POISSON ENVELOPING ALGEBRAS AND THE POINCARÉ-BIRKHOFF-WITT THEOREM

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Abstract. Poisson algebras are, just like Lie algebras, particular cases of Lie-Rinehart algebras. The latter were introduced by Rinehart in his seminal 1963 paper, where he also introduces the notion of an enveloping algebra and proves — under some mild conditions — that the enveloping algebra of a Lie-Rinehart algebra satisfies a Poincaré-Birkhoff-Witt theorem (PBW theorem). In the case of a Poisson algebra $(\mathcal{A},\cdot,\{\cdot\,,\cdot\})$ over a commutative ring R (with unit), Rinehart's result boils down to the statement that if $\mathcal{A}$ is *smooth* (as an algebra), then $\mathrm{gr}(U(\mathcal{A}))$ and $\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$ are isomorphic as graded algebras; in this formula, $U(\mathcal{A})$ stands for the Poisson enveloping algebra of $\mathcal{A}$ and $\Omega(\mathcal{A})$ is the $\mathcal{A}$-module of Kähler differentials of $\mathcal{A}$ (viewing $\mathcal{A}$ as an R-algebra). In this paper, we give several new constructions of the Poisson enveloping algebra in some general and in some particular contexts. Moreover, we show that for an important class of *singular* Poisson algebras, the PBW theorem still holds. In geometrical terms, these Poisson algebras correspond to (singular) Poisson hypersurfaces of arbitrary smooth affine Poisson varieties.

Contents

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1. Introduction

Poisson brackets first appeared in classical mechanics as a tool for constructing new constants of motion from given ones. Their importance was soon emphasized by the discovery that Poisson-commutativity is the key ingredient of (Liouville) integrability. Since then, they also play a mayor rôle in quantization theories (geometric quantization, deformation quantization, quantum groups, ...) and in many other parts of mathematical physics, such as string theory. Formalizing the properties of the Poisson bracket leads on the geometrical side to the definition of a Poisson manifold as being a manifold equipped with a bivector field whose induced operation on smooth functions (the Poisson bracket!) is a Lie bracket. In purely algebraic terms, a Poisson algebra $(\mathcal{A},\cdot,\{\cdot\,,\cdot\})$ over some ring R (always assumed commutative and unitary) is an R-module, equipped with a commutative unitary algebra structure $''\cdot''$ and with a Lie algebra structure $\{\cdot\,,\cdot\}$, satisfying the following compatibility relation, valid for all $a_1, a_2, a_3 \in \mathcal{A}$:

$$\{a_1 \cdot a_2, a_3\} = a_1 \cdot \{a_2, a_3\} + a_2 \cdot \{a_1, a_3\} \ .$$

The algebra of smooth functions on a Poisson manifold is an important class of Poisson algebras, but even in the realm of geometry, in particular in Lie theory and in algebraic geometry, one is soon led to considering singular varieties, which are equipped with a Poisson structure. The simplest example of a singular Poisson variety is the cone $X_1^2 + X_2^2 + X_3^2 = 0$ in $\mathbb{C}^3$, whose algebra of regular functions

$$\mathcal{B} := \frac{\mathbb{C}[X_1, X_2, X_3]}{(X_1^2 + X_2^2 + X_3^2)}$$

is equipped with the Poisson structure, defined by the following three Poisson brackets:

$$\{X_1, X_2\} = X_3, \quad \{X_2, X_3\} = X_1, \quad \{X_3, X_1\} = X_2 \ . \tag{1.1}$$

It was observed by Weinstein [22, 7] that every Poisson manifold is in a natural way a Lie algebroid; the algebraic version of this relationship is that every Poisson algebra is in a natural way a Lie-Rinehart algebra. Stated briefly, a *Lie-Rinehart algebra* is a pair $(\mathcal{A}, L)$ where $\mathcal{A}$ is a commutative algebra and L is a Lie algebra, with the following extra structure: L is an $\mathcal{A}$-module and $\mathcal{A}$ is a Lie L-module; denoting the latter structure by the so-called *anchor map* $\omega : L \to \mathrm{End}(\mathcal{A})$, it is moreover demanded that ω is

$\mathcal{A}$-linear and takes values in $\mathrm{Der}(\mathcal{A})$, and that

$$[x, a \cdot y] = a \cdot [x, y] + \omega_x(a)y$$

for every $a \in \mathcal{A}$ and $x, y \in L$. It can be seen as a far-reaching generalization of the notion of a Lie algebra over a ring R: for $\mathcal{A} = R$ the notion of a Lie-Rinehart algebra boils down to the notion of a Lie algebra (over R). Poisson algebras can also be seen as a particular case of Lie-Rinehart algebras: for any Poisson algebra $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ the pair $(\mathcal{A}, \Omega(\mathcal{A}))$ is a Lie-Rinehart algebra, where $\Omega(\mathcal{A})$ stands for the $\mathcal{A}$-module of Kähler differentials on $\mathcal{A}$; the bracket and the anchor map ω are defined by

$$[a\mathrm{d}F, b\mathrm{d}G] := a\{F, b\}\mathrm{d}G + b\{a, G\}\mathrm{d}F + ab\mathrm{d}\{F, G\}, \qquad \omega_{a\mathrm{d}F}(b) := a\{F, b\}$$

for every $a, b, F, G \in \mathcal{A}$.

Rinehart shows in his seminal paper [19], in which he introduces the notion of a Lie-Rinehart algebra, that every Lie-Rinehart algebra $(\mathcal{A}, L)$ has an enveloping algebra $U(\mathcal{A}, L)$ (a notion that he also introduces) and that there is a natural surjective $\mathcal{A}$-algebra morphism $\mathrm{Sym}_{\mathcal{A}}(L) \to \mathrm{gr}(U(\mathcal{A}, L))$ which generalizes the classical PBW map (Poincaré-Birkhoff-Witt map). Moreover, he shows the following fundamental theorem:

Theorem 1.1 (Rinehart)**.** *If L is projective as an $\mathcal{A}$-module, then the PBW map is an isomorphism of graded algebras.*

In the case of a Lie algebra, one recovers the classical PBW theorem (in its modern form). For Poisson algebras, asking that $\Omega(\mathcal{A})$ is a projective $\mathcal{A}$-module is tantamount to demanding that $\mathcal{A}$ be a *smooth* algebra which, in geometrical contexts (for example when $R = \mathbb{C}$) is in turn equivalent to demanding that $\mathcal{A}$ is the algebra of regular functions on a non-singular variety. Thus, the upshot of Rinehart's theorem, applied to the case of Poisson algebras, is that for smooth Poisson algebras the PBW theorem holds.

The main result of this paper is a generalization of Rinehart's theorem to a large class of Poisson algebras, including the algebra of functions of any irreducible Poisson hypersurface (possibly singular) of an arbitrary smooth Poisson variety. Before stating the result, let us recall that a Poisson ideal I of a Poisson algebra $\mathcal{A}$ is a submodule which is both an ideal and a Lie ideal; the quotient $\mathcal{B} := \mathcal{A}/I$ then inherits a unique Poisson structure from $\mathcal{A}$ such that the canonical surjection $\pi : \mathcal{A} \to \mathcal{B}$ is a morphism of Poisson algebras.

Theorem 1.2. *Suppose that $\mathcal{A}$ is a smooth Poisson algebra and that I is a Poisson ideal of $\mathcal{A}$, which is generated (as an ideal) by a single element. If the quotient $\mathcal{B} := \mathcal{A}/I$ is an integral domain then the PBW theorem holds for $\mathcal{B}$.*

An important ingredient in our proof is a new construction of the Poisson enveloping algebra $U(\mathcal{B})$ of $\mathcal{B} := \mathcal{A}/I$ in terms of the Poisson enveloping algebra $U(\mathcal{A})$ of $\mathcal{A}$. This construction, which is valid for an arbitrary Poisson ideal I in an arbitrary Poisson algebra $\mathcal{A}$, has some similarities with smash products, which are known to provide a construction of the Poisson enveloping algebra of any Poisson algebra (see Section 2.3), but is yet quite different. In the case of the above singular example, the quadratic polynomial $X_1^2 + X_2^2 + X_3^2$ generates a Poisson ideal I of $\mathcal{A} := \mathbb{C}[X_1, X_2, X_3]$, equipped with the Poisson bracket (1.1), so the theorem applies. Here,

$$\Omega(\mathcal{B}) = \frac{\mathcal{B}\mathrm{d}X_1 + \mathcal{B}\mathrm{d}X_2 + \mathcal{B}\mathrm{d}X_3}{\langle X_1\mathrm{d}X_1 + X_2\mathrm{d}X_2 + X_3\mathrm{d}X_3\rangle}$$

and the theorem says that the graded Poisson enveloping algebra of $\mathcal{B}$ is isomorphic to the symmetric algebra $\mathrm{Sym}_{\mathcal{B}}\,\Omega(\mathcal{B})$. For the precise definition of the isomorphism, which is given by the PBW map, see Section 3.2.

The PBW theorem has important applications to deformation theory and to Poisson and Hochschild (co-) homology; we will discuss this in a future publication.

The structure of the paper is the following. After quickly recalling the definition of a Poisson algebra and of a Poisson module over a Poisson algebra we will give the definition of a Poisson enveloping algebra and show that modules over the latter algebra are in one-to-one correspondence with Poisson modules over the underlying Poisson algebra. We discuss a few examples of Poisson enveloping algebras of increasing complexity: the ones corresponding to a null Poisson bracket, to a polynomial algebra, to a general Poisson algebra and to a quotient of a general Poisson algebra. The latter case is important for Section 3, in which we discuss the PBW theorem. First we introduce the PBW map and state what it means for a Poisson algebra to satisfy the PBW theorem. We pick up our list of examples again and show at the end of the section our main result (Theorem 1.2). We finish the papers with some examples and consequences.

In this paper, all rings are assumed to be unitary and all ring morphism are assumed to preserve the unit. Similarly, without the adjectives *Lie* or *Poisson*, the word *algebra* stands for an associative algebra with unit and every algebra morphism preserves the unit. Let R be a commutative ring. For an R-module M we denote the tensor algebra of M by $T_R(M)$ or $T(M)$ and the symmetric algebra by $\mathrm{Sym}_R(M)$ or $\mathrm{Sym}(M)$; both are graded R-algebras, with the latter being commutative. For a Lie algebra $\mathfrak{g}$ over R, its universal enveloping algebra is denoted by $U_{Lie}(\mathfrak{g})$. For any algebra U over R we denote by U_L the corresponding Lie algebra over R, where the bracket is defined by the commutator in U: $[u, v] := uv - vu$ for all $u, v \in U = U_L$. For a graded (resp. filtered) algebra U we will denote the factor of U consisting of all homogeneous elements of degree n by U^n (resp. the submodule of all elements of filtered degree at most n by U_n). Without any further specification all algebras are R-algebras and $\otimes$ stands for $\otimes_R$.

2. Poisson enveloping algebras

In this section, we first recall the definition of a Poisson module and of a Poisson enveloping algebra. We give a few constructions of the Poisson enveloping algebra for the cases of a Lie-Poisson or, more generally, a polynomial Poisson algebra, the main construction being the construction of the Poisson enveloping algebra of a quotient of a Poisson algebra by a Poisson ideal. These constructions will turn out to be very useful in the next section, when we study the PBW theorem for Poisson algebras.

2.1. **The enveloping algebra of a Poisson algebra.** Recall that a *Poisson algebra* (over R) is an R-module $\mathcal{A}$ equipped with two multiplications $(F,G)\mapsto F\cdot G$ and $(F,G)\mapsto\{F,G\}$, such that

(1) $(\mathcal{A},\cdot)$ is a commutative algebra (over R);
(2) $(\mathcal{A},\{\cdot\,,\cdot\})$ is a Lie algebra (over R);
(3) The two multiplications are compatible in the sense that the following derivation property is satisfied:

$$\{a_1\cdot a_2,a_3\}=a_1\cdot\{a_2,a_3\}+a_2\cdot\{a_1,a_3\}\ , \tag{2.1}$$

where a_1,a_2 and a_3 are arbitrary elements of $\mathcal{A}$.

The bilinear map $\{\cdot\,,\cdot\}$ is called the *Poisson bracket* (of $\mathcal{A}$). When dealing with the product in a Poisson algebra, we will always write a_1a_2 for $a_1\cdot a_2$. *Morphisms* of Poisson algebras are linear maps which are both morphisms of algebras and of Lie algebras. A *Poisson ideal* I of $\mathcal{A}$ is a submodule which is both an ideal and a Lie ideal of $\mathcal{A}$; the quotient $\mathcal{B}:=\mathcal{A}/I$ then has a unique Poisson structure, making the canonical surjection $\pi:\mathcal{A}\to\mathcal{B}$ into a morphism of Poisson algebras.

The following three examples will be discussed several times in what follows.

Example 2.1. Let M be an R-module and let σ be a skew-symmetric bilinear form on M. On $\mathrm{Sym}(M)$ a Poisson bracket is defined setting $\{x,y\}:=\sigma(x,y)$, for all $x,y\in M$, and extending $\{\cdot\,,\cdot\}$ to a biderivation of $\mathrm{Sym}(M)$. Explicitly, this yields for monomials $\underline{x}=x_1x_2\dots x_k$ and $\underline{y}=y_1y_2\dots y_\ell$ of $\mathrm{Sym}(M)$,

$$\{\underline{x},\underline{y}\}=\sum_{i=1}^{k}\sum_{j=1}^{\ell}\check{\underline{x}}^i\,\check{\underline{y}}^j\,\sigma(x_i,y_j)\ , \tag{2.2}$$

where $\check{\underline{x}}^i$ stands for the monomial $\underline{x}$ with x_i omitted. When $M=V$ is a finite-dimensional real vector space and σ is a non-degenerate skew-symmetric bilinear form on V, then (V,σ) is a symplectic vector space and the above Poisson bracket yields on the algebra $C^\infty(V)$ a Poisson bracket, which is precisely Poisson's original bracket (see [15, Ch. 6]).

