Partitioning a triangle-free planar graph into a forest and a forest of bounded degree

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Abstract

An $(\mathcal{F}, \mathcal{F}_d)$-partition of a graph is a vertex-partition into two sets F and F_d such that the graph induced by F is a forest and the one induced by F_d is a forest with maximum degree at most d. We prove that every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_5)$-partition. Moreover we show that if for some integer d there exists a triangle-free planar graph that does not admit an $(\mathcal{F}, \mathcal{F}_d)$-partition, then it is an NP-complete problem to decide whether a triangle-free planar graph admits such a partition.

1 Introduction

We only consider finite simple graphs, with neither loops nor multi-edges. Planar graphs we consider are supposed to be embedded in the plane. Consider i classes of graphs $\mathcal{G}_1, \dots, \mathcal{G}_i$. A $(\mathcal{G}_1, \dots, \mathcal{G}_i)$-partition of a graph G is a vertex-partition into i sets $V_1, \dots, V_i$ such that, for all $1 \le j \le i$, the graph $G[V_j]$ induced by V_j belongs to $\mathcal{G}_j$. In the following we will consider the following classes of graphs:

- $\mathcal{F}$ the class of forests,
- $\mathcal{F}_d$ the class of forests with maximum degree at most d,
- $\mathcal{D}_d$ the class of d-degenerate graphs (recall that a *d-degenerate graph* is a graph such that all subgraphs have a vertex of degree at most d),
- Δ_d the class of graphs with maximum degree at most d,
- $\mathcal{I}$ the class of empty graphs (i.e. graphs with no edges).

For example, an $(\mathcal{I}, \mathcal{F}, \mathcal{D}_2)$-partition of G is a vertex-partition into three sets V_1, V_2, V_3 such that $G[V_1]$ is an empty graph, $G[V_2]$ is a forest, and $G[V_3]$ is a 2-degenerate graph.

The Four Colour Theorem [1, 2] states that every planar graph G admits a proper 4-colouring, that is that G can be partitioned into four empty graphs, i.e. G has an $(\mathcal{I}, \mathcal{I}, \mathcal{I}, \mathcal{I})$-partition. Borodin [3] proved that every planar graph admits an acyclic colouring with at most five colours (an acyclic colouring is a proper colouring in which every two colour classes induce a forest). This implies that every planar graph admits an $(\mathcal{I}, \mathcal{F}, \mathcal{F})$-partition. Poh [8] proved that every planar graph admits an $(\mathcal{F}_2, \mathcal{F}_2, \mathcal{F}_2)$-partition. Thomassen proved that every planar graph admits an $(\mathcal{F}, \mathcal{D}_2)$-partition [10], and an $(\mathcal{I}, \mathcal{D}_3)$-partition [11]. However,

Classes	Vertex-partitions	References
Planar graphs	$(\mathcal{I},\mathcal{I},\mathcal{I},\mathcal{I})$	The Four Color Theorem [1, 2]
	$(\mathcal{I},\mathcal{F},\mathcal{F})$	Borodin [3]
	$(\mathcal{F}_2,\mathcal{F}_2,\mathcal{F}_2)$	Poh [8]
	$(\mathcal{F},\mathcal{D}_2)$	Thomassen [10]
	$(\mathcal{I},\mathcal{D}_3)$	Thomassen [11]
Planar graphs with girth 4	$(\mathcal{I},\mathcal{I},\mathcal{I})$	Grötzsch [6]
	$(\mathcal{F},\mathcal{F})$	Folklore
	$(\mathcal{F}_5,\mathcal{F})$	Present paper (Theorem 3)
	$(\mathcal{I},\mathcal{F})$	Open question (Question 1)
Planar graphs with girth 5	$(\mathcal{I},\mathcal{F})$	Borodin and Glebov [4]

Table 1: Known results.

there are planar graphs that do not admit any $(\mathcal{F},\mathcal{F})$-partition [5]. Borodin and Glebov [4] proved that every planar graph of girth at least 5 (that is every planar graph with no triangles nor cycles of length 4) admits an $(\mathcal{I},\mathcal{F})$-partition.

We focus on triangle-free planar graphs. Raspaud and Wang [9] proved that every planar graph with no triangles at distance at most 2 (and thus in particular every triangle-free planar graph) admits an $(\mathcal{F},\mathcal{F})$-partition. However, it is not known whether every triangle-free planar graph admits an $(\mathcal{I},\mathcal{F})$-partition. We pose the following questions:

Question 1. *Does every triangle-free planar graph admit an $(\mathcal{I},\mathcal{F})$-partition?*

Question 2. *More generally, what is the lowest d such that every triangle-free planar graph admits an $(\mathcal{F},\mathcal{F}_d)$-partition?*

Note that proving $d = 0$ in Question 2 would prove Question 1. The main result of this paper is the following:

Theorem 3. *Every triangle-free planar graph admits an $(\mathcal{F},\mathcal{F}_5)$-partition.*

This implies that $d \leq 5$ in Question 2. Our proof uses the discharging method. It is constructive and immediately yields an algorithm for finding an $(\mathcal{F},\mathcal{F}_5)$-partition of a triangle-free planar graph in quadratic time.

Note that Montassier and Ochem [7] proved that not every triangle-free planar graph can be partitioned into two graphs of bounded degree (which shows that our result is tight in some sense).

Finally, we show that if for some d, there exists a triangle-free planar graph that does not admit an $(\mathcal{F},\mathcal{F}_d)$-partition, then deciding whether a triangle-free planar graph admits such a partition is NP-complete. That is, if the answer to Question 2 is some $k > 0$, then for all $0 \leq d < k$, deciding whether a triangle-free planar graph admits an $(\mathcal{F},\mathcal{F}_d)$-partition is NP-complete. We prove this by reduction to Planar 3-Sat.

All presented results on vertex-partition of planar graphs are summarized in Table 1.

Theorem 3 will be proved in Section 2. Section 3 is devoted to complexity results.

Notation

Let $G = (V, E)$ be a plane graph (i.e. planar graph together with its embedding).

For a set $S \subset V$, let $G - S$ be the graph constructed from G by removing the vertices of S and all the edges incident to some vertex of S. If $x \in V$, then we denote $G - \{x\}$ by $G - x$. For a set S of vertices such that $S \cap V = \emptyset$, let $G + S$ be the graph constructed from G by adding the vertices of S. If $x \notin V$, then we denote $G + \{x\}$ by $G + x$. For a set E' of pairs of vertices of G such that $E' \cap E = \emptyset$, let $G + E'$ be the graph constructed from G by adding the edges of E'. If e is a pair of vertices of G and $e \notin E$, then we denote $G + \{e\}$ by $G + e$. For a set $W \subset V$, we denote by $G[W]$ the subgraph of G induced by W.

