and a Bernoulli domination argument, with which we begin.

2.1. **The random environment of pinned points.** Let us now prove a simple Lemma on partition functions for the measure P_N^ε. We denote as f_{φ_E} the density of φ_x with respect to the measure $\prod_{x\in E} \mathrm{d}\varphi_x \prod_{x\in E^c} \delta_0(\mathrm{d}\varphi_x)$ and Z_E its partition function.

Lemma 2.6. *In $d \geq 5$ there exist constants $0 < C_\ell, C_r < +\infty$ such that for every $E \subseteq V_N$ and $x \in E$*

$$C_\ell \leq \frac{Z_E}{Z_{E\setminus\{x\}}} \leq C_r. \tag{2.6}$$

Proof.

$$\frac{Z_E}{Z_{E\setminus\{x\}}} = \frac{f_{\varphi_E}(0,\ldots,0)}{f_{\varphi_{E\setminus\{x\}}}(0,\ldots,0)} = f_{\varphi_x|\varphi_{E\setminus\{x\}}}(0|0,\ldots,0)$$

where the latter is the conditional density of φ_x given that the field $\{\varphi_x,\, x \in E\setminus\{x\}\}$ is zero. We know already that φ_x conditioned on $\{\varphi_x,\, x\in E\setminus\{x\}\}$ is a well-defined normal variable $\mathcal{N}(0,\sigma_x^2)$ by Proposition 2.1, with $\sigma_x^2 \leq \gamma$ because of Lemma 2.3. Therefore

$$0 < C_\ell := \frac{1}{\sqrt{2\pi\gamma}} \leq \frac{1}{\sqrt{2\pi\sigma_x^2}} = f_{\varphi_x|\varphi_{E\setminus\{x\}}}(0|0,\ldots,0) \leq 1 =: C_r.$$

□

Lemma 2.7. *In $d \geq 5$ there exist constants $0 < C_\ell, C_r < +\infty$ such that for every $E \subseteq V_N$ and $x \in E$*

$$C_\ell \frac{1}{\sqrt{\log N}} \leq \frac{Z_E}{Z_{E\setminus\{x\}}} \leq C_r. \tag{2.7}$$

Proof. The proof is similar to Lemma 2.6, using the fact that $\sigma_x^2 \leq \gamma \log N$, see Lemma 2.4. □

Our target now is to control the pinning measure ζ_N^ε through a natural distribution of sites on the discrete lattice, that is through independent site percolation. We will briefly recall here two definitions.

Definition 2.8 (Stochastic and strong stochastic domination)**.** *Given two probability measures μ and ν on the set $\mathcal{P}(\Lambda)$, $|\Lambda| < +\infty$, we will say that μ* dominates ν strongly stochastically *if for all x, $C \subseteq \Lambda\setminus\{x\}$,*

$$\mu(A:\, x\in A \,|\, A\setminus\{x\} = C) \geq \nu(A:\, x\in A\,|\, A\setminus\{x\} = C). \tag{2.8}$$

When (2.8) *holds we will write* $\mu \succ \nu$. *We will say that* μ dominates ν stochastically, $\mu \succeq \nu$, *if for all increasing functions* f,

$$\mu(f) \geq \nu(f).$$

Note that strong stochastic domination implies stochastic domination.

Let now ν_Λ^ρ be the Bernoulli site percolation measure on V_N with intensity ρ. We would like to prove that our Gaussian free fields restricted to the pinned set are "sandwiched" between two such Bernoullian in the stochastic ordering. This argument is similar to the one in Velenik (2006, Section 5.3).

Proposition 2.9. *Let* $d \geq 5$. *There exist constants* $0 < c_-(d) < c_+(d) < \infty$ *such that for* ε *small enough,*

$$\nu_N^{\rho_-(d)} \prec \zeta_N^\varepsilon \prec \nu_N^{\rho_+(d)}$$

where $\rho_\pm(d) = c_\pm(d)\varepsilon$.

Proof. In the following we will omit the subscript N as we will be always working on the d-dimensional box of side-length N. The first step is to notice that for all $i \in V_N$, $C \subseteq V_N \setminus \{i\}$,

$$\zeta^\varepsilon(A : i \in A \,|\, A \setminus \{i\} = C) = \frac{\zeta^\varepsilon(C \cup \{i\})}{\zeta^\varepsilon(C)}$$

and by (2.6)

$$C_\ell \leq \frac{Z_{C\cup\{i\}}}{Z_C} \leq C_r.$$

Therefore stochastic domination is achieved for two Bernoulli measures of parameter $\rho_-(d) := C_\ell\varepsilon$, $\rho_+(d) := C_r\varepsilon$. □

Proposition 2.10. *Let* $d = 4$. *There exist constants* $0 < c_-(4) < c_+(4) < \infty$ *such that for* ε *small enough,*

$$\nu_N^{\rho_-(4)} \prec \zeta_N^\varepsilon \prec \nu_N^{\rho_+(4)}$$

where $\rho_+(4) = c_+(4)\varepsilon$, *and*

$$\rho_-(4) = \frac{c_-(4)\varepsilon}{\sqrt{\log N}}.$$

Remark 2.11. Observe that $\rho_-(4)$ converges to 0 as $N \to +\infty$.