Example 2.2. Let $(\mathfrak{g}, [\cdot\,,\cdot])$ be a Lie algebra over R and let σ be a 2-cocycle in the trivial Lie algebra cohomology of $(\mathfrak{g}, [\cdot\,,\cdot])$. The latter means that σ is a skew-symmetric bilinear form on $\mathfrak{g}$, such that

$$\sigma([x,y],z) + \sigma([y,z],x) + \sigma([z,x],y) = 0\ ,$$

for all $x, y, z \in \mathfrak{g}$. A Poisson bracket is defined on $\mathrm{Sym}(\mathfrak{g})$ by setting $\{x,y\}_\sigma := [x,y] + \sigma(x,y)$ for all $x, y \in \mathfrak{g}$, and extending $\{\cdot\,,\cdot\}_\sigma$ to a biderivation of $\mathrm{Sym}(\mathfrak{g})$. Explicitly, it is given as in (2.2), with $\sigma(x_i, y_j)$ replaced by $[x_i, y_j] + \sigma(x_i, y_j)$. When the Lie bracket $[\cdot\,,\cdot]$ is the trivial bracket, the present example reduces to Example 2.1. When σ is trivial, $\{\cdot\,,\cdot\}_\sigma$ is a linear Poisson structure, usually referred to as a *Lie-Poisson structure* and $(\mathrm{Sym}(\mathfrak{g}), \{\cdot\,,\cdot\})$ is called a *Lie-Poisson algebra*; in general, we refer to it as a *modified Lie-Poisson algebra* (see [15, Ch. 7]).

Example 2.3. Let P and Q be two polynomials in three variables. They define a Poisson structure on $R[X_1, X_2, X_3]$ by setting

$$\{X_1, X_2\} := Q\frac{\partial P}{\partial X_3}\ , \quad \{X_2, X_3\} := Q\frac{\partial P}{\partial X_1}\ , \quad \{X_3, X_1\} := Q\frac{\partial P}{\partial X_2}\ ,$$

which is again extended to a biderivation of $R[X_1, X_2, X_3]$. Notice that P is a *Casimir* of this Poisson structure, i.e., it belongs to the center of $\{\cdot\,,\cdot\}$. The above Poisson structure on $R[X_1, X_2, X_3]$ is called a *Nambu-Poisson structure* (see [15, Ch. 8.3]).

Example 2.4. Suppose that M is an R-module and that its symmetric algebra $\mathrm{Sym}(M)$ is equipped with a skew-symmetric biderivation $\{\cdot\,,\cdot\}$, satisfying the Jacobi identity for all triplets of elements from M. Then $\{\cdot\,,\cdot\}$ satisfies the Jacobi identity for all triplets of elements from $\mathrm{Sym}(M)$, hence makes $\mathrm{Sym}(M)$ into a Poisson algebra. Such an algebra is called a *polynomial Poisson algebra* (see [15, Ch. 1.4, 8.1]).

Let $\mathcal{A}$ be a Poisson algebra (over R). A *Poisson module* over $\mathcal{A}$ is an R-module E which is both a module and a Lie module over $\mathcal{A}$, satisfying supplementary derivation (Leibniz) rules (see [4, 17]). To be precise, $\mathcal{A}$ is equipped with two maps $\alpha_E, \beta_E : \mathcal{A} \to \mathrm{End}(E)$, such that, for all $a_1, a_2 \in \mathcal{A}$,

(1) $\alpha_E(a_1a_2) = \alpha_E(a_1)\alpha_E(a_2)$
(2) $\beta_E(\{a_1, a_2\}) = \beta_E(a_1)\beta_E(a_2) - \beta_E(a_2)\beta_E(a_1)$
(3) $\alpha_E(\{a_1, a_2\}) = \alpha_E(a_1)\beta_E(a_2) - \beta_E(a_2)\alpha_E(a_1)$,
(4) $\beta_E(a_1a_2) = \alpha_E(a_1)\beta_E(a_2) + \alpha_E(a_2)\beta_E(a_1)$.

In the right hand side of these formulas, the product is composition of elements of $\mathrm{End}(E)$. Item (1) resp. (2) says that α_E is an algebra morphism, resp. that β_E is a Lie algebra morphism; item (4) says that β_E is an α_E-derivation.

Examples of Poisson modules include $\mathcal{A}$ itself and any of its powers, any Poisson ideal of $\mathcal{A}$, the dual of $\mathcal{A}$ and so on. With the natural notion of morphism between Poisson modules (where one asks that the morphism is both a morphism of modules and of Lie modules), the Poisson modules over $\mathcal{A}$ form a category which, as we will see later, is an Abelian category.

We are now ready for defining the notion of a Poisson enveloping algebra.

Definition 2.5. Let $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ be a Poisson algebra (over R). A *Poisson enveloping algebra* for $\mathcal{A}$ is an algebra U, equipped with two maps:

(1) An algebra morphism $\alpha : (\mathcal{A}, \cdot) \to U$,
(2) A Lie algebra morphism $\beta : (\mathcal{A}, \{\cdot\,,\cdot\}) \to U_L$,

such that, for any $a_1, a_2 \in \mathcal{A}$,

(3) $\alpha(\{a_1, a_2\}) = \alpha(a_1).\beta(a_2) - \beta(a_2).\alpha(a_1)$,
(4) $\beta(a_1 a_2) = \alpha(a_1).\beta(a_2) + \alpha(a_2).\beta(a_1)$,

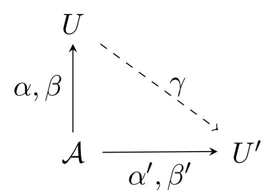
and such that the following universal property holds: if U' is any algebra and $\alpha' : (\mathcal{A}, \cdot) \to U'$ and $\beta' : (\mathcal{A}, \{\cdot\,,\cdot\}) \to U'_L$ are any algebra (resp. Lie algebra) morphisms, satisfying the following properties: for any $a_1, a_2 \in \mathcal{A}$,

(3') $\alpha'(\{a_1, a_2\}) = \alpha'(a_1).\beta'(a_2) - \beta'(a_2).\alpha'(a_1)$,
(4') $\beta'(a_1 a_2) = \alpha'(a_1).\beta'(a_2) + \alpha'(a_2).\beta'(a_1)$,

then there exists a unique algebra morphism $\gamma : U \to U'$, such that

$$\gamma \circ \alpha = \alpha', \qquad \gamma \circ \beta = \beta' .$$

The two equalities are summarized in the following commutative diagram:



(2.3)

Notice that U is not only an R-algebra, but is also in a natural way an $\mathcal{A}$-module, as we may define, for $a \in \mathcal{A}$ and for $u \in U$, $a \cdot u := \alpha(a)u$, which we often write simply as au; in general, we often identify $a \in \mathcal{A}$ with $\alpha(a) \in U(\mathcal{A})$, which is without danger because the algebra morphism α is always an injection (see [19, p. 198], and also [18, Prop. 2.2]). Notice that the algebra morphism $\gamma : U \to U'$ in diagram (2.3) is a morphism of $\mathcal{A}$-modules, when U and U' are viewed as $\mathcal{A}$-modules.

Theorem 2.6. *Let $(\mathcal{A}, \cdot, \{\cdot\,,\cdot\})$ be a Poisson algebra (over R). There exists a Poisson enveloping algebra for $\mathcal{A}$ and it is unique up to isomorphism: if (U, α, β) and (U', α', β') are two Poisson enveloping algebras for $\mathcal{A}$, then there exists an algebra isomorphism $\gamma : U \to U'$, such that $\gamma \circ \alpha = \alpha'$ and $\gamma \circ \beta = \beta'$. The Poisson enveloping algebra of $\mathcal{A}$, which is unique up to isomorphism, is denoted by $U(\mathcal{A})$ and its accompanying maps are denoted by α and β (or by $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ when more than one Poisson enveloping algebra is considered).*

Uniqueness of the Poisson enveloping algebra is clear. A few different existence proofs can be found in [19, 12, 17]. We will give in the next subsections a few alternative constructions for the cases which we will consider in the next section. In the present subsection, we only treat the example of a Poisson algebra $\mathcal{A}$ whose Poisson bracket is the zero bracket. This case is quite simple, but very instructive, as it will provide the natural candidate for the source of the PBW map and give a first instance of a Poisson algebra which satisfies the PBW theorem (stated and treated in general in Section 3). For any algebra $\mathcal{A}$ we denote by $\Omega(\mathcal{A})$ the $\mathcal{A}$-module of Kähler differentials on $\mathcal{A}$ (see [9, Ch. 16]).

Proposition 2.7. *Let $\mathcal{A}$ be any algebra which we make into a Poisson algebra by adding the zero Poisson bracket. Denote by $\alpha : \mathcal{A} \to \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$ the canonical inclusion map and let $\beta := \mathrm{d} : \mathcal{A} \to \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$. The triplet $(\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})), \alpha, \beta)$ is a Poisson enveloping algebra for $\mathcal{A}$.*

Proof. Since the Poisson bracket on $\mathcal{A}$ is null and since $\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$ is commutative, the verification of properties (1) – (3) in Definition 2.5 is immediate; (4) is just the relation $\mathrm{d}(a_1a_2) = a_1\mathrm{d}a_2 + a_2\mathrm{d}a_1$ (which holds in $\Omega(\mathcal{A})$), rewritten in terms of α and β. Suppose now that U' is any algebra and that $\alpha', \beta' : \mathcal{A} \to U'$ are any algebra (resp. Lie algebra) morphisms, satisfying (3') and (4') in Definition 2.5. This means in particular that all elements in the image of α' and β' commute. If there exists an algebra morphism $\gamma : \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) \to U'$, such that $\gamma \circ \alpha = \alpha'$ and $\gamma \circ \beta = \beta'$, then it is given by

$$\gamma(a\mathrm{d}a_1\mathrm{d}a_2 \dots \mathrm{d}a_k) = \alpha'(a).\beta'(a_1).\beta'(a_2)\dots\beta'(a_k) ,$$

for $a, a_1, \dots, a_k \in \mathcal{A}$. This shows that γ is unique, if it exists. Clearly, γ is well-defined by this formula and satisfies $\gamma \circ \alpha = \alpha'$ and $\gamma \circ \beta = \beta'$. Finally, γ is an algebra morphism because all elements in the image of α' and β' commute. □

Let E be a Poisson module over $\mathcal{A}$, with structure maps α' and β'. In view of the universal property of the Poisson enveloping algebra, there exists an algebra morphism $\gamma : U(\mathcal{A}) \to \mathrm{End}(E)$ as in (2.3) (with $U' = \mathrm{End}(E)$), in particular E has a natural structure of $U(\mathcal{A})$-module. Conversely, composition with α and β transforms any $U(\mathcal{A})$ module into a Poisson module over $\mathcal{A}$. The upshot of this natural (and functorial) correspondence is that the category of Poisson modules over $\mathcal{A}$ is equivalent to the category of modules over $U(\mathcal{A})$. It follows that the category of Poisson modules over a given Poisson algebra is an Abelian category, as we announced earlier.

2.2. **The Poisson enveloping algebra of a polynomial Poisson algebra.** In this subsection we give a new construction of the Poisson enveloping algebra of a polynomial Poisson algebra, which we also specialize to the case of a Lie-Poisson algebra. For doing this, we use smash product algebras, which are constructed from module algebras, two notions which we first recall (for more details on these notions and for proofs, see [16], for example).

2.2.1. *Module algebras and smash product algebras.* Let H be a Hopf algebra and let $\mathcal{A}$ be an algebra (both over R). One says that $\mathcal{A}$ is a *(left) H-module algebra* if $\mathcal{A}$ has the structure of a left H-module, with the following properties: for all $u \in H$ and for all $a_1, a_2 \in \mathcal{A}$,

(1) $u \cdot (a_1 a_2) = \sum_{(u)} (u_{(1)} \cdot a_1)(u_{(2)} \cdot a_2)$;
(2) $u \cdot 1 = \epsilon(u)1$.

In (1) we have used Sweedler's notation, i.e., we have written the coproduct of $u \in H$ as $\Delta(u) = \sum_{(u)} u_{(1)} \otimes u_{(2)}$. Also, ϵ denotes the counit of H. The *smash product algebra* of $\mathcal{A}$ by H, denoted $\mathcal{A}\#H$ is as an R-module $\mathcal{A} \otimes H$, with elements denoted by $a\#u$, and with product defined for all $a_1, a_2 \in \mathcal{A}$ and $u, v \in H$ by

$$(a_1\#u) \odot (a_2\#v) := \sum_{(u)} a_1(u_{(1)} \cdot a_2)\#u_{(2)}v \; . \tag{2.4}$$

This product is associative with unit $1\#1$. For $a_1, a_2 \in \mathcal{A}$ and $u, v \in H$ it follows from definition (2.4) that $(a_1\#1) \odot (a_2\#1) = a_1(1 \cdot a_2)\#1 = a_1 a_2\#1$, so that the inclusion map $\imath_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}\#H$ is a morphism of algebras. It can be used to define an $\mathcal{A}$-module structure on $\mathcal{A}\#H$ by setting, for $a_1, a_2 \in \mathcal{A}$ and $u \in H$,

$$a_1 \cdot (a_2\#u) := \imath_{\mathcal{A}}(a_1) \odot (a_2\#u) = a_1 a_2\#u \; .$$

In the sequel, we write $a_1(a_2\#u)$ for $a_1 \cdot (a_2\#u)$. It also follows from the definitions that

$$\begin{aligned} (a\#u) \odot (1\#v) &= \sum_{(u)} a(u_{(1)} \cdot 1)\#u_{(2)}v = \sum_{(u)} a\epsilon(u_{(1)})\#u_{(2)}v \\ &= a\# \left(\sum_{(u)} \epsilon(u_{(1)})u_{(2)} \right) v = a\#uv \; , \end{aligned} \tag{2.5}$$

where we used in the last equality that ϵ is the counit of H. This shows in particular that the inclusion map $\imath_H : H \to \mathcal{A}\#H$ is also a morphism of algebras. Notice that every element of $\mathcal{A}\#H$ of the form $a\#u$ can be written as the product of an element of $\mathrm{Im}(\imath_{\mathcal{A}})$ with an element of $\mathrm{Im}(\imath_H)$, namely for all $a \in \mathcal{A}$ and $u \in H$,

$$(a\#1) \odot (1\#u) = a\#u \; . \tag{2.6}$$

It leads, in view of the above properties, to a simple proof that $1\#1$ is the unit of $\mathcal{A}\#H$, as we said above. We give two typical examples; they will be used later, besides others which will be introduced as we need them.