We call a vertex of degree k, at least k and at most k, a *k-vertex*, a *k^+-vertex* and a *k^--vertex* respectively, and by extension, for any fixed vertex v, we call a neighbour of v of degree k, at least k and at most k, a *k-neighbour*, a *k^+-neighbour*, and a *k^--neighbour* of v respectively. When there is some ambiguity on the graph, we call a neighbour of v in G a *G-neighbour* of v. We call a cycle of length ℓ, at least ℓ and at most ℓ a *ℓ-cycle*, a *ℓ^+-cycle*, and a *ℓ^--cycle* respectively, and by extension a face of length ℓ, at least ℓ and at most ℓ a *ℓ-face*, a *ℓ^+-face*, and a *ℓ^--face* respectively. We say that a vertex of G is *big* if it is a 8^+-vertex, and *small* otherwise. By extension, a big neighbour of a vertex v is a 8^+-neighbour of v, and a small neighbour of v is a 7^--neighbour of v.

Two neighbours u and w of a vertex v are *consecutive* if uvw forms a path on the boundary of a face.

2 Proof of Theorem 3

We prove Theorem 3 by contradiction. Let $G = (V, E)$ be a counter-example to Theorem 3 of minimum order.

Graph G is connected, otherwise at least one of its connected components would be a counter-example to Theorem 3, contradicting the minimality of G.

Let us consider any plane embedding of G. Let us prove a series of lemmas on the structure of G, that correspond to forbidden configurations in G.

Lemma 4. *There are no 2^--vertices in G.*

Proof. Suppose there is a 2^--vertex v in G. By minimality of G, $G - v$ admits an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D). If v is a 1^--vertex, then $G[F \cup \{v\}] \in \mathcal{F}$. Suppose v is a 2-vertex. If both of its neighbours are in F, then $G[D \cup \{v\}] \in \mathcal{F}_5$. Otherwise, $G[F \cup \{v\}] \in \mathcal{F}$. In all cases, one can obtain an $(\mathcal{F}, \mathcal{F}_5)$-partition of G, a contradiction. ☐

Lemma 5. *Every 3-vertex in G has at least one big neighbour.*

Proof. Suppose there is a 3-vertex v in G that has three small neighbours. By minimality of G, $G - v$ admits an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D). If at least two neighbours of v are in D, then $G[F \cup \{v\}] \in \mathcal{F}$. If no neighbour of v is in D, then $G[D \cup \{v\}] \in \mathcal{F}_5$. Suppose exactly one neighbour u of v is in D. If at most one of the neighbours of u is in F, then $G[F \cup \{u\}] \in \mathcal{F}$, and $G[D \backslash \{u\} \cup \{v\}] \in \mathcal{F}_5$. Otherwise, since u is small, at most four of the neighbours of u are in D, thus $G[D \cup \{v\}] \in \mathcal{F}_5$. In all cases, one can obtain an $(\mathcal{F}, \mathcal{F}_5)$-partition of G, a contradiction. ☐

Lemma 6. *Every 4-vertex or 5-vertex in G has at least one 4^+-neighbour.*

Proof. Suppose there is a 4-vertex or 5-vertex v in G that has no 4^+-neighbour. Let the u_i be the neighbours of v, for $i \in \{0, ..., 3\}$ or $i \in \{0, ..., 4\}$. Let $G' = G - v - \bigcup_i \{u_i\}$. By minimality of G, G' admits an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D). Add v to D, and for all u_i, add u_i to D if its two neighbours distinct from v are in F, and add u_i to F otherwise. Vertex v has at most five neighbours in D, and each of the u_i that is in D has one neighbour in D. Each of the u_i that is in F has at most one neighbour in F. We have an $(\mathcal{F}, \mathcal{F}_5)$-partition of G, a contradiction. ☐

We will need the following observation in the next two lemmas.

Observation 7. *Let $v_0v_1v_2v_3$ be a face of G, u_0 a neighbour of v_0 and u_1 a neighbour of v_1. Either u_0 and v_2 are at distance at least 3, or u_1 and v_3 are at distance at least 3.*

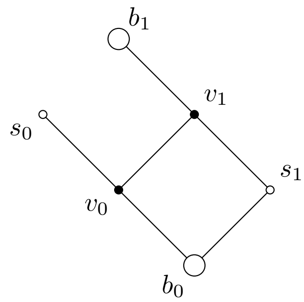

Figure 1: The forbidden configuration of Lemma 8. The big vertices are represented with big circles, and the small vertices with small circles. The filled circles represent vertices whose incident edges are all represented.

By contradiction, suppose that u_0 and v_2 are at distance at most two, and that u_1 and v_3 are at distance at most two. Since G is triangle-free, a shortest path from u_0 to v_2 (resp. from u_1 to v_3) does not contain any of the u_i and v_i except for its extremities. Then by planarity there exists a vertex w adjacent to u_0, v_2, u_1 and v_3. In particular v_2v_3w is a triangle, a contradiction.

Lemma 8. *The following configuration does not occur in G: two adjacent 3-vertices v_0 and v_1 such that for $i \in \{0,1\}$, v_i has a big neighbour b_i and a small neighbour s_i, and such that $v_0v_1s_1b_0$ bounds a face of G.*

Proof. Suppose such a configuration exists in G. See Figure 1 for an illustration of this configuration. Observe that all the vertices defined in the statement are distinct (since G is triangle-free). By Observation 7, either b_0 and b_1 are at distance at least 3, or s_0 and s_1 are at distance at least 3. For the remaining of the proof, we no longer need the fact that $b_0s_1 \in E(G)$. We forget this assumption, and only remember that either b_0 and b_1 are at distance at least 3, or s_0 and s_1 are at distance at least 3. This provides some symmetry in the graph.

Let $G_0 = G - \{v_0, v_1\} + b_0b_1$ and $G_1 = G - \{v_0, v_1\} + s_0s_1$. By what precedes, either G_0 or G_1 is triangle-free, thus there exists a j such that G_j is a triangle-free planar graph. By minimality of G, G_j admits an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D).

Let us first prove that if we do not have b_0 and b_1 in D, and s_0 and s_1 in F, then the conditions $G[F] \in \mathcal{F}$ and $G[D] \in \mathcal{F}_5$ lead to a contradiction. We will see that we can always extend the $(\mathcal{F}, \mathcal{F}_5)$-partition of G_j to G.

- If at least three of the b_i and s_i are in D, then $G[F \cup \{v_0, v_1\}] \in \mathcal{F}$.
- If all of the b_i and s_i are in F, then $G[D \cup \{v_0, v_1\}] \in \mathcal{F}_5$.
- Suppose now that exactly three of the b_i and s_i are in F. W.l.o.g., $b_0 \in D$ or $s_0 \in D$. We have $G[F \cup \{v_0\}] \in \mathcal{F}$ and $G[D \cup \{v_1\}] \in \mathcal{F}_5$.
- Suppose now that exactly two of the b_i and s_i are in F. If b_0 and s_0 are in F (resp. b_1 and s_1 are in F), then $G[D \cup \{v_0\}] \in \mathcal{F}_5$ and $G[F \cup \{v_1\}] \in \mathcal{F}$ (resp. $G[F \cup \{v_0\}] \in \mathcal{F}$ and $G[D \cup \{v_1\}] \in \mathcal{F}_5$).

  Now w.l.o.g. $b_0 \in F$ and $s_0 \in D$. If s_0 has at most one G-neighbour in F, then $G[F \cup \{s_0\}] \in \mathcal{F}$, we can replace F by $F \cup \{s_0\}$ and D by $D \backslash \{s_0\}$, and we fall into a previous case. We can thus assume that s_0 has at least two of its G-neighbours in F, and thus it has at most four of its G-neighbours in D. Therefore $G[D \cup \{v_0\}] \in \mathcal{F}_5$, and $G[F \cup \{v_1\}] \in \mathcal{F}$.