Proof. The argument is the same of Prop. 2.9 where the conclusion is this time drawn from (2.7). □

3. Proof of the main results

3.1. **Equivalence of norms.** For a function $f : \mathbb{Z}^d \to \mathbb{R}$ we define the derivative in the i-th coordinate direction, $i \in \{1, ..., d\}$ by

$$D_i f(x) := f(x + e_i) - f(x), \; x \in \mathbb{Z}^d, \; i = 1, \ldots, d,$$

where e_i is the unit vector in direction i. Define the discrete gradient as

$$\nabla f(x) := (D_1 f(x), \ldots, D_d f(x)).$$

It will be convenient to introduce $D_{-i} f(x) := f(x - e_i) - f(x) = -D_i f(x - e_i)$, for $i = 1, \ldots, d$. The second discrete derivatives of a function are

$$D_{ij} f(x) := D_i D_j f(x), \quad i, j \in \{\pm 1, \ldots, \pm d\}.$$

With this notation, the discrete Laplacian is then given by

$$\Delta f(x) = -\frac{1}{2d}\sum_{i=1}^{d} D_{i,-i} f(x)$$

and the Bilaplacian assumes the form

$$\Delta^2 f(x) = \frac{1}{4d^2}\sum_{i,j=1}^{d} D_{i,-i}D_{j,-j} f(x). \tag{3.1}$$

We have the following summation by parts formula whose proof is an elementary calculation:

Lemma 3.1. *Let f, g be such that $\sum_{x\in\mathbb{Z}^d} f(x)g(x) < +\infty$ and $\sum_{x\in\mathbb{Z}^d} f(x)g(x+e_i) < +\infty$ for all $i \in \{\pm 1 \ldots \pm d\}$. Then for all $i \in \{\pm 1, \ldots, \pm d\}$ we have*

$$\sum_{x\in\mathbb{Z}^d} D_i f(x) g(x) = \sum_{x\in\mathbb{Z}^d} f(x) D_{-i} g(x).$$

Moreover

Lemma 3.2. *For $u : \mathbb{Z}^d \to \mathbb{R}$ we have*

$$\sum_{x\in\mathbb{Z}^d}\sum_{i,j=1}^{d} (D_i D_j u(x))^2 = 4d^2 \sum_{x\in\mathbb{Z}^d} u(x)\,\Delta^2 u(x).$$

Proof. Follows from Lemma 3.1 and (3.1). □

The standard discrete Sobolev norms on $E \subseteq \mathbb{Z}^d$ associated to the discrete Sobolev space $H^k(E)$ are given by

$$\|f\|^2_{H^k(E)} = \sum_{\ell=0}^{k}\left(\sum_{i_1,\ldots,i_\ell=1}^{d}\sum_{x\in E} |D_{i_1}\ldots D_{i_\ell} f(x)|^2\right). \tag{3.2}$$

We also introduce the norms

$$\|\nabla_k f\|^2_{L^2(E)} := \sum_{i_1,\ldots,i_k}\sum_{x\in E} |D_{i_1}\ldots D_{i_k} f(x)|^2. \tag{3.3}$$

We obviously have

$$\|\nabla_k f\|_{L^2(E)} \le \|f\|_{H^\ell(E)}, \quad k \le \ell \tag{3.4}$$

and

$$\|\Delta f(x)\|_{L^2(E)} \le C\|\nabla_2 f\|_{L^2(E)} \tag{3.5}$$

for some C depending only on d. The next Lemma will show that the above norms are equivalent on subsets where "groups" of pinned points are not too spread out. Let $A \subset \mathbb{Z}^d$. Set

$$\widehat{A} := \{x \in A : \text{ for all } y \sim x,\, y \in A\}.$$

We can think of $\widehat{A}$, which obviously is a subset of A, as the interiour of "pinned clusters". We introduce the notation

$$\mathrm{d}_E(x,y) := \min\{\ell : \exists\{x_0 = x, x_1, \ldots, x_\ell = y\} \subseteq E,\, x_i \sim x_{i+1}\,\forall 0 \le i \le \ell-1,\, x_i \ne x_j\,\forall i \ne j\}$$

for the graph distance on $E \subset \mathbb{Z}^d$, $x, y \in E$. In the rest of the paper, $c = c(d)$ denotes a constant depending from the dimension which may vary from line to line.