Example 2.8. Let M be an R-module and let $\{\cdot\,,\cdot\}$ be a Poisson bracket on $\mathrm{Sym}(M)$, making it into a polynomial Poisson algebra. It is well-known (see [14, Ch. 3]) that the tensor algebra $T(M)$ has a natural structure of a

Hopf algebra, where the comultiplication $\Delta : T(M) \to T(M) \otimes T(M)$ is the unique algebra morphism, given for $x \in M$ by $\Delta(x) = 1 \otimes x + x \otimes 1$ and the counit $\epsilon : T(M) \to R$ picks the constant (degree zero) term of a tensor. Using the Poisson bracket, $\mathrm{Sym}(M)$ becomes a $T(M)$-module algebra upon setting, for $x_1 \otimes x_2 \otimes \cdots \otimes x_k \in T(M)$ and $a \in \mathrm{Sym}(M)$,

$$(x_1 \otimes x_2 \otimes \cdots \otimes x_k) \cdot a := \{x_1, \{x_2, \dots \{x_k, a\} \dots\}\} \ . \tag{2.7}$$

It is understood that this definition specializes for $k = 0$ to $1 \cdot a := a$. Since both Δ and ϵ are algebra morphisms, and since items (1) and (2) in the above definition of a module algebra are obviously satisfied for $u = 1$, it suffices to check them for $u = x \in M$ (and $a_1, a_2 \in \mathrm{Sym}(M)$). Since $\{\cdot\,,\cdot\}$ is a derivation in each argument, we have

$$x\cdot(a_1a_2) = \{x, a_1a_2\} = a_1\{x, a_2\} + \{x, a_1\}a_2 = (1\cdot a_1)(x\cdot a_2) + (x\cdot a_1)(1\cdot x_2) \ ,$$

which proves (1), since $\Delta(m) = 1 \otimes m + m \otimes 1$. Also, (2) holds because $m \cdot 1 = \{m, 1\} = 0$ and $\epsilon(m) = 0$. This shows that $\mathrm{Sym}(M)$ is a $T(M)$-module algebra. We can therefore form the smash product algebra $\mathrm{Sym}(M)\#T(M)$. According to definition (2.4), the product is given, for $a_1, a_2 \in \mathrm{Sym}(M)$ and $x \in M$ and $u \in T(M)$ by

$$(a_1\#x) \odot (a_2\#u) = a_1a_2\#(x \otimes u) + a_1\{x, a_2\}\#u \ . \tag{2.8}$$

Example 2.9. Let $\mathcal{A}$ be an arbitrary Poisson algebra. We show that $\mathcal{A}$ is a $U_{Lie}(\mathcal{A})$ module algebra. To do this, we first recall that the standard Hopf algebra structure of $U_{Lie}(\mathcal{A})$ is induced by the Hopf algebra structure on $T(\mathcal{A})$, recalled in the previous example (see [14, Ch. 5]). It means that $\Delta : U_{Lie}(\mathcal{A}) \to U_{Lie}(\mathcal{A}) \otimes U_{Lie}(\mathcal{A})$ is the unique algebra morphism which is defined for all $a \in \mathcal{A} \subset U_{Lie}(\mathcal{A})$ by $\Delta(a) := 1 \otimes a + a \otimes 1$. Consider the linear map $X : \mathcal{A} \to \mathrm{End}(\mathcal{A})$, defined by $a \mapsto X_a = \{a, \cdot\}$. In view of the Jacobi identity, it makes $\mathcal{A}$ into a Lie module over $\mathcal{A}$, hence makes $\mathcal{A}$ into a module over $U_{Lie}(\mathcal{A})$. To check that it makes $\mathcal{A}$ into a (left) $U_{Lie}(\mathcal{A})$ module algebra, it suffices to verify (1) and (2) in the above definition of a module algebra for $u \in \mathcal{A}$ and for $u \in R$ (that is for $u \in U_{Lie}(\mathcal{A})$ of degree at most 1). For example, when $u \in \mathcal{A}$, so that $\Delta(u) = 1 \otimes u + u \otimes 1$, then $u \cdot (a_1a_2) = \{x, a_1a_2\}$, while

$$\sum_{(u)}(u_{(1)}\cdot a_1)(u_{(2)}\cdot a_2) = (1\cdot a_1)(u\cdot a_2) + (u\cdot a_1)(1\cdot a_2) = a_1\{u, a_2\} + a_2\{u, a_1\} \ ,$$

which is the same as $\{u, a_1b_1\}$ because $\{\cdot\,,\cdot\}$ is a biderivation. The other verifications are even simpler. Thus, $\mathcal{A}$ is a left $U_{Lie}(\mathcal{A})$-module algebra and we can form the smash product $\mathcal{A}\#U_{Lie}(\mathcal{A})$ of $\mathcal{A}$ by $U_{Lie}(\mathcal{A})$. For future use, let us point out that the product in $\mathcal{A}\#U_{Lie}(\mathcal{A})$ is given, for $a_1, a_2, a_3 \in \mathcal{A}$ and $u \in U_{Lie}(\mathcal{A})$ by

$$(a_1\#a_3) \odot (a_2\#u) = a_1a_2\#a_3.u + a_1\{a_3, a_2\}\#u \ , \tag{2.9}$$

where $a_3.u$ stands for the product of a_3 and u in $U_{Lie}(\mathcal{A})$.

2.2.2. *The Poisson enveloping algebra of a (modified) Lie-Poisson algebra as a smash product algebra.* We show in this paragraph that the Poisson enveloping algebra of the modified Lie-Poisson algebra $\mathrm{Sym}_\sigma(\mathfrak{g}) = (\mathrm{Sym}(\mathfrak{g}), \{\cdot\,,\cdot\}_\sigma)$ (where $\mathfrak{g}$ is a Lie algebra and σ is a 2-cocycle in the trivial Lie algebra cohomology of $\mathfrak{g}$, see Example 2.2) is the smash product algebra $\mathrm{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g})$, with accompanying maps α and β which will be defined below.

First, we need to explain how we turn $\mathrm{Sym}(\mathfrak{g})$ into a $U_{Lie}(\mathfrak{g})$-module algebra. The construction is very similar to the one given in Example 2.9: $\mathrm{Sym}(\mathfrak{g})$ is a Lie module over $\mathfrak{g}$, when setting $x \cdot a := \{x, a\}_\sigma$, for $x \in \mathfrak{g}$ and $a \in \mathrm{Sym}(\mathfrak{g})$, so that $\mathrm{Sym}(\mathfrak{g})$ is a module over $U_{Lie}(\mathfrak{g})$ and one verifies like in Example 2.9 that $\mathrm{Sym}(\mathfrak{g})$ is a $U_{Lie}(\mathfrak{g})$-module algebra. We can therefore form the smash product algebra $\mathrm{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g})$. By construction, the product in this algebra is given, for $a_1, a_2 \in \mathrm{Sym}_\sigma(\mathfrak{g})$ and $x \in \mathfrak{g} \subset U_{Lie}(\mathfrak{g})$ and $u \in U_{Lie}(\mathfrak{g})$ by

$$(a_1\#x) \odot (a_2\#u) = a_1a_2\#x.u + a_1\{x, a_2\}_\sigma \#u\ . \tag{2.10}$$

The maps α and β are defined by

$$\begin{array}{rccl} \alpha & : & \mathrm{Sym}_\sigma(\mathfrak{g}) & \to & \mathrm{Sym}_\sigma(\mathfrak{g})\#U_{Lie}(\mathfrak{g}) \\ & & a & \mapsto & a\#1\ , \\ \beta & : & \mathrm{Sym}_\sigma(\mathfrak{g}) & \to & \mathrm{Sym}_\sigma(\mathfrak{g})\#U_{Lie}(\mathfrak{g}) \\ & & \underline{x} & \mapsto & \sum_i \check{\underline{x}}^i\#x_i\ . \end{array}$$

Notice that β can be defined as the unique α-derivation such that $\beta(x) = 1\#x$ for all $x \in \mathfrak{g}$.

Proposition 2.10. $(\mathrm{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g}), \alpha, \beta)$ *is a Poisson enveloping algebra of the Lie-Poisson algebra* $\mathrm{Sym}_\sigma(\mathfrak{g})$*.*

Proof. We first verify items (1) – (4) in Definition 2.5. We know that α is an algebra morphism and that β is an α-derivation, which is the content of (1) and (4). We move to item (3): for a monomial $\underline{x} \in \mathrm{Sym}_\sigma(\mathfrak{g})$ and for any $a \in \mathrm{Sym}_\sigma(\mathfrak{g})$ we have

$$\begin{aligned} [\alpha(a), \beta(\underline{x})] &= a\#1 \odot \sum_i \check{\underline{x}}^i\#x_i - \sum_i \check{\underline{x}}^i\#x_i \odot a\#1 \\ &= \sum_i a\,\check{\underline{x}}^i\#x_i - \sum_i \check{\underline{x}}^i a\#x_i - \sum_i \check{\underline{x}}^i\{x_i, a\}_\sigma\#1 \\ &= \{a, \underline{x}\}_\sigma\#1 = \alpha(\{a, \underline{x}\}_\sigma)\ . \end{aligned}$$

We still need to prove (2). In view of the items just proved, it is easily shown by recursion that it suffices to prove that $\beta(\{x, y\}_\sigma) = [\beta(x), \beta(y)]$ for all $x, y \in \mathfrak{g}$. Since $\{x, y\}_\sigma = [x, y] + \sigma(x, y)$ and since β is null on constants, we have

$$\begin{aligned} \beta(\{x, y\}_\sigma) &= \beta([x, y]) = 1\#[x, y] = 1\#(x.y - y.x) \\ &= (1\#x) \odot (1\#y) - (1\#y) \odot (1\#x) = [\beta(x), \beta(y)]\ . \end{aligned}$$

We now prove the universal property of $\operatorname{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g})$. Let U' be any algebra and suppose that we are given any algebra morphism $\alpha' : \operatorname{Sym}_\sigma(\mathfrak{g}) \to U'$ and any Lie algebra morphism $\beta' : \operatorname{Sym}_\sigma(\mathfrak{g}) \to U'_L$, satisfying properties (3') and (4') (of Definition 2.5). We show that there is a unique algebra morphism $\gamma : \operatorname{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g}) \to U'$ such that $\gamma \circ \alpha = \alpha'$ and $\gamma \circ \beta = \beta'$. Since every element $a\#u$ of $\operatorname{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g})$ can be written as the product of an element of $\operatorname{Im}(\alpha)$ with a product of elements of $\operatorname{Im}(\beta)$, the morphism γ is unique, if it exists; moreover, it leads to the formulas

$$\begin{aligned} \gamma(a\#1) &= \alpha'(a) , \\ \gamma(a\#(x_1.x_2 \dots x_k)) &= \alpha'(a).\beta'(x_1).\beta'(x_2)\dots\beta'(x_k) , \end{aligned}$$

where $a \in \operatorname{Sym}_\sigma(\mathfrak{g})$ and $x_1, \dots, x_k \in \mathfrak{g}$. In order to show that the map γ is well-defined by this formula, we use the R-linear map $\gamma' : \operatorname{Sym}_\sigma(\mathfrak{g})\#T(\mathfrak{g}) \to U'$, defined by

$$\gamma'(a \otimes x_1 \otimes \dots \otimes x_k) := \alpha'(a).\beta'(x_1)\dots\beta'(x_k) ,$$

where $a \in \operatorname{Sym}_\sigma(\mathfrak{g})$ and $x_1, \dots, x_k \in \mathfrak{g}$. For $X = x_1 \otimes x_2 \otimes \dots \otimes x_k$ and $Y = y_1 \otimes y_2 \otimes \dots \otimes y_\ell$ in $T(\mathfrak{g})$, and for $x, y \in \mathfrak{g}$, we have

$$\begin{aligned} &\gamma'(a \otimes X \otimes (x \otimes y - y \otimes x - [x,y]) \otimes Y) = \\ &\quad \alpha'(a).\beta'(x_1)\dots\beta'(x_k).([\beta'(x),\beta'(y)] - \beta'([x,y])).\beta'(y_1)\dots\beta'(y_\ell) = 0 , \end{aligned}$$

because β' is a Lie morphism. It follows that the R-linear map $\gamma : \operatorname{Sym}_\sigma(\mathfrak{g}) \otimes U_{Lie}(\mathfrak{g}) \to U'$ is well-defined. We need to show that γ is an algebra morphism, i.e. that $\gamma((a_1\#u) \odot (a_2\#v)) = \gamma(a_1\#u).\gamma(a_2\#v)$ for all $a_1, a_2 \in \operatorname{Sym}(\mathfrak{g})$ and for all homogeneous elements u, v of $U_{Lie}(\mathfrak{g})$. We do this by recursion on the filtered degree k of u and we write $v = v_1.v_2\dots v_\ell$, where all v_i belong to $\mathfrak{g}$. For $u = 1$, we have

$$\gamma((a_1\#1) \odot (a_2\#v)) = \alpha'(a_1a_2).\beta'(v_1)\dots\beta'(v_\ell) = \gamma(a_1\#1).\gamma(a_2\#v) .$$

We next take $u = x \in \mathfrak{g} \subset U_{Lie}(\mathfrak{g})$. Then, using (3'),

$$\begin{aligned} \gamma((a_1\#x) \odot (a_2\#v)) &= \gamma(a_1a_2\#x.v + a_1\{x, a_2\}\#v) \\ &= \alpha'(a_1).\left(\alpha'(a_2).\beta'(x) - \alpha'(\{a_2, x\})\right).\beta'(v_1)\dots\beta'(v_\ell) \\ &= \alpha'(a_1).\beta'(x).\alpha'(a_2).\beta'(v_1)\dots\beta'(v_\ell) \\ &= \gamma(a_1\#x).\gamma(a_2\#v) . \end{aligned}$$

Suppose now that $\gamma((a_1\#u) \odot (a_2\#v)) = \gamma(a_1\#u).\gamma(a_2\#v)$ holds for any u of degree at most k. Then, using the associativity of $\odot$, the recursion hypothesis and (2.5),

$$\begin{aligned} \gamma((a_1\#u.x) \odot (a_2\#v)) &= \gamma(a_1\#u \odot (1\#x \odot a_2\#v)) \\ &= \gamma(a_1\#u).\gamma(1\#x \odot a_2\#v) \\ &= \gamma(a_1\#u).\gamma(1\#x).\gamma(a_2\#v) \\ &= \gamma(a_1\#u \odot 1\#x).\gamma(a_2\#v) \\ &= \gamma(a_1\#u.x).\gamma(a_2\#v) . \end{aligned}$$

This shows that the formula also holds for u of degree at most $k+1$, and hence that γ is an algebra morphism. Finally, we need to check that $\gamma\circ\alpha=\alpha'$ and $\gamma\circ\beta=\beta'$. The first equality is immediate from the above explicit formula for γ, so we only prove the second one. For any monomial $\underline{x}\in \mathrm{Sym}_\sigma(\mathfrak{g})$,

$$\gamma(\beta(\underline{x}))=\sum_i \gamma(\check{\underline{x}}^i \# x_i)=\sum_i \alpha'(\check{\underline{x}}^i).\beta'(x_i)=\beta'(\underline{x})\ .$$

The last equality is valid because β' is an α'-derivation. □

2.2.3. *The Poisson enveloping algebra of a (modified) Lie-Poisson algebra as a (modified) Lie enveloping algebra.* We give in this paragraph a different description of the Poisson enveloping algebra of a (modified) Lie-Poisson algebra. In the unmodified case, the result is that for any Lie algebra $\mathfrak{g}$, the Lie enveloping algebra of a certain double $\mathfrak{g}^+$ of $\mathfrak{g}$ (known as a Takiff algebra, see [21]) is a Poisson enveloping algebra of the Lie-Poisson algebra $\mathrm{Sym}(\mathfrak{g})$. In the modified case, the same result holds, upon using the notion of a modified Lie enveloping algebra, also known as a Sridharan algebra ([20]).