In all cases, G has an $(\mathcal{F}, \mathcal{F}_5)$-partition, a contradiction.

Remains the case where b_0 and b_1 are in D, and s_0 and s_1 are in F. In the case where we added the edge b_0b_1 (i.e. the case $j = 0$), we have $G[D \cup \{v_0, v_1\}] \in \mathcal{F}_5$, since $G[D \cup \{v_0, v_1\}]$ is equal to $G_0[D]$ where an edge is subdivided twice. Similarily, in the case where we added the edge s_0s_1 (i.e. the case $j = 1$), we have $G[F \cup \{v_0, v_1\}] \in \mathcal{F}$, since $G[F \cup \{v_0, v_1\}]$ is equal to $G_0[F]$ where an edge is subdivided twice. Again, G has an $(\mathcal{F}, \mathcal{F}_5)$-partition, a contradiction. ∎

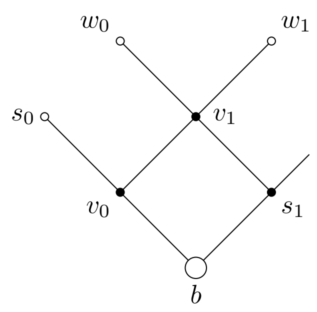

Figure 2: The forbidden configuration of Lemma 9.

Lemma 9. *The following configuration does not occur in G: a 3-vertex v_0 adjacent to a 4-vertex v_1 such that v_0 has a big neighbour b and a small neighbour s_0, and v_1 has three other small neighbours s_1, w_0, and w_1 such that $v_0v_1s_1b$ bounds a face of G and s_1 has degree 3.*

Proof. Suppose such a configuration exists in G. See Figure 2 for an illustration of this configuration. Observe that all the vertices defined in the statement are distinct (since G is triangle-free). By Observation 7, either b and w_0 are at distance at least 3, or s_0 and s_1 are at distance at least 3. Let $G_0 = G - \{v_0, v_1\} + bw_0$ and $G_1 = G - \{v_0, v_1\} + s_0s_1$. By what precedes, either G_0 or G_1 is triangle-free, thus there exists a j such that G_j is a triangle-free planar graph. By minimality of G, G_j has an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D).

Let us first prove that except in the case where $\{b, w_0, w_1\} \subset D$ and $\{s_0, s_1\} \subset F$, the conditions $G[F] \in \mathcal{F}$ and $G[D] \in \mathcal{F}_5$ lead to a contradiction. We will see that we can always extend the $(\mathcal{F}, \mathcal{F}_5)$-partition of G_j to G.

If at least four among the w_i, the s_i and b are in D, then $G[F \cup \{v_0, v_1\}] \in \mathcal{F}$.

Suppose now that at most three among the w_i, the s_i and b are in D. Suppose $x \in \{b, s_0, s_1, w_0, w_1\}$ is in D. If x has at most one G-neighbour in F, then $G[F \cup \{x\}] \in \mathcal{F}$, and we could consider $F \cup \{x\}$ instead of F and $D \backslash \{x\}$ instead of D. Note that this cannot lead to the case we excluded ($\{b, w_0, w_1\} \subset D$ and $\{s_0, s_1\} \subset F$) unless at least four among the w_i, the s_i and b are in D. Thus we can assume that for any x among the w_i and s_i such that $x \in D$, x has at most four G-neighbours in D, and thus adding one neighbour of x in D cannot cause x to have at least six neighbours in D. We consider two cases according to b:

- Suppose $b \in F$. If at least three of the w_i and s_i are in F, then $G[D \cup \{v_0, v_1\}] \in \mathcal{F}_5$.

  If at least two among the w_i and s_1 are in D, then $G[F \cup \{v_1\}] \in \mathcal{F}$ and $G[D \cup \{v_0\}] \in \mathcal{F}_5$. Else, at least two among the w_i and s_1 are in F, and we may assume that s_0 is in D (otherwise we fall into a previous case), so $G[D \cup \{v_1\}] \in \mathcal{F}_5$ and $G[F \cup \{v_0\}] \in \mathcal{F}$.

- Suppose now that $b \in D$. As s_1 has degree 3, it has at most one G-neighbour in F, and thus as previously we could consider $F \cup \{s_1\}$ instead of F and $D \backslash \{s_1\}$ instead of

D. Again, this cannot lead to the case we excluded ($\{b, w_0, w_1\} \subset D$ and $\{s_0, s_1\} \subset F$) unless at least four among the w_i, the s_i and b are in D. Therefore we can assume that $s_1 \in F$. The w_i are not both in D (otherwise we fall into the case we excluded). We have $G[D \cup \{v_1\}] \in \mathcal{F}_5$ and $G[F \cup \{v_0\}] \in \mathcal{F}$.

In all cases, G has an $(\mathcal{F}, \mathcal{F}_5)$-partition, a contradiction.

Remains the case $\{b, w_0, w_1\} \subset D$ and $\{s_0, s_1\} \subset F$. In the case where we added the edge bw_0 (i.e. the case $j = 0$), b has at most five G_0-neighbours in D, and thus at most four G-neighbours in D, so $G[D \cup \{v_0\}] \in \mathcal{F}_5$, and $G[F \cup \{v_1\}] \in \mathcal{F}$. In the case where we added the edge $s_0 s_1$ (i.e. the case $j = 1$), we have $G[F \cup \{v_0, v_1\}] \in \mathcal{F}$, since $G[F \cup \{v_0, v_1\}]$ is equal to $G_0[F]$ where an edge is subdivided twice. Again, G has an $(\mathcal{F}, \mathcal{F}_5)$-partition, a contradiction. □

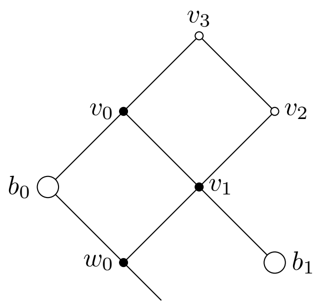

Figure 3: Configuration 10.

We define a specific configuration:

Configuration 10. *Two 4-faces $b_0 v_0 v_1 w_0$ and $v_0 v_1 v_2 v_3$, such that b_0 is a big vertex, v_0 and w_0 are 3-vertices, v_1 is a 4-vertex, v_2 and v_3 are small vertices, and the fourth neighbour of v_1, say b_1, is a big vertex. See Figure 3 for an illustration of this configuration.*

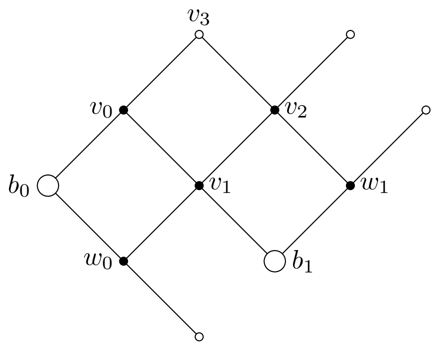

Figure 4: The forbidden configuration of Lemma 11.