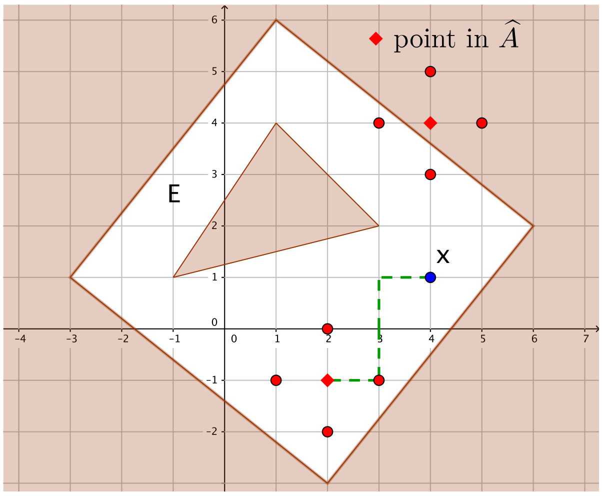

FIGURE 1. E in white. A in red. The length of the green path is $d_E(x, \widehat{A} \cap E)$.

Lemma 3.3. *Let $E \subset \mathbb{Z}^d$ be connected in the ℓ^1-topology. Assume there exists $M < +\infty$ such that $\sup_{x \in E} d_E\left(x, \widehat{A} \cap E\right) \leq M/2$.[iii]. Let $u : \mathbb{Z}^d \to \mathbb{R}$ be a function in $H^2(E)$ such that $u(x) = 0$ for all $x \in A$. Then there exists a constant $c = c(d)$ such that*

$$\|u\|_{H^2(E)} \leq cM^{2d-2}\|\nabla_2 u\|_{L^2(E)}.$$

Proof. Consider a partition $\mathcal{E}$ of E made up by sets of diameter at most M such that each set $B \in \mathcal{E}$ has no-empty intersection with $\widehat{A}$. In other words, every set in this partition contains at least one point $x_0 \in \widehat{A}$. Fix $B \in \mathcal{E}$, and fix $y \in B$. Then we can find a path $z_0, ..., z_K$ inside B, such that $z_0 = x_0$, $z_K = y$, $z_n \neq z_m$ for $n \neq m$, $\|z_{n+1} - z_n\| = 1$ for all n, and $K \leq M$. Then, since $u(x_0) = 0$,

$$|u(y)|^2 = \left|\sum_{n=1}^{K} u(z_n) - u(z_{n-1})\right|^2 \leq K\sum_{n=1}^{K} |u(z_n) - u(z_{n-1})|^2 \leq K\sum_{z \in B} |\nabla u(z)|^2.$$

We can do this for every $y \in B$. Thus

$$\sum_{y \in B} |u(y)|^2 \leq M^{d+1} \sum_{z \in B} |\nabla u(z)|^2.$$

Hence, summing over all $B \in \mathcal{E}$, we obtain

$$\|u\|^2_{L^2(E)} = \sum_{B \in \mathcal{E}} \sum_{y \in B} |u(y)|^2 \leq M^{d+1} \sum_{B \in \mathcal{E}} \sum_{y \in B} |\nabla u(z)|^2 = M^{d+1}\|\nabla u\|^2_{L^2(E)}.$$

Now we want to use the same type of argument on $|D_i u(y)|$ (resp. $|\nabla u(y)|$). Since $x_0 \in \widehat{A}$, we have $\nabla u(x_0) = 0$. So our argument gives

$$|\nabla u(y)|^2 \quad = \left|\nabla u(x_0) + \textstyle\sum_{n=1}^{K}(\nabla u(z_n) - \nabla u(z_{n-1}))\right|^2 \leq K \textstyle\sum_{z \in B}\sum_{i,j} |D_{ij}u(z)|^2$$

[iii] For $\tau \in E$, $\Lambda \subset \mathbb{Z}^d$, $d_E(\tau, \Lambda) := \inf_{\lambda \in \Lambda} d_E(\tau, \lambda)$.

which leads to

$$\sum_{y\in B} |\nabla u(y)|^2 \quad \leq M^{d+1} \textstyle\sum_{z\in B} \sum_{i,j=1}^{d} |D_{ij}u(z)|^2.$$

Thus

$$\|\nabla u\|^2_{L^2(E)} \leq M^{d+1} \|\nabla_2 u\|^2_{L^2(E)}.$$

Finally

$$\begin{aligned}\|u\|^2_{H^2(E)} &= \|u\|^2_{L^2(E)} + \|\nabla u\|^2_{L^2(E)} + \|\nabla_2 u\|^2_{L^2(E)} \leq \|\nabla u\|^2_{L^2(E)} \left(M^{d+1}+1\right) + \|\nabla_2 u\|^2_{L^2(E)} \\ &\leq \left(M^{2(d+1)} + M^{d+1} + 1\right) \|\nabla_2 u\|^2_{L^2(E)} \leq c(d) M^{2(d+1)} \|\nabla_2 u\|^2_{L^2(E)}.\end{aligned}$$

This completes the proof. □

For fixed y, let $B_k = B_{k,y} := \left\{x \in \mathbb{Z}^d : \|x-y\|_1 \leq k\right\}$ denote the ball with radius k and center y on the lattice, $k \geq 0$. We denote by G_A^y a solution to (2.2) for fixed y. Recall in $d = 4$ we are extending G_A^y to 0 outside $V_N \times V_N$.

3.2. **Deterministic pinning.** The equivalence of norms of Lemma 3.3 can be applied as follows.