Suppose, as in the previous paragraph, that $\mathfrak{g}$ is a Lie algebra and that σ is a 2-cocycle in the trivial Lie algebra cohomology of $\mathfrak{g}$. Let us denote by $\mathfrak{g}^0$ the abelian Lie algebra, whose underlying module is $\mathfrak{g}$. Consider $\mathfrak{g}^+ := \mathfrak{g}^0\oplus\mathfrak{g}$, in which $\mathfrak{g}^0$ and $\mathfrak{g}$ are naturally embedded. For $x\in\mathfrak{g}$ we will write x^0, respectively x^1, for its canonical image in $\mathfrak{g}^0$, respectively in $\mathfrak{g}$, viewed as a subspace of $\mathfrak{g}^+$. Thus, we can write every element x^+ of $\mathfrak{g}^+$ uniquely as $x^+=y^0+z^1$, with $y,z\in\mathfrak{g}$. A Lie bracket is defined on $\mathfrak{g}^+$ by

$$[y^0,z^0]^+=0,\quad [y^0,z^1]^+=[y,z]^0,\quad [y^1,z^1]^+=[y,z]^1\ ,$$

where $y,z\in\mathfrak{g}$. The Lie algebra $\mathfrak{g}^+$ is a semi-direct product of $\mathfrak{g}^0$ and $\mathfrak{g}$: for elements $x_1^+=y_1^0+z_1^1$ and $x_2^+=y_2^0+z_2^1$ of $\mathfrak{g}^+$, we have

$$[x_1^+,x_2^+]^+=[y_1,z_2]^0+[z_1,y_2]^0+[z_1,z_2]^1\ .$$

The cocycle σ becomes a cocycle σ^+ of $\mathfrak{g}^+$ upon setting for all $x,y\in\mathfrak{g}$:

$$\sigma^+(x^0,y^0):=\sigma^+(x^1,y^1):=0,\ \sigma^+(x^0,y^1):=\sigma^+(x^1,y^0):=\sigma(x,y)\ ,$$

and extending these definitions by bilinearity. Since

$$\sigma^+([x_1^+,x_2^+],x_3^+)=\sigma([y_1,z_2],z_3)+\sigma([z_1,y_2],z_3)+\sigma([z_1,z_2],y_3)$$

and since σ is a cocycle, σ^+ is indeed a cocycle. The modified Lie enveloping algebra (or Sridharan algebra) of $\mathrm{Sym}_\sigma(\mathfrak{g})$ is given by $U_{Lie,\sigma^+}(\mathfrak{g}^+) := T(\mathfrak{g})/I_\sigma$ where I_σ is the two-sided ideal of $T(\mathfrak{g})$ generated by all elements of the form

$$\{x^+\otimes y^+ - y^+\otimes x^+ - [x^+,y^+]^+ - \sigma^+(x^+,y^+)\cdot 1\}\ , \tag{2.11}$$

where $x^+,y^+\in\mathfrak{g}^+$. Let ι denote the canonical inclusion $\iota:\mathfrak{g}^+\hookrightarrow U_{Lie,\sigma^+}(\mathfrak{g}^+)$. For $x\in\mathfrak{g}$, let $\alpha'(x):=\iota(x^0)$ and $\beta'(x):=\iota(x^1)$. For $x,y\in\mathfrak{g}$ we have, in view of (2.11), that $\iota(x^0)\iota(y^0)-\iota(y^0)\iota(x^0)=\iota([x^0,y^0]^+)+\sigma^+(x^0\otimes y^0)\cdot 1=0$. We can therefore uniquely extend α' to an algebra morphism

$$\alpha' : \mathrm{Sym}_\sigma(\mathfrak{g}) \to U_{Lie,\sigma^+}(\mathfrak{g}^+)\ .$$

By a slight abuse of notation, we will also write $\iota(a^0)$ for $\alpha'(a)$, where $a\in\mathrm{Sym}_\sigma(\mathfrak{g})$. As for β', it extends uniquely to an α'-derivation

$$\beta' : \mathrm{Sym}_\sigma(\mathfrak{g}) \to U_{Lie,\sigma^+}(\mathfrak{g}^+)\ .$$

Explicitly, β' is given for a monomial $\underline{x}\in\mathrm{Sym}_\sigma(\mathfrak{g})$ by $\beta'(\underline{x})=\sum_j \iota(\check{\underline{x}}^j)^0\iota(x_j^1)$.

Proposition 2.11. *The modified Lie enveloping algebra* $(U_{Lie,\sigma^+}(\mathfrak{g}^+),\alpha',\beta')$ *is a Poisson enveloping algebra of the modified Lie-Poisson algebra* $\mathrm{Sym}_\sigma(\mathfrak{g})$.

Proof. By construction, α' is an algebra morphism and α' and β' satisfy property (4') of Definition 2.5. We show that they also satisfy property (3') of the latter definition and that β' is a morphism of Lie algebras. For a monomial $\underline{x}\in\mathrm{Sym}_\sigma(\mathfrak{g})$ and for any $a\in\mathrm{Sym}_\sigma(\mathfrak{g})$ we have

$$\begin{aligned}\left[\alpha'(a),\beta'(\underline{x})\right] &= \iota(a^0).\sum_j \iota((\check{\underline{x}}^j)^0).\iota(x_j^1)-\sum_j \iota((\check{\underline{x}}^j)^0).\iota(x_j^1).\iota(a^0)\\ &= -\sum_j \iota((\check{\underline{x}}^j)^0).\iota(\{x_j,a\}_\sigma^0)=\iota(\{a,\underline{x}\}_\sigma^0)=\alpha'(\{a,\underline{x}\}_\sigma)\ .\end{aligned}$$

As in the proof of Proposition 2.10, β' is a Lie morphism as soon as it has the Lie morphism property when applied to elements of $\mathfrak{g}$. Therefore, let $x,y\in\mathfrak{g}$. On the one hand,

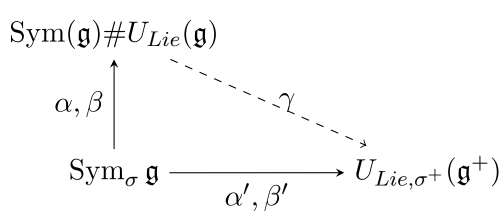
$$\beta'(\{x,y\}_\sigma)=\beta'([x,y]+\sigma(x,y))=\beta'([x,y])=\iota([x,y]^1),$$

while on the other hand,

$$[\beta'(x),\beta'(y)]=\iota(x^1).\iota(y^1)-\iota(y^1).\iota(x^1)=\iota([x,y]^1)+\sigma^+(x^1,y^1)\cdot 1=\iota([x,y]^1).$$

This shows that β' is a Lie algebra morphism.

We can now apply the universal property of the Poisson enveloping algebra $(\mathrm{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g}),\alpha,\beta)$ (see Proposition 2.10): there exists a (unique) algebra morphism γ, making the following diagram commutative:



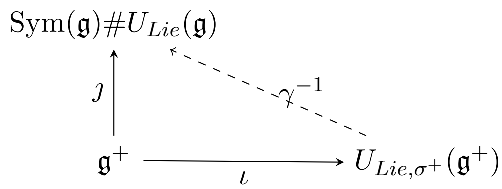
In order to show that γ is an isomorphism, we construct the inverse map. We use for it the universal property of the modified Lie enveloping algebra $U_{Lie,\sigma^+}(\mathfrak{g}^+)$. Denote by $\jmath:\mathfrak{g}^+\to\mathrm{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g})$ the linear map, defined by $\jmath(x^+) = \jmath(y^0 + z^1) := \alpha(y) + \beta(z)$. For $x_1^+ = y_1^0 + z_1^1$ and $x_2^+ = y_2^0 + z_2^1$ in $\mathfrak{g}^+$ we have the following three equalities:

$$
\begin{aligned}
\left[\jmath(x_1^+), \jmath(x_2^+)\right] &= [\alpha(y_1) + \beta(z_1), \alpha(y_2) + \beta(z_2)] \\
&= [\alpha(y_1), \beta(z_2)] + [\alpha(z_1), \beta(y_2)] + [\beta(z_1), \beta(z_2)] \ , \\
\jmath\left(\left[x_1^+, x_2^+\right]^+\right) &= \alpha([y_1, z_2] + [z_1, y_2]) + \beta([z_1, z_2]) \\
&= \alpha(\{y_1, z_2\} - \sigma(y_1, z_2) + \{z_1, y_2\} - \sigma(z_1, y_2)) + \beta(\{z_1, z_2\}) \\
&= [\alpha(y_1), \beta(z_2)] + [\alpha(z_1), \beta(y_2)] + [\beta(z_1), \beta(z_2)] \\
&\quad -\alpha(\sigma(y_1, z_2) + s(z_1, y_2)) \ , \\
\sigma^+(x_1^+, x_2^+)\#1 &= \sigma(y_1, z_2)\#1 + \sigma(z_1, y_2)\#1 = \alpha(\sigma(y_1, z_2) + \alpha(\sigma(z_1, y_2)) \ .
\end{aligned}
$$

This shows that

$$\left[\jmath(x_1^+), \jmath(x_2^+)\right] = \gamma(\left[x_1^+, x_2^+\right]^+) + \sigma^+(x_1^+, x_2^+)(1\#1) \ ,$$

for all $x_1^+, x_2^+ \in \mathfrak{g}^+$. By the universal property of the modified Lie enveloping algebra, there exists a (unique) algebra morphism γ^{-1} making the following diagram commutative:



On generators of these algebras, one checks that γ and γ^{-1} are inverse to each other, showing that γ is an algebra isomorphism (with inverse γ^{-1}). $\square$

Remark 2.12. Being a (modified) Lie enveloping algebra, $U_{Lie,\sigma^+}(\mathfrak{g}^+)$ has a natural filtration, where every element of $\mathfrak{g}^+$, viewed as an element of $U_{Lie,\sigma^+}(\mathfrak{g}^+)$, has filtered degree 1. As a Poisson enveloping algebra, it also has a natural filtration; in the latter filtration, all elements of $\alpha'(\mathfrak{g})$ have degree 0 and all elements of $\beta'(\mathfrak{g})$ have degree 1; said differently, for this filtration, every element of $\mathfrak{g}^0 \subset \mathfrak{g}^+$, viewed as an element of $U_{Lie,\sigma^+}(\mathfrak{g}^+)$, has filtered degree 0, while every element of $\mathfrak{g} \subset \mathfrak{g}^+$, viewed as an element of $U_{Lie,\sigma^+}(\mathfrak{g}^+)$, has filtered degree 1. This issue has important consequences, as we will see when discussing the PBW theorem (see Remark 3.6 below).

Example 2.13. Let (V, ω) be a symplectic vector space of dimension $2n$ over a field $\mathbb{F}$. There exists a symplectic basis $(X_1, Y_1, \ldots, X_n, Y_n)$ of V such that $\omega(X_i, Y_j) = \delta_{i,j}$ and $\omega(X_i, X_j) = \omega(Y_i, Y_j) = 0$, for all i, j. Viewing V as a trivial Lie algebra, ω is a 2-cocycle in the trivial Lie algebra cohomology of V. The symmetric algebra $\mathrm{Sym}(V) \simeq \mathbb{F}[X_1, Y_1, \ldots, X_n, Y_n]$ is a modified Lie-Poisson algebra, whose Poisson bracket is given by $\{X_i, Y_i\}_\omega = 1$ for $1 \leqslant i \leqslant n$ and all other brackets between basis elements are zero. The double, $V^+ = V^0 \oplus V$, has as basis $(p_i, q_i)_{1 \leqslant i \leqslant 2n}$ with $q_i = X_i^0$, $q_{n+i} = Y_i^0$,

$p_i = -Y_i$, $p_{n+i} = X_i$ for $1 \leqslant i \leqslant 2n$. It follows that the Poisson enveloping algebra $U_{Lie,\omega}(V^+)$ of $(\mathrm{Sym}(V), \{\cdot\,,\cdot\}_\omega)$ is the Weyl algebra $A(V) \simeq A_{4n}(\mathbb{F})$.