Lemma 11. *The following configuration is forbidden: Configuration 10 with the added condition that there is a 4-face $b_1 v_1 v_2 w_1$ with w_1 a 3-vertex, v_2 a 4-vertex, and the fourth neighbour of v_2, the third neighbour of w_1, and the third neighbour of w_0 are small vertices.*

Proof. Suppose such a configuration exists in G. See Figure 4 for an illustration of this configuration. Observe that all the vertices named in the statement are distinct since G is triangle-free and w_1 is a small vertex whereas b_0 is a big one.

Let us prove that either b_0 and b_1 are at distance at least 3, or w_0 and w_1, and w_0 and v_3 are at distance at least 3. By contradiction, suppose that b_0 and b_1 are at distance at most two, and that either w_0 and w_1 are at distance at most two, or w_0 and v_3 are at distance at most 2. Since G is triangle-free, a shortest path from b_0 to b_1, from w_0 to w_1 or from w_0 to v_3 does not go through any of the vertices defined in the statement. Then by planarity there exists a vertex w adjacent to b_0, b_1, w_0 and either w_1 or v_3. In particular b_0w_0w is a triangle, a contradiction.

Let $G_0 = G - \{v_0, v_1\} + b_0b_1$ and $G_1 = G - \{v_0, v_1\} + w_0w_1 + w_0v_3$. By what precedes, either G_0 or G_1 is triangle-free, thus there exists a j such that G_j is a triangle-free planar graph. By minimality of G, G_j has an $(\mathcal{F}, \mathcal{F}_5)$-partition (F, D).

Let s_0 be the third neighbour of w_0, s_1 be the third neighbour of w_1 and s_2 be the fourth neighbour of v_2. They are all small vertices, but there may be some that are equal between themselves, or equal to some vertices we defined previously. However, if one of the s_i is in $\{v_0, v_1, v_2, w_0, w_1\}$, then this s_i is a 4^--vertex in G (and in particular it has at most 4 neighbours in D).

Suppose first that b_0 and b_1 are both in D.

1. Suppose w_0 is in D. Here we only consider (F, D) as an $(\mathcal{F}, \mathcal{F}_5)$-partition of $G - \{v_0, v_1\}$.

   If v_3 is also in D, then adding v_0 and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Suppose v_3 is in F. We show now that we can assume that v_2 is in D. By contradiction, suppose v_2 is in F. We remove v_2 from F.

   Observe that we can assume that v_2 has no G-neighbour in D with five G-neighbours in D. Indeed, suppose v_2 has a G-neighbour in D with five G-neighbours in D. This G-neighbour is a 5^+-vertex, so it is s_2. Moreover, s_2 is not equal to v_3 (because v_3 is in F), and is not equal to any of the other vertices named in the statement (because of the degree conditions). As s_2 is a small D-vertex, has at least five G-neighbours in D and is adjacent to v_2 that is neither in F nor in D, s_2 has at most one neighbour in F. Therefore we can put s_2 in F.

   Observe that we can assume that v_2 has at most one G-neighbour in D. Suppose v_2 has two G-neighbours in D. These G-neighbours are s_2 and w_1. Vertex w_1 has at most one neighbour in F, that is s_1, so we can put w_1 in F.

   Now v_2 has at most one G-neighbour in D, and no G-neighbour of v_2 in D has five G-neighbours in D, so we can put v_2 in D. Therefore we can always assume that v_2 is in D. Note that we do not need to change where s_2 is in the partition if it is equal to one of the vertices named in the statement. Adding v_0 and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

2. Suppose w_0 is in F, v_3 is in D and w_1 is in D. If s_2 is in D, then putting v_0, v_1 and v_2 in F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Suppose s_2 is in F. We put v_0, v_1 and w_1 in F, and v_2 in D. If this increases the number of G-neighbours of v_3 in D above five, then since v_3 is small, v_3 has at most one neighbour in F, which is v_0, and we put v_3 in F. This leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

3. Suppose w_0 is in F, v_3 is in D and w_1 is in F. Suppose s_2 is in F. We put v_0 and v_1 in F, and v_2 in D. If this increases the number of G-neighbours of v_3 in D above five, then since v_3 is small, v_3 has at most one neighbour in F, which is v_0, and we put v_3 in F. This leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Suppose s_2 is in D. If v_2 is not in F, we may put it in F, since it has only one G_j-neighbour in F, that is w_1. Therefore we can assume that v_2 is in F. If $j = 0$, then b_1 has at most 4 G-neighbours in D (since it has at most 5 such G_0-neighbours), so adding v_0 to F and v_1 to D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. If $j = 1$, then adding v_0 and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

4. Suppose w_0 is in F and v_3 is in F. Suppose $j = 0$. The vertex b_0 has at most 4 G-neighbours in D (since it has at most 5 such G_0-neighbours), so we can add v_0 to D. If v_2 is in D, then adding v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. If v_2 is in F, then adding v_1 to D makes $G[D]$ equal to $G_0[D]$ with an edge subdivided twice, and this leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Suppose $j = 1$. Here we only consider (F, D) as an $(\mathcal{F}, \mathcal{F}_5)$-partition of $G - \{v_0, v_1\} + w_0v_3$. As in 1, we can suppose, up to changing where s_2 and w_1 are in the partition, that v_2 is in D. Note that if s_2 is equal to one of the vertices named in the statement, we do not need to move s_2 in the partition. Adding v_0 and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

Now we may assume that at least one of b_0 and b_1 is in F. From now on we only consider (F, D) as an $(\mathcal{F}, \mathcal{F}_5)$-partition of $G - \{v_0, v_1\}$.

- Suppose b_0 is in F and b_1 is in D. In that case we put v_0 and w_0 in D, and v_1 in F. Adding v_0 in D (resp. w_0 in D) may violate the degree condition of $G[D]$; however, if it happens, one can put v_3 (resp. s_0) in F. In any case, we obtain an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.

- Suppose b_0 is in D and b_1 is in F. If at least one of w_0 and v_2 is in F, then adding v_0 in F and v_1 in D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Assume w_0 and v_2 are both in D. If v_3 is in D, then adding v_0 and v_1 in F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Assume v_3 is in F. We consider three cases:

  - Suppose s_2 and w_1 are in F. Adding v_0 in F and v_1 in D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - Suppose s_2 is in F and w_1 is in D. If s_1 is in D, then we can put w_1 in F and we fall into the previous case. If s_1 is in F, then adding v_0 in F and v_1 in D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - Suppose s_2 is in D. If s_1 is in D and has five G-neighbours in D distinct from w_1, then as s_1 is small, it is distinct from all the vertices named in the statement, and we can put it in F. Therefore we can put w_1 in D and v_2 in F. We fall into a previous case (at least one of w_0 and v_2 is in F).

- Suppose b_0 and b_1 are in F. If s_0 is in D and has five G-neighbours in D distinct from w_0, then as s_0 is small, it is distinct from all the vertices named in the statement aside from v_3, and we can put it in F. Therefore we can put w_0 in D. We consider the following cases:

  - If v_2 and v_3 are in F, then adding v_0 and v_1 to D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - If v_2 is in F and v_3 is in D, then adding v_0 to F and v_1 to D leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - If v_2 is in D and v_3 is in F, then adding v_0 to D and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G.
  - If v_2 and v_3 are in D, then adding v_0 to D and v_1 to F leads to an $(\mathcal{F}, \mathcal{F}_5)$-partition of G. Adding v_0 to D may violate the degree condition of $G[D]$, but in that case we can put v_3 in F.