Lemma 3.4. *Let $d \geq 1$ and let $A \subset \mathbb{Z}^d$. Fix $k \geq 5$. For any connected subset $D_k \subseteq B_k^c$ for which there exists $M = M(D_k, A) < +\infty$ such that $\sup_{x\in D_k} \mathrm{d}_{D_k}(x, \widehat{A} \cap D_k) \leq M/2$, there exists $c = c(d)$ such that*

$$\|G_A^y\|^2_{H^2(D_k)} \leq cM^{2d+2} \|G_A^y\|_{H^2(B_k \setminus B_{k-5})}.$$

Proof. Let $(\eta_k)_{k\geq 1}$ be a family of cutoff functions such that

$$\begin{cases} \eta_k(x) = 1 & x \in B_{k-2}^c, \\ \eta_k(x) = 0 & x \in B_{k-3}, \\ 0 \leq \eta_k(x) \leq 1 & x \in \mathbb{Z}^d. \end{cases}$$

We also denote by $\eta_k G_A^y(x) := \eta_k(x) G_A^y(x)$ the pointwise product of the two functions. Since we have $\eta_k G_A^y = G_A^y$ on B_{k-2}^c, we obtain from Lemma 3.3

$$\begin{aligned}\|G_A^y\|^2_{H^2(D_k)} &= \|\eta_k G_A^y\|^2_{H^2(D_k)} \leq c(d) M^{2d+2} \|\nabla_2(\eta_k G_A^y)\|^2_{L^2(D_k)} \\ &\leq c(d) M^{2d+2} \|\nabla_2(\eta_k G_A^y)\|^2_{L^2(\mathbb{Z}^d)}.\end{aligned} \tag{3.6}$$

On the other hand we have, using Lemma 3.2, the properties of η_k, and the fact that $G_A^y(x)$ is biharmonic,

$$\begin{aligned}\|\nabla_2(\eta_k G_A^y)\|^2_{L^2(\mathbb{Z}^d)} &= \sum_{x\in\mathbb{Z}^d} \sum_{i,j=1}^{d} |D_i D_j(\eta_k G_A^y(x))|^2 = 4d^2 \sum_{x\in\mathbb{Z}^d} \left(\eta_k G_A^y(x)\right) \left(\Delta^2 \eta_k G_A^y(x)\right) \\ &= 4d^2 \sum_{x\in B_{k-1}\setminus B_{k-3}} \left(\eta_k G_A^y(x)\right) \left(\Delta^2 \eta_k G_A^y(x)\right) \\ &\leq C_2(d) \|\eta_k G_A^y\|^2_{H^2(B_k\setminus B_{k-5})} \leq C_2(d) \|G_A^y\|^2_{H^2(B_{k+1}\setminus B_{k-5})}.\end{aligned} \tag{3.7}$$

Putting (3.6) and (3.7) together gives the desired result. Observe that the above Lemma holds for any dimension $d \geq 1$, in particular for $d = 4$, if we set

$$G_{A,N}^y := G_{A\cap V_N}(\cdot, y), \tag{3.8}$$

which we extend to $\mathbb{Z}^d$ by setting it 0 outside V_N. The statement is obviously interesting only if $D_k \cap V_N \neq \emptyset$, but it is trivially true otherwise. $\square$

With this preparation, we can prove the following deterministic version of our main result, whose proof illustrates the ideas behind our approach.

Theorem 3.5 (Deterministic pinned set). *Let $d \geq 5$, and let $A \subset \mathbb{Z}^d$ be such that there exists $M < +\infty$ such that $\sup_{x\in A^c} \mathrm{d}(x, \widehat{A}) \leq M/2$. Then there exist $s = s(d, M) \in (0, +\infty)$ and $c = c(d, M) \in (0, +\infty)$ such that for all $x, y \in A^c$*

$$|G_A(x, y)| \leq c\mathrm{e}^{-s\|x-y\|}$$

Moreover $\|G_A^y\|^2_{H^2(\mathbb{Z}^d)} \leq \gamma$.

Proof. From Lemma 3.4 we obtain, choosing $D_k = B_k^c$, and observing $\|G_A^y\|^2_{H^2(A\cup B)} = \|G_A^y\|^2_{H^2(A)} + \|G_A^y\|^2_{H^2(B)}$ for any disjoint sets $A, B \subset \mathbb{Z}^d$,

$$\|G_A^y\|^2_{H^2\left(B_k^c\right)} \leq cM^{2d+2}\left(\|G_A^y\|^2_{H^2\left(B_{k-5}^c\right)} - \|G_A^y\|^2_{H^2\left(B_k^c\right)}\right)$$

which yields

$$\|G_A^y\|^2_{H^2\left(B_k^c\right)} \leq \frac{cM^{2d+2}}{1+cM^{2d+2}}\|G_A^y\|^2_{H^2\left(B_{k-5}^c\right)}.$$