2.2.4. *The Poisson enveloping algebra of a polynomial Poisson algebra as a quotient of a smash product algebra.* Let M be an R-module and suppose that its symmetric algebra $\mathrm{Sym}(M)$ is equipped with a Poisson bracket $\{\cdot\,,\cdot\}$, making it into a polynomial Poisson algebra (see Example 2.4). We give in this paragraph a construction of its Poisson enveloping algebra, using the smash product algebra $\mathrm{Sym}(M)\#T(M)$, constructed in Example 2.8. Let $\psi_M : \mathrm{Sym}(M) \to \mathrm{Sym}(M)\#T(M)$ denote the unique (R-linear) derivation of $\mathrm{Sym}(M)$ with values in $\mathrm{Sym}(M)\#T(M)$, defined by $\psi_M(x) := 1\#x$ for all $x \in M$. For a monomial $\underline{x} = x_1x_2\dots x_k \in \mathrm{Sym}(M)$,

$$\psi_M(\underline{x}) = \sum_{i=1}^{k} \check{\underline{x}}^i \# x_i \ . \tag{2.12}$$

Notice that ψ_M actually takes values in $\mathrm{Sym}(M) \otimes M$; it will sometimes be convenient to view ψ_M as a map $\mathrm{Sym}(M) \to \mathrm{Sym}(M) \otimes M$, but we will always use the same notation ψ_M, because there is no risk of confusion.

We denote by J_M the two-sided ideal of $\mathrm{Sym}(M)\#T(M)$, generated by all elements $1\#[x,y]_\otimes - \psi_M(\{x,y\})$, where x and y both run through M, and where $[x,y]_\otimes := x\otimes y - y\otimes x$. Let $\pi_M : \mathrm{Sym}(M)\#T(M) \to \mathrm{Sym}(M)\#T(M)/J_M$ denote the canonical surjection and let α and β denote the maps, defined by

$$\begin{array}{cccl} \alpha & : & \mathrm{Sym}(M) & \to & \mathrm{Sym}(M)\#T(M)/J_M \\ & & a & \mapsto & \pi_M(a\#1)\ , \end{array}$$

$$\begin{array}{cccl} \beta & : & \mathrm{Sym}(M) & \to & \mathrm{Sym}(M)\#T(M)/J_M \\ & & a & \mapsto & \pi_M(\psi_M(a))\ . \end{array}$$

Theorem 2.14. $(\mathrm{Sym}(M)\#T(M)/J_M, \alpha, \beta)$ *is a Poisson enveloping algebra of the polynomial Poisson algebra* $\mathrm{Sym}(M)$.

Proof. The verification of items (1) – (4) in Definition 2.5 is very similar to the verification in the proof of Theorem 2.10, so we skip it here. We prove the universal property of $\mathrm{Sym}(M)\#T(M)/J_M$. Let U' be any algebra and suppose that we are given any algebra (resp. Lie algebra) morphisms $\alpha' : \mathrm{Sym}(M) \to U'$ and $\beta' : \mathrm{Sym}(M) \to U'_L$, satisfying properties (3') and (4') (of Definition 2.5). We show that there is a unique algebra morphism $\gamma : \mathrm{Sym}(M)\#T(M)/J_M \to U'$ such that $\gamma \circ \alpha = \alpha'$ and $\gamma \circ \beta = \beta'$. As in the case of the proof of Theorem 2.10, the fact that every element of $\mathrm{Sym}(M)\#T(M)/J_M$ can be written as a finite sum, where every term is the product of an element of $\mathrm{Im}(\alpha)$ with elements of $\mathrm{Im}(\beta)$, implies the uniqueness of the morphism γ, if it exists; moreover, it leads to the following formula:

$$\gamma(\pi_M(a\#(x_1 \otimes x_2 \otimes \cdots \otimes x_k))) = \alpha'(a).\beta'(x_1).\beta'(x_2)\dots\beta'(x_k)\ .$$

We need to prove that γ is well-defined by this formula and that it is a morphism of algebras. To do this, we first define a map $\gamma' \colon \mathrm{Sym}(M)\#T(M) \to U'$ by setting

$$\gamma'(a\#(x_1 \otimes x_2 \otimes \cdots \otimes x_k)) := \alpha'(a).\beta'(x_1).\beta'(x_2)\ldots\beta'(x_k)\ .$$

The verification that γ' is a morphism of algebras is exactly the same as the verification given in the proof of Theorem 2.10 that γ is a morphism of algebras. It follows that, in order to show that γ is well-defined, it suffices to show that γ' vanishes on elements of the form $(1\#x) \odot (1\#y) - (1\#y) \odot (1\#x) - \psi_M(\{x, y\})$, where $x, y \in M$:

$$\begin{aligned}\gamma'(1\#\,[x,y]_\otimes - \psi_M(\{x,y\})) &= \beta'(x).\beta'(y) - \beta'(y).\beta'(x) - \gamma'(\psi_M(\{x,y\})) \\ &= \beta'(\{x,y\}) - \gamma'(\psi_M(\{x,y\}))\ .\end{aligned}$$

In order to show that the latter expression is zero, we show that $\beta'(\underline{x}) - \gamma'(\psi_M(\underline{x})) = 0$ for any monomial $\underline{x} \in M$. Since β' is an α'-derivation, we have

$$\beta'(\underline{x}) \;=\; \sum_i \alpha'(\check{\underline{x}}^i).\beta'(x_i) = \sum_i \gamma'(\check{\underline{x}}^i\#x_i) = \gamma'(\psi_M(\underline{x}))\ .$$

Since γ' vanishes on the ideal J_M there exists a unique algebra morphism γ, such that $\gamma \circ \pi_M = \gamma'$. In particular, γ is defined by the above formula and satisfies $\gamma \circ \alpha = \alpha'$ and $\gamma \circ \beta = \beta'$. □

2.3. **The Poisson enveloping algebra of a general Poisson algebra.** We consider in this subsection the construction of the Poisson enveloping algebra of a general Poisson algebra and derive from it the natural filtration of the Poisson enveloping algebra.

2.3.1. *The Poisson enveloping algebra of a general Poisson algebra as a quotient of a smash product algebra.* The idea of the construction that we give is due to Huebschmann [12], who gives an alternative construction to Rinehart's construction of the enveloping algebra of a Lie-Rinehart algebra in terms of Massey-Peterson algebras. We give the construction and only sketch the proof, because it is very similar to the proof of Theorem 2.14.

We start from the smash product algebra $\mathcal{A}\#U_{Lie}(\mathcal{A})$, which was constructed in Example 2.9. Let K denote the two-sided ideal of $\mathcal{A}\#U_{Lie}(\mathcal{A})$ generated by all elements[1] of the form $a\#b + b\#a - 1\#ab$ for $a, b \in \mathcal{A}$. Let π_K denote the canonical surjection $\pi_K : \mathcal{A}\#U_{Lie}(\mathcal{A}) \to \mathcal{A}\#U_{Lie}(\mathcal{A})/K$ and consider the maps α and β, defined by

$$\begin{aligned}\alpha \ : \ \mathcal{A} &\to \mathcal{A}\#U_{Lie}(\mathcal{A})/K \\ a &\mapsto \pi_K(a\#1)\ ,\end{aligned}$$

[1] To be precise, if we denote by $\imath$ the canonical injection of $\mathcal{A}$ in its Lie enveloping algebra and by $1_\mathcal{A}$ the unit of $\mathcal{A}$, then K is the ideal generated by all elements of the form $a\#\imath(b) + b\#\imath(a) - 1_\mathcal{A}\#\imath(ab)$, for $a, b \in \mathcal{A}$.

$$\begin{array}{rcl}\beta & : & \mathcal{A} \rightarrow \mathcal{A}\#U_{Lie}(\mathcal{A})/K \\ & & a \mapsto \pi_K(1\#a)\ .\end{array}$$

Theorem 2.15. *$(\mathcal{A}\#U_{Lie}(\mathcal{A})/K, \alpha, \beta)$ is a Poisson enveloping algebra of $\mathcal{A}$.*

Proof. The verification of properties (1) - (4) of Definition 2.5 is not quite the same as the proof of these properties in Theorem 2.10, but does not pose any real difficulty. For example, (4) is now a consequence of the definition of the ideal K; also, the verification of (2) is now even quicker, because it amounts to the equality of the right hand sides of the following two formulas, valid for $a_1, a_2 \in \mathcal{A}$:

$$\begin{aligned}\beta(\{a_1, a_2\}) &= \pi_K(1\#\{a_1, a_2\}) = \pi_K(1\#(a_1.a_2 - a_2.a_1))\ ,\\ [\beta(a_1), \beta(a_2)] &= \pi_K((1\#a_1) \odot (1\#a_2) - (1\#a_2) \odot (1\#a_1))\ .\end{aligned}$$

The proof that $\mathcal{A}\#U_{Lie}(\mathcal{A})/K$ satisfies the universal property is essentially the same as the proof which we gave of Theorem 2.10. □

2.3.2. *The filtration of the Poisson enveloping algebra.* One immediate consequence of the construction in the previous paragraph is that the Poisson enveloping algebra $U(\mathcal{A})$ of any Poisson algebra $\mathcal{A}$ is generated, as an R-algebra, by the images of the maps α and β. For $k \in \mathbb{N}$, we denote by $U_k(\mathcal{A})$ the $\mathcal{A}$-submodule of $U(\mathcal{A})$, generated by all products of at most k elements of $\beta(\mathcal{A})$.

Proposition 2.16. *Let $\mathcal{A}$ be any Poisson algebra. Its Poisson enveloping algebra is a filtered R-algebra, $U(\mathcal{A}) = \bigcup_{i\in\mathbb{N}} U_i(\mathcal{A})$, where the filtration is given by $\mathcal{A}$-submodules. Moreover, this filtration coincides with the filtration which is induced by the canonical filtration of $U_{Lie}(\mathcal{A})$ (taking on the first component $\mathcal{A}$ of $\mathcal{A}\#U_{Lie}(\mathcal{A})$ the trivial filtration).*

Proof. It follows from Theorem 2.15 that $U(\mathcal{A}) = \bigcup_{i\in\mathbb{N}} U_i(\mathcal{A})$. A key property is that the elements in the images of α and β commute, modulo elements in the image of α. Indeed, according to item (3) in Definition 2.5 we have that $[\alpha(a_1), \beta(a_2)] = \alpha(\{a_1, a_2\})$ for any $a_1, a_2 \in \mathcal{A}$. The property implies on the one hand that $U_k U_\ell \subset U_{k+\ell}$ for all $k, \ell \in \mathbb{N}$. On the other hand, it implies that $U_k(\mathcal{A}) = \pi_K(\mathcal{A} \otimes U_{Lie,k}(\mathcal{A}))$, since $U_{Lie,k}(\mathcal{A})$ is by definition the R-module generated by products of at most k elements of $\mathcal{A}$. □

Proposition 2.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be Poisson algebras with Poisson enveloping algebras $(U(\mathcal{A}), \alpha_\mathcal{A}, \beta_\mathcal{A})$ and $(U(\mathcal{B}), \alpha_\mathcal{B}, \beta_\mathcal{B})$. For every morphism of Poisson algebras $f : \mathcal{A} \to \mathcal{B}$, there exists a unique morphism of filtered algebras $U(f) : U(\mathcal{A}) \to U(\mathcal{B})$, making the following diagram commutative:*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha_{\mathcal{A}},\beta_{\mathcal{A}}} & U(\mathcal{A}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle U(f)} \\ \mathcal{B} & \xrightarrow{\alpha_{\mathcal{B}},\beta_{\mathcal{B}}} & U(\mathcal{B}) \end{array}$$

Proof. Uniqueness of the map $U(f)$ is clear, because $U(\mathcal{A})$ is generated by the images of $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$. For its construction, consider the maps $\alpha_{\mathcal{B}} \circ f$, $\beta_{\mathcal{B}} \circ f : \mathcal{A} \to U(\mathcal{B})$. They are algebra, resp. Lie algebra morphisms satisfying the conditions (3') and (4') in Definition 2.5 because $\alpha_{\mathcal{B}}$ and $\beta_{\mathcal{B}}$ have these properties. Thus, the universal property of $U(\mathcal{A})$ yields the unique algebra morphism $U(f)$ which completes the diagram into a commutative one. It is explicitly given by

$$U(f)(a\beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k)) = f(a)\beta_{\mathcal{B}}(f(a_1)).\beta_{\mathcal{B}}(f(a_2))\dots\beta_{\mathcal{B}}(f(a_k)),$$

where $a, a_1, a_2, \dots, a_k \in \mathcal{A}$. From this formula it is clear that $U(f)$ is filtered. □

The proposition implies that there is a covariant functor U between the category of Poisson algebras over R and the category of filtered algebras over R, which assigns to each Poisson algebra $\mathcal{A}$ the Poisson enveloping algebra $U(\mathcal{A})$, given by Theorem 2.15 and to each morphism of Poisson algebras $f : \mathcal{A} \to \mathcal{B}$ the induced morphism $U(f) : U(\mathcal{A}) \to U(\mathcal{B})$, given by Proposition 2.17.

2.4. **The Poisson enveloping algebra of a quotient of a general Poisson algebra.** Suppose that $\mathcal{A}$ is any Poisson algebra and that I is a Poisson ideal of $\mathcal{A}$. We give in this subsection a description of the Poisson enveloping algebra of the Poisson algebra $\mathcal{B} := \mathcal{A}/I$ in terms of the Poisson enveloping algebra $(U(\mathcal{A}), \alpha_{\mathcal{A}}, \beta_{\mathcal{A}})$ of $\mathcal{A}$. To do this, we first construct a new algebra out of $\mathcal{B}$ and $U(\mathcal{A})$. Using the canonical surjection $\pi : \mathcal{A} \to \mathcal{B}$ and the commutativity of $\mathcal{A}$ and $\mathcal{B}$, we make $\mathcal{B}$ into a symmetric $\mathcal{A}$-module by setting $b \cdot a = a \cdot b := \pi(a)b$, for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Consider the $\mathcal{B}$-module $\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})$. If we denote the unit of $\mathcal{B}$ by $1_{\mathcal{B}}$, then in $\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})$ we have the equality $\pi(a) \otimes u = 1_{\mathcal{B}} \otimes au = 1_{\mathcal{B}} \otimes \alpha_{\mathcal{A}}(a).u$, valid for all $a \in \mathcal{A}$ and $u \in U(\mathcal{A})$.