□

We now apply a discharging procedure: first, for all j, every j-vertex v has a charge equal to $c_0(v) = j - 4$, and every j-face f has a charge equal to $c_0(f) = j - 4$. By Euler's formula, the total charge is negative (equal to -8). Observe that, since G is triangle-free, every face has a non-negative initial charge, and by Lemma 4, the vertices that have negative initial charges are exactly the 3-vertices of G, and they have an initial charge of -1. Here is our discharging procedure:

Discharging procedure:

- *Step 1*: Every big vertex gives $\frac{1}{2}$ to each of its small neighbours. Furthermore, for every 4-face $uvwx$ where u and v are big, and w and x are small, v gives $\frac{1}{4}$ to x (and u gives $\frac{1}{4}$ to w).
- *Step 2*: Consider a 4-vertex v that does not correspond to v_1 in Configuration 10. Vertex v gives $\frac{1}{4}$ to each of its small neighbours that are consecutive (as neighbours of v) to exactly one big vertex, and $\frac{1}{2}$ to each of its small neighbours that are consecutive (as neighbours of v) to two big vertices.

  Consider the case where v corresponds to v_1 in Configuration 10. We use the notations of Configuration 10. If w_0 has two big neighbours, then v_1 gives $\frac{1}{4}$ to v_0 and $\frac{1}{4}$ to v_2. Otherwise, it gives $\frac{1}{4}$ to w_0 and $\frac{1}{4}$ to v_0.

  Every small 5^+-vertex that has a big neighbour gives $\frac{1}{4}$ to each of its small neighbours, and an additional $\frac{1}{4}$ for each that is consecutive (as neighbours of v) to at least one big vertex. Every small 5^+-vertex that has no big neighbour gives $\frac{1}{4}$ to each of its 3-neighbours.
- *Step 3*: For every 4-face $uvwx$, with u a big vertex, v a 3-vertex, w a 4-vertex, and x a small vertex such that x gave charge to w in Step 2, w gives $\frac{1}{4}$ to v.
- *Step 4*: Every 5^+-face that has a big vertex in its boundary gives $\frac{1}{4}$ to each of the small vertices in its boundary. Every 5^+-face that has no big vertex in its boundary gives $\frac{1}{5}$ to each of the vertices in its boundary.
- *Step 5*: For every 4-face $uvwx$, with u a big vertex, v a 3-vertex, w a 4-vertex and x a 3-vertex such that the other face that has vw in its boundary is a 5^+-face, w gives $\frac{1}{5}$ to v.

For every vertex or face x of G, for every $i \in \{1, 2, 3, 4, 5\}$, let $c_i(x)$ be the charge of x at the end of Step i. Observe that during the procedure, no charges are created and no charges disappear; hence the total charge is kept fixed.

We now prove that every vertex and every face has a non-negative charge at the end of the procedure. That leads to the following contradiction:

$$0 \leq \sum_{x \in V(G) \cup F(G)} c_5(x) = \sum_{x \in V(G) \cup F(G)} c_0(x) = -8$$

Lemma 12. *Every face has non-negative charge at the end of the procedure.*

Proof. At the beginning of the procedure, for every j-face f we have $c_0(f) = j - 4 \geq 0$ (as $j \geq 4$). The procedure does not involve 4-faces. Hence if $j = 4$, then $c_5(f) = c_0(f) = 0$. If $j = 5$, then f gives at most four times $\frac{1}{4}$ if it is incident to a big vertex and at most five times $\frac{1}{5}$ otherwise in Step 4. It follows that $c_5(f) \geq 0$. If $j \geq 6$, then f can give $\frac{1}{3}$ to each of its incident vertices (and so $\frac{1}{4}$ or $\frac{1}{5}$) during Step 4, and $c_5(f) \geq j - 4 - \frac{j}{3} \geq 0$. ◻

Lemma 13. *A 4^+-vertex never has negative charge.*

Proof. Consider a j-vertex z with $j \geq 4$. At the beginning, $c_0(z) = j - 4 \geq 0$. We will show that $c_i(z) \geq 0$ for $i = 1, ..., 5$.

- Suppose z is a big vertex. Such a vertex only loses charge in Step 1. Since $j \geq 8$, we have $c_0(z) \geq \frac{j}{2}$. In Step 1, vertex z loses $\frac{1}{2}$ for each of its small neighbours, and at most $\frac{1}{2}$ for each of its big neighbours. Therefore it has more charge than what it gives, and thus it keeps a non-negative charge.

- Suppose z is a small 5^+-vertex. It does not lose charge in Steps 1, 3, 4 and 5.

  Suppose z has a big neighbour. It has at most $j-1$ small neighbours, and it has charge at least $\frac{1}{4}(j-1)$ at the beginning of the procedure, since $j \geq 5$. Moreover, it receives $\frac{1}{2}$ from each of its big neighbours in Step 1. Therefore it does not give more charge that it has in Step 2.

  Suppose now that z has no big neighbour. If z is a 5-vertex, then by Lemma 6, it has at most four 3-vertices, and $c_2(z) \geq 1 - 4\frac{1}{4} \geq 0$. If z is a 6^+-vertex, then $c_2(z) \geq j - 4 - j\frac{1}{4} \geq 0$.

- Suppose z is a 4-vertex. It does not lose charge in Steps 1 and 4. Suppose z gives charge in Step 2. Consider first that z does not correspond to v_1 in Configuration 10. If z is adjacent to a small vertex that is consecutive (as a neighbour of z) to two big neighbours, then z gives at most twice $\frac{1}{2}$ in Step 2 and received twice $\frac{1}{2}$ in Step 1; hence $c_2(z) \geq 0$. Otherwise, z gives at most twice $\frac{1}{4}$ in Step 2, and received at least once $\frac{1}{2}$ in Step 1; hence $c_2(z) \geq 0$. Let us now consider the case where z corresponds to v_1 in Configuration 10. The vertex z has a big neighbour that gave $\frac{1}{2}$ to z in Step 1, and z gives $\frac{1}{4}$ to two of its neighbours in Step 2. Therefore z received in Step 1 at least as much as what it gives in Step 2.

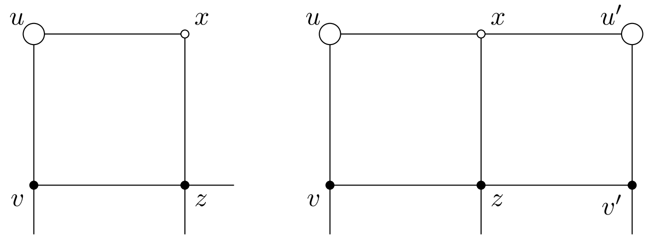

Figure 5: Some configurations that appear in Lemma 13.