Since $\|G_A^y\|^2_{H^2(B_i^c)} \geq \|G_A^y\|^2_{H^2(B_{i+1}^c)}$ for all $i \geq 0$, iteration yields, setting $C = cM^{2d+2}$,

$$\begin{aligned}\|G_A^y\|^2_{H^2\left(B_k^c\right)} \leq \left(\frac{C}{C+1}\right)^{\lfloor k/5 \rfloor}\|G_A^y\|^2_{H^2(B_0^c)} \leq \left(\left(\frac{C}{C+1}\right)^{1/5}\right)^{k-1}\|G_A^y\|^2_{H^2(B_0^c)} \\ \leq \mathrm{e}^{-s(k-1)}\|G_A^y\|^2_{H^2\left(B_0^c\right)}\end{aligned} \tag{3.9}$$

for

$$s = \frac{1}{5}\log\frac{1+C}{C} > 0.$$

Note that for some $c(d) < +\infty$ we have

$$\|G_A^y\|^2_{H^2(B_0^c)} \leq \|G_A^y\|^2_{H^2(\mathbb{Z}^d)} = \sum_{z\in\mathbb{Z}^d}\left(\sum_{i,j=1}^{d} D_iD_jG_A^y(z)\right)^2 \leq c(d)\sum_{z\in\mathbb{Z}^d}\sum_{i,j=1}^{d}\left(D_iD_jG_A^y(z)\right)^2.$$

Lemma 3.2 and the fact that $G_A^y(z) = 0$ for $z \in A$ give

$$\begin{aligned}\sum_{z\in\mathbb{Z}^d}\sum_{i,j=1}^{d}\left(D_iD_jG_A^y(z)\right)^2 &= \sum_{z\in\mathbb{Z}^d}\Delta^2G_A^y(z)G_A^y(z) = \sum_{z\in A^c}\Delta^2G_A^y(z)G_A^y(z) \\ &= \sum_{z\in A^c}\delta_y(z)G_A^y(z) = G_A^y(y) \leq G(y, y) = \gamma \overset{(2.3)}{<} +\infty.\end{aligned} \tag{3.10}$$

Hence $\|G_A^y\|^2_{H^2(B_0^c)} < +\infty$ for all A, and we get for $\|x-y\| > k$ from (3.9)

$$\left|G_A^y(x)\right|^2 \leq \|G_A^y\|^2_{H^2(B_k^c)} \leq C\mathrm{e}^{-sk} \tag{3.11}$$

which is what we wanted to prove. $\square$

We now pass on proving the 4-dimensional case as follows.

Theorem 3.6 (Deterministic pinned set). *Let $d = 4$, and let $A \subset \mathbb{Z}^d$ be such that there exists $M < +\infty$ such that $\sup_{x \in A^c} \mathrm{d}(x, \widehat{A}) \leq M/2$. Then there exist $s = s(d, M) \in (0, +\infty)$ and $c = c(d, M) \in (0, +\infty)$ such that for all $x, y \in A^c$*

$$|G_{A \cap V_N}(x, y)| \leq c \log N e^{-s\|x-y\|}.$$

Moreover $\|G_A^y\|^2_{H^2(\mathbb{Z}^d)} \leq \gamma \log N$.

Proof. This works exactly as for Theorem 3.5, using $G^y_{A,N}$ defined in (3.8) instead of G^y_A, except that (3.10) has to be replaced by

$$\begin{aligned}\sum_{z \in \mathbb{Z}^d} \sum_{i,j=1}^{d} \left(D_i D_j G_A^y(z)\right)^2 &= \sum_{z \in \mathbb{Z}^d} \Delta^2 G^y_{A,N}(z) G^y_{A,N}(z) = \sum_{z \in A^c \cap V_N} \Delta^2 G^y_{A,N}(z) G^y_{A,N}(z) \\ &= \sum_{z \in A^c \cap V_N} \delta_y(z) G^y_{A,N}(z) = G^y_{A,N}(y) \leq G_N(y,y) \overset{(2.3)}{\leq} \gamma \log N.\end{aligned} \tag{3.12}$$

In the last inequality we have used Lemma 2.4. This leads to the statement. □

3.3. **From deterministic to random pinning.** In this subsection we explain how to "transfer" the decay of covariances from the deterministic case to the random situation. Now the point is that in the random situation there is no fixed M that we can take as in the previous proofs. Fix $k > 5, A \subset \mathbb{Z}^d$. The idea is now to choose sets $D_\ell^{(k)}$, $0 \leq \ell \leq \lfloor k/5 \rfloor$, in the right way such that there are suitable $M_\ell^{(k)} = M\left(D_\ell^{(k)}\right)$ for which we can adjust the iteration procedure. We make the following choices:

$$D_\ell^{(k)} := B_{k+1} \setminus B_{k-5\ell}, \quad 0 \leq \ell \leq \lfloor k/5 \rfloor, \tag{3.13}$$

and

$$M_\ell^{(k)} = M_\ell^{(k)}(A) := \max_{x \in D_\ell^{(k)}} \mathrm{d}_{D_\ell^{(k)}}\left(x, \widehat{A} \cap D_\ell^{(k)}\right),$$

where the distance is taken on the lattice. If $D_\ell^{(k)} \cap \widehat{A} = \emptyset$, let $M_\ell^{(k)} := +\infty$. The following Lemma, albeit deterministic, shows that if we wish to obtain a strong decay of correlations, one needs to control appropriately the maximal distance between a point and the clusters of pinned points.