Proposition 2.18. *The $\mathcal{B}$-module $\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})$ is a unitary algebra over R with the product defined for $\pi(a_i) \otimes u_i \in \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})$, $i = 1, 2$, by*

$$(\pi(a_1) \otimes u_1) \cdot (\pi(a_2) \otimes u_2) := 1_{\mathcal{B}} \otimes (a_1 u_1).(a_2 u_2) = 1_{\mathcal{B}} \otimes \alpha_{\mathcal{A}}(a_1).u_1.\alpha_{\mathcal{A}}(a_2).u_2 \,. \tag{2.13}$$

Proof. If the product is well-defined then it is clear that it is associative and has $1_{\mathcal{B}} \otimes 1_{U(\mathcal{A})}$ as unit. To prove that the product is well-defined it is sufficient to show that if $\jmath \in I$ and $u, v \in U(\mathcal{A})$ then, in the $\mathcal{B}$-module $\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})$, one has $1_{\mathcal{B}} \otimes u.\alpha_{\mathcal{A}}(\jmath).v = 0$. In view of Proposition 2.16, it suffices to show that $1_{\mathcal{B}} \otimes u.\alpha_{\mathcal{A}}(\jmath).v = 0$ for all u of the form $\beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k)$, where $k \in \mathbb{N}$ and all a_i belong to $\mathcal{A}$. We do this by recursion on k.

Since $1_{\mathcal{B}} \otimes \alpha_{\mathcal{A}}(\jmath).v = 1_{\mathcal{B}} \otimes \jmath v = \pi(\jmath) \otimes v = 0$ for any $\jmath \in I$, the result is clair for $k = 0$. Let us assume it to be true up to order $k-1$. Using successively the relation $\beta_{\mathcal{A}}(a_i).\alpha_{\mathcal{A}}(\jmath) = \alpha_{\mathcal{A}}(\jmath).\beta_{\mathcal{A}}(a_i) + \alpha_{\mathcal{A}}(\{a_i, \jmath\})$ for $i = k, k-1, \ldots, 1$ allows us to permute $\alpha_{\mathcal{A}}(\jmath)$ with all $\beta_{\mathcal{A}}(a_i)$, because $\{a_i, \jmath\} \in I$ (recall that I is a Poisson ideal), and so each one of the correction terms $1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2) \ldots .\beta_{\mathcal{A}}(a_{i-1}).\alpha_{\mathcal{A}}(\{a_i, \jmath\}).u'.v$ is zero, in view of the recursion hypothesis. It follows that

$$1_{\mathcal{B}} \otimes u.\alpha_{\mathcal{A}}(\jmath).v = 1_{\mathcal{B}} \otimes \jmath u.v = \pi(\jmath) \otimes u.v = 0\ .$$

□

Let $I_{\mathcal{B}}$ denote the (two-sided) ideal of $\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})$, generated by $1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(I)$ and let $\pi_{\mathcal{B}} : \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A}) \to \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})/I_{\mathcal{B}}$ denote the canonical surjection. Consider the maps α and β, defined by

$$\begin{array}{rcrcl} \alpha & : & \mathcal{B} & \to & \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})/I_{\mathcal{B}} \\ & & \pi(a) & \mapsto & \pi_{\mathcal{B}}(1_{\mathcal{B}} \otimes \alpha_{\mathcal{A}}(a))\ , \end{array} \tag{2.14}$$

$$\begin{array}{rcrcl} \beta & : & \mathcal{B} & \to & \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})/I_{\mathcal{B}} \\ & & \pi(a) & \mapsto & \pi_{\mathcal{B}}(1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(a))\ . \end{array} \tag{2.15}$$

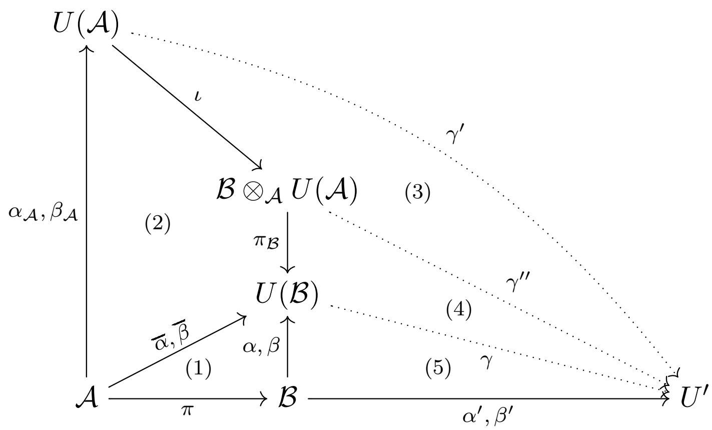
Notice that both maps all well-defined: the first one in view of the proof of proposition 2.18 and the second one in view of the definition of the ideal $I_{\mathcal{B}}$.

Theorem 2.19. *Let $\mathcal{A}$ be any Poisson algebra, I a Poisson ideal of $\mathcal{A}$, and $(U(\mathcal{A}), \alpha_{\mathcal{A}}, \beta_{\mathcal{A}})$ a Poisson enveloping algebra of $\mathcal{A}$. A Poisson enveloping algebra of the quotient Poisson algebra $\mathcal{B} = \mathcal{A}/I$ is $(\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})/I_{\mathcal{B}}, \alpha, \beta)$.*

Proof. The proof that α and β satisfy the properties (1) – (4) in Definition 2.5 is an immediate consequence of the fact that $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ satisfy these properties, in combination with the following three formulas, which are a direct consequence of definition (2.13): for any $a_1, a_2 \in \mathcal{A}$ and $u_1, u_2 \in U(\mathcal{A})$,

$$\begin{array}{rcl} (\pi(a_1) \otimes 1_{U(\mathcal{A})}) \cdot (\pi(a_2) \otimes 1_{U(\mathcal{A})}) & = & \pi(a_1 a_2) \otimes 1_{U(\mathcal{A})}\ , \\ (1_{\mathcal{B}} \otimes u_1) \cdot (1_{\mathcal{B}} \otimes u_2) & = & 1_{\mathcal{B}} \otimes u_1.u_2\ , \\ (\pi(a_1) \otimes 1_{U(\mathcal{A})}) \cdot (1_{\mathcal{B}} \otimes u_2) & = & \pi(a_1) \otimes u_2\ . \end{array}$$

In order to show that $U(\mathcal{B}) := \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})/I_{\mathcal{B}}$ satisfies the universal property, suppose that U' is any algebra and that $\alpha' : \mathcal{B} \to U', \beta' : \mathcal{B} \to U'_L$ are algebra (resp. Lie algebra) morphisms, satisfying (3') and (4') in Definition 2.5. We will prove that there exists a unique algebra morphism $\gamma : U(\mathcal{B}) \to U'$ such that $\gamma \circ \alpha = \alpha'$ and $\gamma \circ \beta = \beta'$. We do this by showing that the following diagram is a commutative diagram: the above relations which γ is ought to satisfy are equivalent to the commutativity of the triangle (5).



The morphisms $\alpha' \circ \pi$ and $\beta' \circ \pi$ are algebra (resp. Lie algebra) morphisms and satisfy the same properties (3') and (4') as α' and β', so by the universal property of $U(\mathcal{A})$ there exists a (unique) algebra morphism $\gamma' : U(\mathcal{A}) \to U'$ which makes the outer diagram commute. Consider the linear map $\iota : U(A) \to \mathcal{B} \otimes_{\mathcal{A}} U(A)$ defined for all $u \in U(A)$ by $\iota(u) = 1_{\mathcal{B}} \otimes u$. Proposition 2.18 shows that ι is an algebra morphism. Consequently, $\overline{\alpha} = \pi_{\mathcal{B}} \circ \iota \circ \alpha_{\mathcal{A}}$ and $\overline{\beta} = \pi_{\mathcal{B}} \circ \iota \circ \beta_{\mathcal{A}}$ are algebra (resp. Lie algebra) morphisms. The definition of α and β implies that the diagrams (1) and (2) commute.

Consider the linear map

$$\begin{array}{cccc} \gamma'' & : & \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A}) & \to & U' \\ & & b \otimes u & \mapsto & \alpha'(b).\gamma'(u) \ . \end{array}$$

For $a \in \mathcal{A}$ we have that

$$\begin{aligned} \gamma''((a \cdot b) \otimes u) &= \gamma''((b \cdot a) \otimes u) = \alpha'(b\pi(a)).\gamma'(u) \\ &= \alpha'(b).\alpha'(\pi(a)).\gamma'(u) = \alpha'(b).\gamma'(\alpha_{\mathcal{A}}(a)).\gamma'(u) \\ &= \gamma''(b \otimes (au)) \ , \end{aligned}$$

so that γ'' is well-defined. If we make U' into a $\mathcal{B}$-module upon using α', then γ'' can be described as the unique morphism of $\mathcal{B}$-modules, which sends $\iota(u) = 1_{\mathcal{B}} \otimes u$ to $\gamma'(u)$. Thus the diagram (3) commutes. Since γ' is an algebra morphism, it follows from this description that γ'' is also an algebra morphism. For $\jmath \in I$ we have

$$\gamma''(1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(\jmath)) = \gamma'(\beta_{\mathcal{A}}(\jmath)) = \beta'(\pi(\jmath)) = 0 \ ,$$

so that γ'' induces an algebra morphism $\gamma : \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})/I_{\mathcal{B}} \to U'$, such that the diagram (4) commutes. The commutativity of the diagrams (1) – (4) and of the outer diagram shows that $\gamma \circ \alpha \circ \pi = \alpha' \circ \pi$ and $\gamma \circ \beta \circ \pi = \beta' \circ \pi$. By surjectivity of π we conclude that the diagram (5) commutes. It remains to be shown that the morphism γ is unique. This follows from the fact that $\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})/I_{\mathcal{B}}$ is generated by the images of α and β, which is in turn a consequence of the fact that $U(\mathcal{A})$ is generated by the images of $\alpha_\mathcal{A}$ and $\beta_\mathcal{A}$. □

Remark 2.20. Let I_P denote the two-sided ideal of $U(\mathcal{A})$, generated by $\alpha_\mathcal{A}(I)$ and $\beta_\mathcal{A}(I)$. It can be shown as above that $(U(\mathcal{A})/I_P, \alpha, \beta)$ is a Poisson enveloping algebra for $\mathcal{B} = \mathcal{A}/I$, where α and β are defined as the unique morphisms which make the following diagram commutative:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\alpha_\mathcal{A},\,\beta_\mathcal{A}} & U(\mathcal{A}) \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \mathcal{B} & \xrightarrow{\alpha,\,\beta} & \frac{U(\mathcal{A})}{I_P} \end{array}$$

In this diagram, the vertical arrows are the canonical surjections.

3. The Poincaré-Birkhoff-Witt theorem

3.1. **The graded algebra associated with $U(\mathcal{A})$.** Let $\mathcal{A}$ be a Poisson algebra (over R) and let $(U(\mathcal{A}), \alpha, \beta)$ be its Poisson enveloping algebra. We recall from Section 2.3.2 that $U(\mathcal{A})$ has a canonical filtration,

$$U(\mathcal{A}) = \bigcup_{i\in\mathbb{N}} U_i(\mathcal{A}) ,$$

where $U_k(\mathcal{A})$ stands for the $\mathcal{A}$-submodule of $U(\mathcal{A})$, generated by all products of at most k elements of $\beta(\mathcal{A})$, where $k \in \mathbb{N}$. The graded algebra (over R) associated with the filtered algebra $U(\mathcal{A})$ is given by

$$\operatorname{gr}(U(\mathcal{A})) = \bigoplus_{i\in\mathbb{N}} \operatorname{gr}^i(U(\mathcal{A})), \quad \text{where} \quad \operatorname{gr}^k(U(\mathcal{A})) := \frac{U_k(\mathcal{A})}{U_{k-1}(\mathcal{A})} .$$

The homogeneous components $\operatorname{gr}^k(U(\mathcal{A}))$ are $\mathcal{A}$-modules, just like the $\mathcal{A}$-submodules $U_k(\mathcal{A})$ of $U(\mathcal{A})$ from which they are constructed. As in the case of Lie algebras, we have the following result:

Proposition 3.1. $\operatorname{gr}(U(\mathcal{A}))$ *is a commutative* $\mathcal{A}$*-algebra.*

Proof. In terms of the canonical surjections

$$\operatorname{gr}_k : U_k(\mathcal{A}) \to \frac{U_k(\mathcal{A})}{U_{k-1}(\mathcal{A})} , \tag{3.1}$$

the product on $\operatorname{gr}(U(\mathcal{A}))$ is given, for $\xi_k \in U_k(\mathcal{A})$ and $\xi_\ell \in U_\ell(\mathcal{A})$ by $\operatorname{gr}_k(\xi_k)\operatorname{gr}_\ell(\xi_\ell) := \operatorname{gr}_{k+\ell}(\xi_k.\xi_\ell)$. The fact that β is a Lie algebra morphism, item (3) in Definition 2.5 and the commutativity of $\mathcal{A}$ imply respectively that

$$\begin{aligned} [\operatorname{gr}_1(\beta(a_1)), \operatorname{gr}_1(\beta(a_2))] &= \operatorname{gr}_2([\beta(a_1), \beta(a_2)]) = \operatorname{gr}_2(\beta(\{a_1, a_2\})) = 0 , \\ [\operatorname{gr}_1(\alpha(a_1)), \operatorname{gr}_1(\beta(a_2))] &= \operatorname{gr}_2([\alpha(a_1), \beta(a_2)]) = \operatorname{gr}_2(\alpha(\{a_1, a_2\})) = 0 , \\ [\operatorname{gr}_1(\alpha(a_1)), \operatorname{gr}_1(\alpha(a_2))] &= 0 , \end{aligned}$$

for all $a_1, a_2 \in \mathcal{A}$. It follows that the product on $\mathrm{gr}(U(\mathcal{A}))$ is $\mathcal{A}$-bilinear and commutative. □

3.2. **The PBW map.** As in the previous section, let $\mathcal{A}$ be a Poisson algebra and $(U(\mathcal{A}), \alpha, \beta)$ its Poisson enveloping algebra. Recall that we denote by $\Omega(\mathcal{A})$ the $\mathcal{A}$-module of Kähler differentials of $\mathcal{A}$. Recall also that (4) in Definition 2.5 says that β is an α-derivation of $\mathcal{A}$ with values in $U(\mathcal{A})$. The universal property of $\Omega(\mathcal{A})$ leads to an $\mathcal{A}$-linear map, defined by

$$\begin{array}{rccc} \psi & : & \Omega(\mathcal{A}) & \to & U(\mathcal{A}) \\ & & \mathrm{d}a & \mapsto & \beta(a) \, . \end{array} \tag{3.2}$$

Notice that ψ actually takes values in $U_1(\mathcal{A})$. Let $\Psi : \Omega(\mathcal{A}) \to \mathrm{gr}(U(\mathcal{A}))$ be the induced map, which is a morphism of graded $\mathcal{A}$-modules with values in a commutative $\mathcal{A}$-algebra. By the universal property of the symmetric algebra $\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$ of $\Omega(\mathcal{A})$ we get a morphism of graded $\mathcal{A}$-algebras $\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) \to \mathrm{gr}(U(\mathcal{A}))$. It is called the *Poincaré-Birkhoff-Witt map*, or *PBW map* for short, and is explicitly given by

$$\begin{array}{rccc} \mathrm{PBW}_{\mathcal{A}} & : & \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) & \to & \mathrm{gr}(U(\mathcal{A})) \\ & & a\mathrm{d}a_1\mathrm{d}a_2\ldots\mathrm{d}a_k & \mapsto & \mathrm{gr}_k(a\beta(a_1).\beta(a_2)\ldots\beta(a_k)) \, , \end{array} \tag{3.3}$$

where $a, a_1, \ldots, a_k \in \mathcal{A}$. The latter image can also be written as the product $a\,\mathrm{gr}_1(\beta(a_1)).\,\mathrm{gr}_1(\beta(a_2))\ldots\mathrm{gr}_1(\beta(a_k))$. It is clear from (3.3) and Proposition 2.16 that the PBW map is surjective. Also, $\mathrm{PBW}_{\mathcal{A}}$ is a map of *graded* $\mathcal{A}$-algebras, because $\mathrm{gr}_1(\beta(a))$ is homogeneous of degree 1 in $\mathrm{gr}(U(\mathcal{A}))$ for any $a \in \mathcal{A}$.