  Suppose z gives charge in Step 3. There is a 4-face $uvzx$ with u a big vertex, v a 3-vertex, and x a small vertex such that x gave charge to z in Step 2. Suppose z is consecutive to exactly one big vertex (as neighbours of x). The vertex x gave at least $\frac{1}{4}$ to z in Step 2, and there is exactly one such face with the same z and x (i.e. there is no pair (u', v') distinct from (u, v) that verifies the properties we stated for (u, v))(see Figure 5, left). Therefore z can give $\frac{1}{4}$ to v in Step 3. Suppose z is consecutive to exactly two big vertices (as neighbours of x). The vertex x gave $\frac{1}{2}$ to z in Step 2, and there are at most two such faces with the same z and x (i.e. there is at most one pair (u', v') distinct from (u, v) that verifies the properties we stated for (u, v)) (see Figure 5, right). Therefore z can give $\frac{1}{4}$ to each of the corresponding v's in Step 3. Therefore z received in Step 2 at least as much as what it gives in Step 3.

  Suppose z gives charge in Step 5. There is a 4-face $uvzx$, with u a big vertex, v a 3-vertex, and x a 3-vertex such that the other face, say f, that has vz in its boundary is a 5^+-face. Vertex z received at least $\frac{1}{5}$ from f in Step 4, and it gives $\frac{1}{5}$ to v. There is a problem only if there is another 4-face $u'v'zx'$, such that vzv' is on the boundary of f, u' is a big vertex, and x' and v' are 3-vertices. But then z would have four 3-neighbours, contradicting Lemma 6. Therefore z received in Step 4 at least as much as what it gives in Step 5.

In all cases, z never has negative charge. □

Lemma 14. *At the end of the procedure, every 3-vertex has non-negative charge.*

Proof. Let z be a 3-vertex. It never loses charge in the procedure, so we only need to prove that it received at least 1 over the whole procedure. Assume by contradiction that it received less than that.

By Lemma 5, vertex z has at least one big neighbour b. Let x_0 and x_1 be its two other neighbours. Vertex b gives $\frac{1}{2}$ to z in Step 1, so z only needs to receive $\frac{1}{2}$ from x_0, x_1, and its surrounding faces. In particular, if one of the x_i is a big vertex, then it gives $\frac{1}{2}$ to z in Step 1, and z receives all the charge it needs, a contradiction. Therefore x_0 and x_1 are small vertices.

Let f be the face that contains x_0zx_1 in its boundary, f_0 be the face that contains x_0zb in its boundary and f_1 the face that contains x_1zb in its boundary. Let y_0 and y_1 be such that bzx_0y_0 and bzx_1y_1 are 4-paths that are in the boundaries of f_0 and f_1 respectively. Let us count the charge that x_0, y_0, and f_0 give to z plus half the charge that f gives to z. If we show that this sum is at least $\frac{1}{4}$, then by symmetry we will know that z received at least $\frac{1}{2}$ from x_0, x_1, y_0, y_1, and the faces f, f_0, and f_1, and that leads to a contradiction.

Observe that f_0 is a 4-face. If it is a 5^+-face, then since it has the big vertex b in its boundary, it gives $\frac{1}{4}$ to z in Step 4, a contradiction.

Observe that y_0 is a small vertex. If y_0 is a big vertex, then y_0 gives $\frac{1}{4}$ to z in Step 1, a contradiction. See Figure 6 for a representation of the vertices we know.

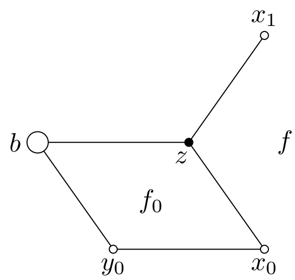

Figure 6: The face f_0 and the vertex x_1.

Observe that x_0 has degree 4. Suppose x_0 is a 5^+-vertex. It gives at least $\frac{1}{4}$ to z in Step 2, a contradiction. Suppose x_0 is a 3-vertex. Then x_0 has a big neighbour by Lemma 5, and it cannot be y_0. This contradicts Lemma 8.

Let a and a' be the neighbours of x_0 distinct from z and y_0, such that a is consecutive to z(as a neighbour of x_0). Suppose a is a big vertex. If x_0 does not correspond to v_1 in Configuration 10, then x_0 gives $\frac{1}{4}$ to z in Step 2. If x_0 corresponds to v_1 in Configuration 10, then z corresponds to w_0 that is not adjacent to two big vertices, so x_0 also gives $\frac{1}{4}$ to z in Step 2. Therefore a is a small vertex.

Observe that y_0 is a 4^+-vertex. Suppose y_0 is a 3-vertex. By Lemma 9, there is at least one big vertex in $\{a, a'\}$, which has to be a'. If f is a 4-face, then x_0 corresponds to v_1 in Configuration 10, and it gives $\frac{1}{4}$ to z in Step 2. Therefore f is a 5^+-face, and it gives at least $\frac{1}{5}$ to z in Step 4, and x_0 gives $\frac{1}{5}$ to z in Step 5. As $\frac{1}{10} + \frac{1}{5} \geq \frac{1}{4}$, this leads to a contradiction.

Suppose first that y_0 corresponds to v_1 in Configuration 10. See Figure 7 for an illustration of the vertices we know, and of the correspondence with vertices of Configuration 10. By Lemma 11, the third neighbour of w_0 is big. Therefore y_0 gives $\frac{1}{4}$ to x_0 in Step 2. It follow that x_0 gives $\frac{1}{4}$ to z in Step 3, a contradiction.

Now y_0 does not correspond to v_1 in Configuration 10. Vertex y_0 gives $\frac{1}{4}$ to x_0 in Step 2, since x_0 is a neighbour of y_0 consecutive (as a neighbour of y_0) to a big neighbour. Therefore x_0 gives $\frac{1}{4}$ to z in Step 3, a contradiction. □

Lemmas 12–14 conclude the proof of Theorem 3.

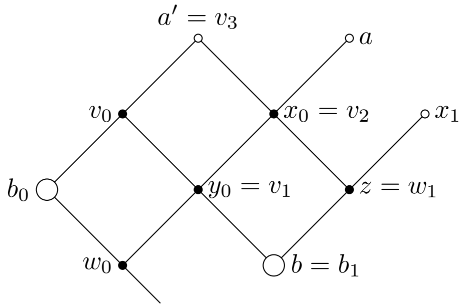

Figure 7: The case in Lemma 14 where y_0 corresponds to v_1 in Configuration 10.

3 NP-completeness

By Theorem 3, there exists a smallest integer $d_0 \leq 5$ such that every triangle-free planar graph has an $(\mathcal{F}, \mathcal{F}_{d_0})$-partition. For all $d \geq d_0$, every triangle-free planar graph has an $(\mathcal{F}, \mathcal{F}_d)$-partition. Let us assume that $d_0 \geq 1$.

In this section, for a fixed d we consider the complexity of the following problem P_d: given a triangle-free planar graph G, does G have an $(\mathcal{F}, \mathcal{F}_d)$-partition? This can be answered positively in constant time for $d \geq d_0$. However, we prove the following:

Theorem 15. *For $d < d_0$, the problem P_d is NP-complete.*

The problem is clearly in NP, since checking that a graph is acyclic and/or has degree at most d can be done in polynomial time. Let us show that the problem is NP-hard.