Lemma 3.7. *Let $d \geq 5$. For $m_k = k^\xi$, $0 < \xi < 1/2(d+1)$, define $a_k = a_k(A) > 0$ as $a_k := |\{\ell \in \{0, ..., \lfloor k/5 \rfloor\} : M_\ell^{(k)} \leq m_k\}|$. Then there exist $c > 0$ dependent only on d such that for $\|x - y\| = k$,*

$$|G_A^y(x)| \leq \gamma e^{-c m_k^{-2(d+1)} a_k}$$

and γ is as in (2.3).

Proof. Observe that we have

$$D_\ell^{(k)} \subseteq B^c_{k-5\ell}, \quad \text{and} \quad D_\ell^{(k)} \cup \left(B_{k-5\ell} \setminus B_{k-5(\ell+1)}\right) = D_{\ell+1}^{(k)}$$

where the last union is disjoint. If $M_\ell^{(k)} < +\infty$ we thus get from Lemma 3.4 that

$$\|G_A^y\|^2_{H^2(D_\ell^{(k)})} \leq c(M_\ell^{(k)})^{2d+2}\|G_A^y\|^2_{H^2(B_{k-5\ell}\setminus B_{k-5\ell})} = c\left(M_\ell^{(k)}\right)^{2d+2}\left(\|G_A^y\|^2_{H^2\left(D_{\ell+1}^{(k)}\right)} - \|G_A^y\|^2_{H^2\left(D_\ell^{(k)}\right)}\right),$$

which leads to

$$\|G_A^y\|^2_{H^2\left(D_\ell^{(k)}\right)} \leq c\frac{\left(M_\ell^{(k)}\right)^{2d+2}}{1+c\left(M_\ell^{(k)}\right)^{2d+2}}\|G_A^y\|^2_{H^2(D_{\ell+1}^{(k)})}.$$

If $M_\ell^{(k)} = +\infty$ we have, since $D_\ell^{(k)} \subset D_{\ell+1}^{(k)}$,

$$\|G_A^y\|^2_{H^2\left(D_\ell^{(k)}\right)} \leq \|G_A^y\|^2_{H^2\left(D_{\ell+1}^{(k)}\right)}.$$

Hence

$$\|G_A^y\|^2_{H^2\left(D_\ell^{(k)}\right)} \leq \left(\mathbf{1}_{\left\{M_\ell^{(k)}<+\infty\right\}}c\frac{(M_\ell^{(k)})^{2d+2}}{1+cM_\ell^{2d+2}} + \mathbf{1}_{\{M_\ell=+\infty\}}\right)\|G_A^y\|^2_{H^2\left(D_{\ell+1}^{(k)}\right)}$$

for all $0 \leq \ell \leq \lfloor k/5 \rfloor$. Iteratively we find

$$\begin{aligned}
\|G_A^y\|^2_{H^2(B_{k+1}\setminus B_k)} &= \|G_A^y\|^2_{H^2\left(D_0^{(k)}\right)} \\
&\leq \prod_{\ell=0}^{\lfloor k/5\rfloor-1}\left(\mathbf{1}_{\left\{(M_\ell^{(k)})<+\infty\right\}}c\frac{(M_\ell^{(k)})^{2d+2}}{1+c(M_\ell^{(k)})^{2d+2}} + \mathbf{1}_{\left\{(M_\ell^{(k)})=+\infty\right\}}\right)\|G_A^y\|^2_{H^2\left(D_{\ell+1}^{(k)}\right)} \\
&\leq \prod_{\ell=0}^{\lfloor k/5\rfloor-1}\left(\mathbf{1}_{\left\{(M_\ell^{(k)})<+\infty\right\}}c\frac{(M_\ell^{(k)})^{2d+2}}{1+c(M_\ell^{(k)})^{2d+2}} + \mathbf{1}_{\left\{(M_\ell^{(k)})=+\infty\right\}}\right)\|G_A^y\|^2_{H^2(\mathbb{Z}^d)}.
\end{aligned} \tag{3.14}$$

With our definition of a_k, we can then rewrite (3.14) as

$$\|G_A^y\|^2_{H^2(B_{k+1}\setminus B_k)} \leq \left(c\frac{m_k^{2(d+1)}}{1+cm_k^{2(d+1)}}\right)^{a_k}\|G_A^y\|^2_{H^2(\mathbb{Z}^d)}.$$

Using the fact that $\log {}^{1+x}/x \geq {}^1/x$, $x > 0$, we obtain for $x \in \mathbb{Z}^d \setminus \{y\}$ and for k such that $x \in B_{k+1} \setminus B_k$,

$$|G_A^y(x)| \leq \|G_A^y\|^2_{H^2(B_{k+1}\setminus B_k)} \leq \left(\frac{cm_k^{2(d+1)}}{1+cm_k^{2(d+1)}}\right)^{a_k}\|G_A^y\|^2_{H^2(\mathbb{Z}^d)} \leq \gamma e^{-c(m_k)^{-2(d+1)}a_k}$$

where we have concluded by means of Theorem 3.5. □

The 4-dimensional case is also at hand as follows:

Lemma 3.8. *Let $d = 4$. For $m_k = k^\xi$, $0 \leq \xi < d-1$, set $a_k = a_k(A) > 0$ such that $a_k := |\{\ell \in \{0,\ldots,\lfloor k/5\rfloor\} : M_\ell^{(k)} \leq m_k\}|$. Then there exist $c > 0$ dependent only on d such that for $\|x-y\| = k$,*

$$|G_A^y(x)| \leq \gamma_d \log N e^{-cm_k^{-2(d+1)}a_k}.$$

Proof. The proof is the same of Lemma 3.7, where in the very last step one uses Theorem 3.6. □

Thus in order to prove our main result, we will try to make $m_k^{-2(d+1)} a_k$ as large as possible. We first have the following auxiliary Lemma:

Lemma 3.9. *Let ν be a Bernoulli site percolation measure on $\mathbb{Z}^d$ with $\nu(x \text{ is open}) = \rho \in (0, 1)$, $x \in \mathbb{Z}^d$. Let $\mathcal{A}$ be the set of open sites. Furthermore let $(m_k)_{k\in\mathbb{N}}$ and $a_k = a_k(\mathcal{A})$ be defined as in Lemma 3.7. Then there exists $C = C(d) \in (0, +\infty)$ independent of $\mathcal{A}$ and k such that*

$$\nu\left(a_k \le \lfloor k/10 \rfloor\right) \le C k^{d+1} (1-\rho^{2d+1})^{\lfloor m_k/4 \rfloor}.$$

Proof. Recall $\widehat{\mathcal{A}} := \{x \in \mathcal{A} : y \in \mathcal{A} \text{ for all } y \sim x\}$. We have $\nu(x \in \widehat{\mathcal{A}}) = \rho^{2d+1}$. We also observe that if $\|x-y\| > 2$, the events $\{x \in \widehat{\mathcal{A}}\}$ and $\{y \in \widehat{\mathcal{A}}\}$ are independent. For any $t \in \mathbb{N}$ with $t \le \left|D_\ell^{(k)}\right|$, we have

$$\begin{aligned}
&\nu\left(\mathrm{d}_{D_\ell^{(k)}}(x, \widehat{\mathcal{A}}) \ge t\right)\\
&\le \nu\left(\exists \{x_0 = x, \ldots, x_t\},\, x_i \in D_\ell^{(k)} \setminus \widehat{\mathcal{A}},\; x_i \sim x_{i+1}\, \forall\, 0 \le i \le t-1,\, x_i \ne x_j\, \forall\, i \ne j\right)\\
&\le \nu\left(x_0 \notin \widehat{\mathcal{A}}, x_3 \notin \widehat{\mathcal{A}}, \ldots, x_{\lfloor t/4 \rfloor} \notin \widehat{\mathcal{A}}\right) = \left(1-\rho^{2d+1}\right)^{\lfloor t/4 \rfloor}
\end{aligned}$$

by independence. By means of the FKG inequality (Grimmett, 2006, Theorem 2.16),

$$\begin{aligned}
\nu\left(\max_{x\in B_\ell^{(k)}} \mathrm{d}_{D_\ell^{(k)}}(x, \widehat{\mathcal{A}}) \ge t\right) &= 1 - \nu\left(\mathrm{d}_{D_\ell^{(k)}}(x_0, \widehat{\mathcal{A}}) < t, \exists\, x_0 \in D_\ell^{(k)}\right)\\
&\le 1 - \left(1 - \left(1-\rho^{2d+1}\right)^{\lfloor t/4 \rfloor}\right)^{\left|D_\ell^{(k)}\right|} \le \left|D_\ell^{(k)}\right| \left(1-\rho^{2d+1}\right)^{\lfloor t/4 \rfloor}\\
&\le |D_{k+1}| \left(1-\rho^{2d+1}\right)^{\lfloor t/4 \rfloor} = \left(\sqrt{2}(k+1)\right)^d \left(1-\rho^{2d+1}\right)^{\lfloor t/4 \rfloor}. \qquad (3.15)
\end{aligned}$$

By the condition imposed on m_k we can choose k large such that $m_k \le \left|D_\ell^{(k)}\right|$ for all $0 \le \ell \le \lfloor k/10 \rfloor$. Then

$$\begin{aligned}
\nu\left(a_k \le \lfloor k/10 \rfloor\right) &\le \nu\left(\max_{0\le\ell\le\lfloor k/10 \rfloor} \max_{x\in D_\ell^{(k)}} \mathrm{d}_{D_\ell^{(k)}}(x, \widehat{\mathcal{A}}) \ge m_k\right)\\
&\le \left\lfloor \frac{k}{10} \right\rfloor (\sqrt{2}(k+1))^d \left(1-\rho^{2d+1}\right)^{\lfloor m_k/4 \rfloor}.
\end{aligned}$$