Definition 3.2. A Poisson algebra $\mathcal{A}$ *satisfies the PBW theorem* if the graded map $\mathrm{PBW}_{\mathcal{A}} : \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) \to \mathrm{gr}(U(\mathcal{A}))$ is injective, hence is an isomorphism of graded $\mathcal{A}$-algebras.

At this moment we do not know of any Poisson algebra which does *not* satisfy the PBW theorem. We give a few examples here and elaborate on some other examples in the subsections that follow.

Example 3.3. Any smooth Poisson algebra $\mathcal{A}$ over a field $\mathbb{F}$ satisfies the PBW theorem: the pair $(\mathcal{A}, \Omega(\mathcal{A}))$ is a Lie-Rinehart algebra (see [12]) and Rinehart shows in [19] that the PBW theorem holds for Lie-Rinehart algebras (A, L) under the condition that L is a projective A-module; in our case, $\Omega(\mathcal{A})$ is a projective $\mathcal{A}$-module because $\mathcal{A}$ is assumed to be a smooth algebra over a field.

Example 3.4. Let $\mathcal{A}$ be any algebra which we make into a Poisson algebra by adding the zero Poisson bracket. We have shown in Proposition 2.7 that $\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$ is a Poisson enveloping algebra of $\mathcal{A}$. In this case, the enveloping algebra is already graded (with grading coming indeed from the canonical filtration of the Poisson enveloping algebra) and the map $\mathrm{PBW}_{\mathcal{A}}$ is just the identity map. In particular, $\mathcal{A}$ satisfies the PBW theorem. Notice that this example is not a particular case of the previous one: here $\mathcal{A}$ can be any algebra, smooth or singular.

3.3. **The PBW theorem for modified Lie-Poisson algebras.** We show in this subsection that the PBW theorem holds for any (modified) Lie-Poisson algebra when the base ring R is a field $\mathbb{F}$ (see Example 2.2). The result is not new, as such an algebra is obviously smooth, so it is covered by Example 3.3, but our proof which is specific to the Lie-Poisson case, has some extra flavors, such as being more direct, more explicit, and it prepares for the singular case, which we will study in the next subsection.

Theorem 3.5. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and let σ be a 2-cocycle in the trivial Lie algebra cohomology of $\mathfrak{g}$. The modified Lie-Poisson algebra $(\mathrm{Sym}(\mathfrak{g}), \{\cdot\,,\cdot\}_\sigma)$ satisfies the PBW theorem.*

Proof. In order to simplify the notation, we denote throughout this proof $\mathrm{Sym}(\mathfrak{g})$ by $\mathcal{A}$, adding a subscript σ to $\mathcal{A}$ when the Poisson structure $\{\cdot\,,\cdot\}_\sigma$ is relevant. We need to show that the PBW map

$$\mathrm{PBW}_{\mathcal{A}_\sigma} : \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) \to \mathrm{gr}(U(\mathcal{A}_\sigma))$$

is an isomorphism of graded $\mathcal{A}$-algebras. It follows from the following chain of isomorphisms of graded $\mathcal{A}$-algebras, each of which will be detailled below:

$$\begin{aligned}\mathrm{gr}(U(\mathcal{A}_\sigma)) &\overset{(1)}{=} \mathrm{gr}(\mathcal{A}\#U_{Lie}(\mathfrak{g})) \overset{(2)}{=} \mathrm{gr}(\mathcal{A}\otimes U_{Lie}(\mathfrak{g}))\\ &\overset{(3)}{\simeq} \mathrm{gr}(U_{Lie}(\mathcal{A}\otimes\mathfrak{g})) \overset{(4)}{\simeq} \mathrm{Sym}_{\mathcal{A}}(\mathcal{A}\otimes\mathfrak{g})\\ &\overset{(5)}{\simeq} \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))\ .\end{aligned}$$

We have shown in Section 2.2.2 that the Poisson enveloping algebra of $\mathcal{A}_\sigma$ is given by

$$U(\mathcal{A}_\sigma) = \mathrm{Sym}(\mathfrak{g})\#U_{Lie}(\mathfrak{g}) = \mathcal{A}\#U_{Lie}(\mathfrak{g})\ . \tag{3.4}$$

This leads to the proof of (1).

Recall that the smash product algebra $\mathcal{A}\#U_{Lie}(\mathfrak{g})$ is the tensor product $\mathcal{A}\otimes U_{Lie}(\mathfrak{g})$, with a special product (dictated by σ and the bracket of $\mathfrak{g}$). Also, the filtration of $\mathcal{A}\#U_{Lie}(\mathfrak{g})$ is induced by the filtration of $U_{Lie}(\mathfrak{g})$, just like the filtration on $\mathcal{A}\otimes U_{Lie}(\mathfrak{g})$. Therefore, the proof of (2) amounts to showing that the product of two elements in $\mathcal{A}\#U_{Lie}(\mathfrak{g})$ is their tensor product, modulo terms of lower degree. To do this, let $a_1, a_2 \in \mathcal{A}$ and let $x, x_1, \ldots, x_k \in \mathfrak{g}$. According to (2.10),

$$(a_1\#x)\odot(a_2\#x_1.x_2\ldots x_k) = a_1a_2\#x.x_1\ldots x_k + a_1\{x,a_2\}_\sigma\#x_1.x_2\ldots x_k\ , \tag{3.5}$$

where the first term is just the tensor product of the two arguments and the second term belongs to $(\mathcal{A}\otimes U_{Lie}(\mathfrak{g}))_k = \mathcal{A}\otimes U_{Lie,k}(\mathfrak{g})$. This shows the claim for $(a_1\#u)\odot(a_2\#v)$ with $u \in \mathfrak{g} \subset U_{Lie}(\mathfrak{g})$; (2) follows from it by writing a general homogeneous element $a\#u$ as the product of elements of the form $a_1\#x$ with $a_1 \in \mathcal{A}$ and $x \in \mathfrak{g}$ and repeatedly using (3.5).

In order to prove (3), first notice that $\mathcal{A} \otimes \mathfrak{g}$ is a Lie algebra over $\mathcal{A}$ by *extension of scalars* (see [3]), namely a Lie bracket on $\mathcal{A} \otimes \mathfrak{g}$ is given for $a_1, a_2 \in \mathcal{A}$ and $x_1, x_2 \in \mathfrak{g}$ by $[a_1 \otimes x_1, a_2 \otimes x_2] := a_1 a_2 \otimes [x_1, x_2]$. The natural inclusion $\mathfrak{g} \to U_{Lie}(\mathfrak{g})$ leads to an inclusion $\mathcal{A} \otimes \mathfrak{g} \to \mathcal{A} \otimes U_{Lie}(\mathfrak{g})$, hence (by the universal property of the enveloping algebra) to an algebra morphism $U_{Lie}(\mathcal{A} \otimes \mathfrak{g}) \to \mathcal{A} \otimes U_{Lie}(\mathfrak{g})$. Its inverse is the unique morphism of algebras which sends $1_{\mathcal{A}} \otimes x \in \mathcal{A} \otimes U_{Lie}(\mathfrak{g})$ to $1_{\mathcal{A}} \otimes x \in U_{Lie}(\mathcal{A} \otimes \mathfrak{g})$. Since the isomorphism respects the (natural) filtrations on $U_{Lie}(\mathcal{A} \otimes \mathfrak{g})$ and $\mathcal{A} \otimes U_{Lie}(\mathfrak{g})$, this shows (3).

The isomorphism (4) is just the classical PBW theorem for the $\mathcal{A}$-Lie algebra $\mathcal{A} \otimes \mathfrak{g}$: the PBW theorem holds for all Lie algebras over $\mathcal{A}$ because $\mathcal{A}$ contains a field (the field $\mathbb{F}$) (see [6]). Finally, for any $\mathbb{F}$-vector space V, the module $\Omega(\mathrm{Sym}(V))$ is a free $\mathrm{Sym}(V)$-module, to wit, $\Omega(\mathrm{Sym}(V)) \simeq \mathrm{Sym}(V) \otimes V$ (see [9, Ch. 16]). Applied to $V = \mathfrak{g}$ we get (5). □

For simplicity, we have assumed in Theorem 3.5 that $\mathfrak{g}$ is a Lie algebra over a field. The assumption was used to assert that the $\mathcal{A}$-Lie algebra $\mathcal{A} \otimes \mathfrak{g}$ satisfies the PBW theorem (for Lie algebras!): the above proof works under the latter, more general, assumption.

Remark 3.6. As we have seen in Section 2.2.3, the Poisson enveloping algebra of $(\mathrm{Sym}(\mathfrak{g}), \{\cdot\,,\cdot\}_\sigma)$ can also be described as a (modified) Lie enveloping algebra and one might be tempted to use the PBW theorem for (modified) Lie algebras to show that $(\mathrm{Sym}(\mathfrak{g}), \{\cdot\,,\cdot\}_\sigma)$ satisfies the PBW theorem. However, as we pointed out in Remark 2.12, the filtration of $U_{Lie,\sigma^+}(\mathfrak{g}^+)$ is different whether we consider it as a Lie enveloping algebra or as a Poisson enveloping algebra. This means that the associated graded algebras and the associated PBW maps are different, and so the classical PBW theorem cannot be applied directly to give a quick proof of Theorem 3.5.

3.4. **The PBW theorem for some singular Poisson algebras.** The purpose of this subsection is to show that if I is a Poisson ideal of a smooth Poisson algebra $\mathcal{A}$, which is generated (as an ideal) by a single element and such that $\mathcal{A}/I$ is an integral domain, then the Poisson algebra $\mathcal{B} := \mathcal{A}/I$ satisfies the PBW theorem. We denote as before by $\pi : \mathcal{A} \to \mathcal{B} = \mathcal{A}/I$ the canonical surjection and we write $U(\mathcal{A})$ for the Poisson enveloping algebra of $\mathcal{A}$, with accompanying maps denoted by $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$. We recall from Section 2.4 that $(\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})/I_{\mathcal{B}}, \alpha_{\mathcal{B}}, \beta_{\mathcal{B}})$ is a Poisson enveloping algebra of $\mathcal{B}$, where the product on $\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})$ is given by (2.13), the ideal $I_{\mathcal{B}}$ is generated by $1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(I)$, and the morphisms $\alpha_{\mathcal{B}}$ and $\beta_{\mathcal{B}}$ are given by (2.14) and (2.15).

Theorem 3.7. *The Poisson algebra $\mathcal{B} = \mathcal{A}/I$ satisfies the PBW theorem.*

Proof. We first outline the proof. Consider the following diagram of graded $\mathcal{B}$-algebras:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{gr}(I_{\mathcal{B}}) & \xrightarrow{\imath_U} & \mathcal{B}\otimes_{\mathcal{A}} \mathrm{gr}(U(\mathcal{A})) & \xrightarrow{\pi_U} & \mathrm{gr}(U(\mathcal{B})) & \longrightarrow & 0\\
 & & \theta\Big\uparrow & & \simeq\Big\uparrow \mathrm{Id}_{\mathcal{B}}\otimes \mathrm{PBW}_{\mathcal{A}} & & \Big\uparrow \mathrm{PBW}_{\mathcal{B}} & & \\
0 & \longrightarrow & (1_{\mathcal{B}}\otimes \mathrm{d}I) & \xrightarrow[\imath_S]{} & \mathcal{B}\otimes_{\mathcal{A}} \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) & \xrightarrow[\pi_S]{} & \mathrm{Sym}_{\mathcal{B}}(\Omega(\mathcal{B})) & \longrightarrow & 0
\end{array}$$

The construction of the different arrows will be discussed below and we will show that the diagram has exact rows and is commutative. We need to show that the rightmost arrow $\mathrm{PBW}_{\mathcal{B}}$ is injective. The middle arrow is an isomorphism (because $\mathcal{A}$ is smooth, so the PBW theorem holds for $\mathcal{A}$). We will show that the leftmost arrow θ is surjective. By a simple diagram chase, this implies that $\mathrm{PBW}_{\mathcal{B}}$ is injective: if $Z \in \mathrm{Sym}_{\mathcal{B}}(\Omega(\mathcal{B}))$ is in the kernel of $\mathrm{PBW}_{\mathcal{B}}$ then there exist elements $Y \in \mathcal{B}\otimes_{\mathcal{A}} \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$ and $X' \in \mathrm{gr}(I_{\mathcal{B}})$, such that $\pi_S(Y) = Z$ and $\mathrm{Id}\otimes \mathrm{PBW}_{\mathcal{A}}(Y) = \imath_U(X')$. By surjectivity of θ, there exists $X \in (1_{\mathcal{B}}\otimes \mathrm{d}I)$ such that $\theta(X) = X'$. By the commutativity and exactness properties of the diagram, we can conclude that $Z = \pi_S(Y) = \pi_S(\imath_S(X)) = 0$, as was to be shown.