Let G be a counter-example to the property that every triangle-free planar graph admits an $(\mathcal{F}, \mathcal{F}_d)$ partition. We consider such a G with minimum number of vertices, and with minimum number of edges among the counter-examples with minimum number of vertices. Let $e = uv$ be an edge of G, and $G' = G - e$. By minimality of G, G' admits an $(\mathcal{F}, \mathcal{F}_d)$-partition. In such a partition (F, D), u and v are either both in F or both in D, and if they are in F, then there is a path from u to v in $G'[F]$ (otherwise it would be an $(\mathcal{F}, \mathcal{F}_d)$-partition of G). Observe that in G', u and v are at distance at least 3, since G is triangle-free. We call a copy of G' an anti-edge uv.

We want to make a gadget H with a vertex x that admits an $(\mathcal{F}, \mathcal{F}_d)$-partition, and such that x is in F for all $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H.

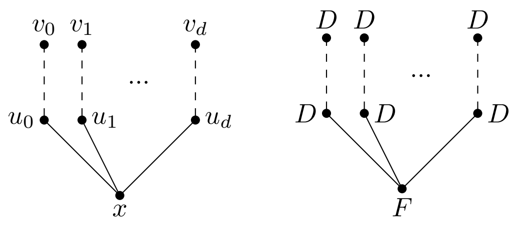

Figure 8: The gadget H in Case 1, and an $(\mathcal{F}, \mathcal{F}_d)$-partition. Dashed lines are anti-edges.

We construct H as follows:

1. Suppose for all $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of G', u and v are in D. See Figure 8 for an illustration of the construction of H and an $(\mathcal{F}, \mathcal{F}_d)$-partition of H in this case. Take $d+1$ copies of G', called G'_0, ..., G'_d, and add a new vertex x adjacent to each copy of u. Consider an $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of G'. This leads to an $(\mathcal{F}, \mathcal{F}_d)$-partition (F_i, D_i) of each G_i, and $(\bigcup_i F_i \cup \{x\}, \bigcup_i D_i)$ is an $(\mathcal{F}, \mathcal{F}_d)$-partition of H.

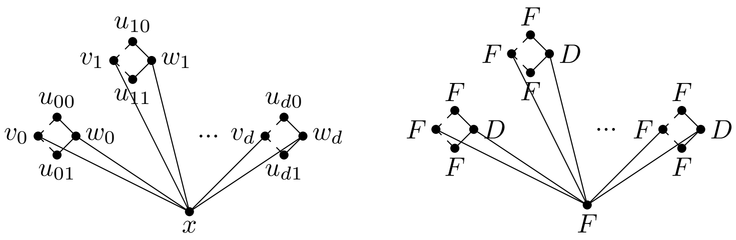

Figure 9: The gadget H in Case 2, and an $(\mathcal{F}, \mathcal{F}_d)$-partition.

Let us now prove that for any $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H, x belongs to F. For any $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H, if $x \in D$, then there exists a u_i that is in F, so the corresponding G'_i admits an $(\mathcal{F}, \mathcal{F}_d)$-partition with $u_i \in F$, a contradiction.

2. Suppose there exists an $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of G' such that u and v are in F. See Figure 9 for an illustration of the construction of H and an $(\mathcal{F}, \mathcal{F}_d)$-partition of H in this case. We construct H as follows. Consider a vertex x. We add new vertices $v_0, ..., v_d$ and $w_0, ..., w_d$ to the graph, adjacent to x. Then for $0 \leq i \leq d$ and $0 \leq j \leq 1$, we add a new vertex u_{ij}, the anti-edge $v_i u_{ij}$, and the edge $u_{ij} w_i$.

   Graph H admits an $(\mathcal{F}, \mathcal{F}_d)$-partition. Indeed, consider an $(\mathcal{F}, \mathcal{F}_d)$-partition of G' with u and v in F, and apply it to every anti-edge of H (as before, we take the union of the F_i and the union of the D_i). Then the v_i and u_{ij} are all in F. Add all the w_i to D. Add x to F. We then have an $(\mathcal{F}, \mathcal{F}_d)$-partition of H.

   Let us now prove that for any $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H, x belongs to F. For any $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H, if $x \in D$, then there exists an i such that v_i and w_i are in F, thus u_{i0} and u_{i1} are in F, so there is a cycle in $H[F]$, a contradiction.

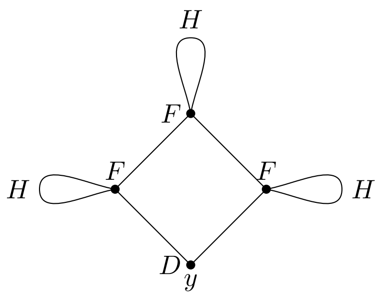

Figure 10: The gadget H' with an $(\mathcal{F}, \mathcal{F}_d)$-partition.

Observe that we can make a gadget H' with a vertex y that admits an $(\mathcal{F}, \mathcal{F}_d)$-partition, and such that y is in D for all $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H' (see Figure 10): we take three copies of H, and make a 4-cycle with the corresponding copies of x and a new vertex y. Taking an $(\mathcal{F}, \mathcal{F}_d)$-partition of H for each copy of H, and adding y to D leads to an $(\mathcal{F}, \mathcal{F}_d)$-partition of H'. Conversely, in an $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D) of H', all the copies of x are in F, so y is in D.

We will first make a reduction from the problem Planar 3-sat to P_0, and then from P_0 to P_d with $d < d_0$.

First reduction: from Planar 3-sat to P_0

Here we will use the gadget H for $d = 0$.

Consider an instance I of Planar 3-sat. The instance I is a boolean formula in conjunctive normal form, associated to a planar graph G_I. For each clause C of I with variables x, y and z, we make a 4-cycle $K_C = x_C y_C z_C a_C$. For each variable x that appears k_x times in the formula, we make the following gadget G_x a path $p_{x,0}...p_{x,2k_x-1}$, and for all $i \in [0, 2k_x - 2]$ we add two adjacent vertices, $q_{x,i}$ and $r_{x,i+1}$, adjacent to $p_{x,i}$ and $p_{x,i+1}$ respectively (see Figure 12). We then add a copy of H for each clause C such that a_C corresponds to the vertex x of H, and a copy of H for each $q_{x,i}$ and each $r_{x,i}$ such that $q_{x,i}$ and $r_{x,i}$ respectively correspond to the vertex x of H. Then for every clause C and every variable x that appears in C, we add an edge from x_C to a $p_{x,i}$, with an even i if the literal associated to x in C is a positive literal and an odd i otherwise, such that no two x_C are adjacent to the same $p_{x,i}$ (see Figure 11). It is possible to do so without breaking planarity, since the graph G_I is planar. We call G'_I the graph we obtain.