□

Proof of Theorem 1.4. Take $x, y \in \mathbb{Z}^d$ and assume $\|x-y\| > k \in \mathbb{N}$. Using the expansion (1.7), Equation (2.3) and Lemma 3.7 we get

$$\begin{aligned}\left|E_N^\varepsilon(\varphi_x\varphi_y)\right| &\le E_{\zeta_N^\varepsilon}\left(\left|G_{\mathcal{A}\cup V_N^c}(x,y)\right|\mathbf{1}_{\{a_k(\mathcal{A})<\lfloor k/10\rfloor\}}\right)+E_{\zeta_N^\varepsilon}\left(\left|G_{\mathcal{A}\cup V_N^c}(x,y)\right|\mathbf{1}_{\{a_k(\mathcal{A})\ge\lfloor k/10\rfloor\}}\right)\\ &\le \gamma\,\zeta_N^\varepsilon\left(a_k(\mathcal{A})<\left\lfloor\frac{k}{10}\right\rfloor\right)+\gamma\sum_{A\subseteq V_N}\zeta_N^\varepsilon\left(\mathcal{A}=A,\,a_k(A)\ge\left\lfloor\frac{k}{10}\right\rfloor\right)\mathrm{e}^{-c\lfloor\frac{k}{10}\rfloor m_k^{-2(d+1)}}\\ &\le \gamma\,\zeta_N^\varepsilon\left(a_k(\mathcal{A})<\left\lfloor\frac{k}{10}\right\rfloor\right)+\gamma\mathrm{e}^{-c\frac{k-10}{10}k^{-2\xi(d+1)}}.\end{aligned}$$

Since $\{a_k(\mathcal{A})<\lfloor k/10\rfloor\}$ is a decreasing event for the percolation realisation, we can use Proposition 2.9 to obtain

$$\zeta_N^\varepsilon\left(a_k(\mathcal{A})<\lfloor k/10\rfloor\right)\le\nu^{\rho_-(d)}\left(a_k(\mathcal{A})<\lfloor k/10\rfloor\right),$$

where due to Lemma 3.9 the right-hand side is bounded by $\mathrm{e}^{-k^{\xi'}}$, for any $\xi'<\xi$. Thus we get the desired result for any $0<\alpha<\min\{\xi,1-2\xi(d+1)\}$. □

Proof of Theorem 1.5. We can proceed as in the previous proof and obtain

$$\begin{aligned}\left|E_N^\varepsilon(\varphi_x\varphi_y)\right| &\le E_{\zeta_N^\varepsilon}\left(\left|G_{\mathcal{A}\cup V_N^c}(x,y)\right|\mathbf{1}_{\{a_k(\mathcal{A})<\lfloor k/10\rfloor\}}\right)+E_{\zeta_N^\varepsilon}\left(\left|G_{\mathcal{A}\cup V_N^c}(x,y)\right|\mathbf{1}_{\{a_k(\mathcal{A})\ge\lfloor k/10\rfloor\}}\right)\\ &\le \gamma\,\zeta_N^\varepsilon\left(a_k(\mathcal{A})<\left\lfloor\frac{k}{10}\right\rfloor\right)+\gamma\log N\sum_{A\subseteq V_N}\zeta_N^\varepsilon\left(\mathcal{A}=A,\,a_k(A)\ge\left\lfloor\frac{k}{10}\right\rfloor\right)\mathrm{e}^{-c\lfloor\frac{k}{10}\rfloor m_k^{-2(d+1)}}\\ &\le \gamma\,\zeta_N^\varepsilon\left(a_k(\mathcal{A})<\left\lfloor\frac{k}{10}\right\rfloor\right)+\gamma\log N\mathrm{e}^{-c\frac{k-10}{10}k^{-2\xi(d+1)}}.\end{aligned}$$

We have to take care of the fact that ρ_- converges to 0 as $N\to+\infty$. From Proposition 2.10 and Lemma 3.9 we have

$$\nu^{\rho_-(d)}\left(a_k(\mathcal{A})<\lfloor k/10\rfloor\right)\le Ck^{d+1}\left(1-\frac{\varepsilon c_-}{\sqrt{\log N}}\right)^{k^\xi/4}.$$

Inserting $k\ge N^\lambda$, we thus get

$$\zeta_N^\varepsilon\left(a_k(\mathcal{A})<\lfloor k/10\rfloor\right)\le\nu^{\rho_-(d)}\left(a_k(\mathcal{A})<\lfloor k/10\rfloor\right)\le\mathrm{e}^{-\lambda\xi'}$$

for any $\xi'<\xi$. Then we conclude by the same arguments as before. □

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Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057, Zurich, Switzerland

E-mail address: eb@math.uzh.ch

Weierstrass Institute, Mohrenstrasse 39, 10117 Berlin, Germany

E-mail address: Alessandra.Cipriani@wias-berlin.de

Technische Universität Berlin, MA 766, Strasse des 17. Juni 136, 10623 Berlin, Germany

E-mail address: kurt@math.tu-berlin.de