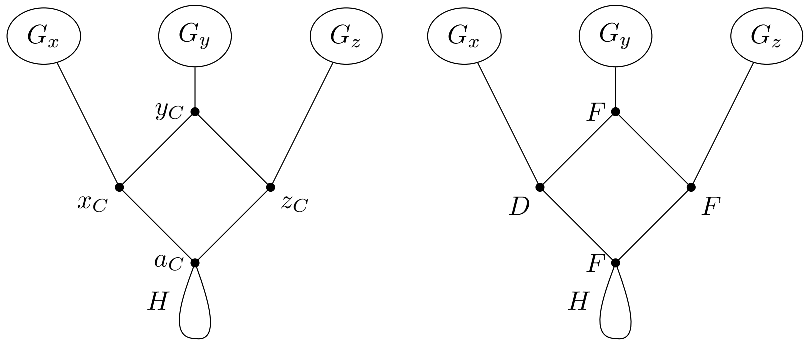

Figure 11: The cycle K_C of a clause C with variables x, y and z, and an $(\mathcal{F}, \mathcal{F}_d)$-partition in the case where variable x satisfies the clause.

Suppose I is satisfiable, and let us consider an assignation σ of the variables that satisfies I. Let us make an $(\mathcal{F}, \mathcal{F}_0)$-partition of G'_I. We first take an $(\mathcal{F}, \mathcal{F}_0)$-partition for each copy of H. All the a_C, $q_{x,i}$ and $r_{x,i}$ are in F. For each variable x, if $\sigma(x) = 1$, then we put all the $p_{x,2i}$ in F and the $p_{x,2i+1}$ in D, else we put all the $p_{x,2i}$ in D and the $p_{x,2i+1}$ in F. Then for each clause C, we choose a variable x of C that satisfies the clause (i.e. x is true if the literal associated to x in C is a positive literal, and false otherwise), we put x_C in D and for the two other variables of C, we put the corresponding y_C in F.

All the vertices are in F or in D. Let v be a G'_I-vertex in D. If v is in a copy of H, then it has no neighbour in D. If v is a x_C, then the three other vertices of K_C are in F. If v is a $p_{x,i}$, then $p_{x,i+1}$ and $p_{x,i-1}$ are in F if they exist, and all the q_j and r_j are in F. Suppose there are two $G'_I[F]$-neighbours in D. One is a x_C and the other is a $p_{x,i}$ (with the same x). Then by construction the variable x satisfies clause C (i.e. x is true if the literal associated to x in C is a positive literal, and false otherwise). If x is associated to a positive literal in clause C, then $\sigma(x) = 1$ and i is even, thus $p_{x,i}$ is in F, a contradiction. If x is associated to a negative literal in clause C, then $\sigma(x) = 0$ and i is odd, thus $p_{x,i}$ is in F, a contradiction. Graph $G'_I[F]$ has no cycle: there is no cycle in the copies of H with every vertex in F; for each clause C, K_C has a vertex in D, and for each $i \in [0, 2k_x - 2]$, $p_{x,2i}$ or $p_{x,2i+1}$ is in D. Therefore (F, D) is an $(\mathcal{F}, \mathcal{F}_0)$-partition of G'_I.

Suppose now that there is an $(\mathcal{F}, \mathcal{F}_0)$-partition (F, D) of G'_I. All the a_C, the $q_{x,i}$ and the $r_{x,i}$ are in F. For all variable x and all $i \in [0, 2k_x - 2]$, either $p_{x,i} \in F$ and $p_{x,i+1} \in D$, or $p_{x,i} \in D$ and $p_{x,i+1} \in F$. Therefore for all x, either all the $p_{x,i}$ are in F for i even and in D for i odd, or all the $p_{x,i}$ are in D for i even and in F for i odd. Let σ be the assignation of the variables x such that $\sigma(x) = 1$ if $p_{x,0}$ is in F, and $\sigma(x) = 0$ otherwise. Let C be a clause

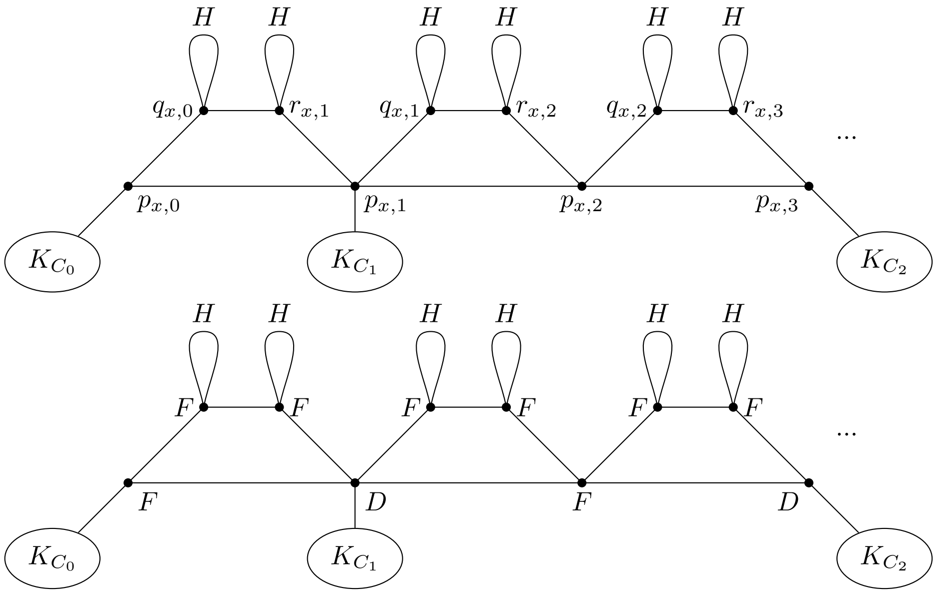

Figure 12: The gadget G_x for a variable x, with an $(\mathcal{F}, \mathcal{F}_d)$-partition that corresponds to the assignation of x to true. Here the literal associated to x in C_0 is positive, and that associated to x in C_1 and C_2 is negative.

of I. At least one of the x_C is in D (otherwise K_C is a cycle with every vertex in F), and it is adjacent to a $p_{x,i}$ with i even if x is positive and i odd if x is negative in C. This $p_{x,i}$ is in F, so if x is positive in C, then $\sigma(x) = 1$, else $\sigma(x) = 0$. Therefore σ satisfies clause C, and this is true for all C, so σ satisfies I.

It is easy to see that the reduction is polynomial, and that G'_I is a triangle-free planar graph. Thus this is a polynomial reduction from Planar 3-sat to P_0.

Second reduction: from P_0 to P_d with $d < d_0$

Consider an instance I of P_0. For each vertex v in I, add d copies of H', such that the corresponding copies of y are adjacent to v. We call I_d the resulting graph.

Suppose I admits an $(\mathcal{F}, \mathcal{F}_0)$-partition. Consider an $(\mathcal{F}, \mathcal{F}_d)$-partition of H'. Apply it to every copy of H' we made in I_d. Complete it with an $(\mathcal{F}, \mathcal{F}_0)$-partition of I. The obtained partition is an $(\mathcal{F}, \mathcal{F}_d)$-partition of I_d.

Suppose now that I_d admits an $(\mathcal{F}, \mathcal{F}_d)$-partition (F, D). In each copy of H', we have $y \in D$, so each vertex in I has exactly d $(I_d - V(I))$-neighbours in D and no $(I_d - V(I))$-neighbours in F. Therefore $(F \cap V(I), D \cap V(I))$ is an $(\mathcal{F}, \mathcal{F}_0)$-partition of I.

It is easy to see that the reduction is polynomial, and that I_d is a triangle-free planar graph. Thus this is a polynomial reduction from P_0 to P_d.

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