We now get to the details of the proof.

Step 1: *Exactness of the bottom line.* The conormal sequence for Kähler differentials (see [9, Proposition 16.3]), applied to the canonical surjection $\pi : \mathcal{A} \to \mathcal{B}$, is the exact sequence of $\mathcal{B}$-modules, given by

$$I/I^2 \to \mathcal{B}\otimes_{\mathcal{A}} \Omega(\mathcal{A}) \to \Omega(\mathcal{B}) \to 0\ ,$$

where the first map sends $j \mod I^2 \in I/I^2$ to $\mathrm{d}j$ and the second map sends $\pi(a_1)\otimes a_2\mathrm{d}a_3$ to $\pi(a_1a_2)\mathrm{d}\pi(a_3)$. Since the image of the first map is the $\mathcal{B}$-submodule $\langle 1_{\mathcal{B}}\otimes \mathrm{d}I\rangle$ of $\mathcal{B}\otimes_{\mathcal{A}}\Omega(\mathcal{A})$, generated by $1_{\mathcal{B}}\otimes \mathrm{d}I$, we have the following short exact sequence of $\mathcal{B}$-modules:

$$0 \to \langle 1_{\mathcal{B}}\otimes \mathrm{d}I\rangle \to \mathcal{B}\otimes_{\mathcal{A}} \Omega(\mathcal{A}) \to \Omega(\mathcal{B}) \to 0\ .$$

Applying the Sym functor, we get according to [2, ¶6.2, Proposition 4] the following short exact sequence

$$0 \to (1_{\mathcal{B}}\otimes \mathrm{d}I) \to \mathrm{Sym}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{A}} \Omega(\mathcal{A})) \to \mathrm{Sym}_{\mathcal{B}}\,\Omega(\mathcal{B}) \to 0\ , \tag{3.6}$$

where we recall that $(1_{\mathcal{B}}\otimes \mathrm{d}I)$ stands for the two-sided ideal (in this case of the $\mathcal{B}$-algebra $\mathrm{Sym}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{A}}\Omega(\mathcal{A}))$), generated by $1_{\mathcal{B}}\otimes \mathrm{d}I$, so it is a homogeneous ideal (generated by elements of degree 1). By extension of the ring of scalars (see [2, ¶6.4, Proposition 7]) we have the following isomorphism of $\mathcal{B}-$modules:

$$\mathrm{Sym}_{\mathcal{B}}(\mathcal{B}\otimes_{\mathcal{A}} \Omega(\mathcal{A})) \simeq \mathcal{B}\otimes_{\mathcal{A}} \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))\ .$$

Substitued in (3.6) we get the desired exactness of the bottom line. For future reference, note that the surjection

$$\pi_S : \mathcal{B}\otimes_{\mathcal{A}} \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) \to \mathrm{Sym}_{\mathcal{B}}(\Omega(\mathcal{B}))$$

is explicitly given by

$$\pi_S(b \otimes a\ \mathrm{d}a_1 \mathrm{d}a_2 \dots \mathrm{d}a_k) = \pi(a)b\ \mathrm{d}\pi(a_1)\mathrm{d}\pi(a_2)\dots \mathrm{d}\pi(a_k)\ ,$$

where $a, a_1, \dots, a_k \in \mathcal{A}$ and $b \in \mathcal{B}$.

Step 2: *Exactness of the top line.* Theorem 2.19 shows exactness of the following sequence of R-algebras and $\mathcal{B}$-modules:

$$0 \longrightarrow I_{\mathcal{B}} \longrightarrow \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A}) \xrightarrow{\pi_{\mathcal{B}}} U(\mathcal{B}) \longrightarrow 0\ .$$

The filtration $U(\mathcal{A}) = \bigcup_{i\in\mathbb{N}} U_i(\mathcal{A})$ by $\mathcal{A}$-modules induces a filtration $\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A}) = \bigcup_{i\in\mathbb{N}} \mathcal{B} \otimes_{\mathcal{A}} U_i(\mathcal{A})$ by $\mathcal{B}$-modules. On $I_{\mathcal{B}}$ we take the induced filtration, i.e., $I_{\mathcal{B},k} = I_{\mathcal{B}} \cap (\mathcal{B} \otimes_{\mathcal{A}} U_k(\mathcal{A}))$, for all k; also, the quotient filtration on $U(\mathcal{B})$ is the canonical filtration of $U(\mathcal{B})$ as a Poisson enveloping algebra, $\pi_{\mathcal{B}}(\mathcal{B} \otimes_{\mathcal{A}} U_k(\mathcal{A})) = U_k(\mathcal{B})$, for all k. Therefore, taking the induced morphism on the graded modules and algebras, we get the exact sequence,

$$0 \longrightarrow \mathrm{gr}(I_{\mathcal{B}}) \longrightarrow \mathrm{gr}(\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})) \xrightarrow{\mathrm{gr}(\pi_{\mathcal{B}})} \mathrm{gr}(U(\mathcal{B})) \longrightarrow 0\ .$$

Finally the graded $\mathcal{B}$-algebra $\mathrm{gr}(\mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A}))$ is naturally isomorphic to $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{gr}(U(\mathcal{A}))$. Therefore we get the exact sequence of the top line of the above diagram.

Step 3: *Commutativity of the diagram.* The commuting diagrams (1) and (2) of Theorem 2.19 show that the map $\pi_{\mathcal{B}} \circ \iota : U(\mathcal{A}) \longrightarrow U(\mathcal{B})$ satisfies the universal property of Proposition 2.17. Uniqueness of the morphism $U(\pi)$ leads to the equality $U(\pi) = \pi_{\mathcal{B}} \circ \iota$. According to the definition of $U(\pi)$,

$$U(\pi)(a\beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k)) = \pi(a)\beta_{\mathcal{B}}(\pi(a_1)).\beta_{\mathcal{B}}(\pi(a_2))\dots\beta_{\mathcal{B}}(\pi(a_k))\ ,$$

while

$$\begin{aligned} \pi_{\mathcal{B}} \circ \iota(a\beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k)) &= \pi_{\mathcal{B}}(1_{\mathcal{B}} \otimes a\beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k)) \\ &= \pi_{\mathcal{B}}(\pi(a) \otimes \beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k)). \end{aligned}$$

From the equality $U(\pi) = \pi_{\mathcal{B}} \circ \iota$, we conclude that

$$\pi_{\mathcal{B}}(\pi(a)\otimes\beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k)) = \pi(a)\beta_{\mathcal{B}}(\pi(a_1)).\beta_{\mathcal{B}}(\pi(a_2))\dots\beta_{\mathcal{B}}(\pi(a_k))\ .$$

Let us denote $\mathrm{gr}'_k : U_k(B) \longrightarrow U_k(B)/U_{k-1}(B)$. For $b \in B$ and $u \in U_k(A)$, the value in $b \otimes \mathrm{gr}_k(u)$ of the map $\pi_U := \mathrm{gr}(\pi_{\mathcal{B}})$ is

$$\pi_U(b \otimes \mathrm{gr}_k(u)) = \mathrm{gr}'_k(\pi_B(b \otimes u))\ .$$

It follows that, for $Y := b\ \otimes\ a\,\mathrm{d}a_1\mathrm{d}a_2\dots\mathrm{d}a_k \in \mathcal{B} \otimes_{\mathcal{A}} \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$,

$$\begin{aligned} \pi_U(\mathrm{Id}_{\mathcal{B}} \otimes \mathrm{PBW}_{\mathcal{A}}(Y)) &= \pi_U(b \otimes a\mathrm{gr}_k(\beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k))) \\ &= \mathrm{gr}'_k(\pi_{\mathcal{B}}(b \otimes a\beta_{\mathcal{A}}(a_1).\beta_{\mathcal{A}}(a_2)\dots\beta_{\mathcal{A}}(a_k))) \\ &= \mathrm{gr}'_k(b\pi(a)\beta_{\mathcal{B}}(\pi(a_1)).\beta_{\mathcal{B}}(\pi(a_2))\dots\beta_{\mathcal{B}}(\pi(a_k))) \\ &= \mathrm{PBW}_{\mathcal{B}}(b\pi(a)\mathrm{d}\pi(a_1).\mathrm{d}\pi(a_2)\dots\mathrm{d}\pi(a_k)) \\ &= \mathrm{PBW}_{\mathcal{B}}(\pi_S(Y))\ . \end{aligned}$$

This proves the commutativity of the rightmost square. As for the leftmost square, we define θ by using the restriction of the isomorphism $\mathrm{Id}_\mathcal{B}\otimes\mathrm{PBW}_\mathcal{A}$ to the ideal $(1_\mathcal{B}\otimes \mathrm{d}I)$: by commutativity of the rightmost square the morphism $\mathrm{Id}_\mathcal{B}\otimes\mathrm{PBW}_\mathcal{A}$ sends $(1_\mathcal{B}\otimes\mathrm{d}I)$ in the image of $\imath_U$. Consider the canonical surjection $\mathrm{gr}''_k : I_\mathcal{B}\cap(\mathcal{B}\otimes U_k(\mathcal{A}))\to I_\mathcal{B}\cap(\mathcal{B}\otimes U_k(\mathcal{A}))/I_\mathcal{B}\cap(\mathcal{B}\otimes U_{k-1}(\mathcal{A}))$. The graded morphism θ is given by

$$\theta(b\otimes a\ \mathrm{d}a_1\mathrm{d}a_2\dots\mathrm{d}a_k)=\mathrm{gr}''_k(b\otimes a\ \beta_\mathcal{A}(a_1)\beta_\mathcal{A}(a_2)\dots\beta_\mathcal{A}(a_k))\ ,$$

where $b\in\mathcal{B}$ and $a,a_1,\dots,a_k\in\mathcal{A}$ are such that $a_i\in I$ for at least one index i. By construction, the leftmost square is commutative.

Step 4: *$I_\mathcal{B}$ as a left ideal.* We claim that $I_\mathcal{B}$, which was defined as the ideal of $\mathcal{B}\otimes_\mathcal{A}U(\mathcal{A})$ generated by $1_\mathcal{B}\otimes\beta_\mathcal{A}(I)$ coincides with $I^L_\mathcal{B}$, the *left* ideal of $\mathcal{B}\otimes_\mathcal{A}U(\mathcal{A})$ generated by $1_\mathcal{B}\otimes\beta_\mathcal{A}(I)$. This property will be useful in the next step when we prove that θ is surjective. Since $U(\mathcal{A})$ is generated by the images of $\alpha_\mathcal{A}$ and $\beta_\mathcal{A}$ it suffices to show that $I^L_\mathcal{B}$ is stable for multiplication on the right by $1_\mathcal{B}\otimes\alpha_\mathcal{A}(a)$ and by $1_\mathcal{B}\otimes\beta_\mathcal{A}(a)$, for all $a\in\mathcal{A}$. For $\jmath\in I$ we have

$$\begin{aligned}(1_\mathcal{B}\otimes\beta_\mathcal{A}(\jmath))\cdot(1_\mathcal{B}\otimes\alpha_\mathcal{A}(a))&=1_\mathcal{B}\otimes\beta_\mathcal{A}(\jmath).\alpha_\mathcal{A}(a)\\&=1_\mathcal{B}\otimes\alpha_\mathcal{A}(a).\beta_\mathcal{A}(\jmath)+1_\mathcal{B}\otimes\alpha_\mathcal{A}(\{j,a\})\\&=(1_\mathcal{B}\otimes\alpha_\mathcal{A}(a))\cdot(1_\mathcal{B}\otimes\beta_\mathcal{A}(\jmath))+\pi(\alpha_\mathcal{A}(\{j,a\})\otimes 1_{U_\mathcal{A}}\\&=(1_\mathcal{B}\otimes\alpha_\mathcal{A}(a))\cdot(1_\mathcal{B}\otimes\beta_\mathcal{A}(\jmath))\ ,\end{aligned}$$

which belongs to $I^L_\mathcal{B}$; we have used in the last step that $\{\jmath,a\}\in I$ (because I is a Poisson ideal). Similarly,

$$(1_\mathcal{B}\otimes\beta_\mathcal{A}(\jmath))\cdot(1_\mathcal{B}\otimes\beta_\mathcal{A}(a))=1_\mathcal{B}\otimes\beta_\mathcal{A}\{\jmath,a\}+(1_\mathcal{B}\otimes\beta_\mathcal{A}(a))\cdot(1_\mathcal{B}\otimes\beta_\mathcal{A}(\jmath))\in I^L_\mathcal{B}\ .$$

This proves our claim.

Step 5: *$\mathcal{B}\otimes_\mathcal{A}\mathrm{gr}(U(\mathcal{A}))$ has no non-trivial zero divisors.* Every $\mathcal{B}$-module becomes an $\mathcal{A}$-module upon using $\pi:\mathcal{A}\to\mathcal{B}$ and similarly for every $\mathcal{B}$-module morphism. Since $\mathcal{A}$ is a smooth algebra, $\Omega(\mathcal{A})$ is a projective $\mathcal{A}$-module. It follows easily that $\mathcal{B}\otimes_\mathcal{A}\Omega(\mathcal{A})$ is a projective $\mathcal{B}$-module. By definition, there exist $\mathcal{B}$-modules L and N, with L free, such that $L\simeq(\mathcal{B}\otimes_\mathcal{A}\Omega(\mathcal{A}))\oplus N$. Since L is free and since $\mathcal{B}$ has no non-trivial zero divisors, $\mathrm{Sym}_\mathcal{B}L$ also has no non-trivial zero divisors. But $\mathrm{Sym}_\mathcal{B}(\mathcal{B}\otimes_\mathcal{A}\Omega(\mathcal{A}))$ is isomorphic to a subalgebra of $\mathrm{Sym}_\mathcal{B}L$, hence also has no non-trivial zero divisors. We can now conclude in view of the isomorphisms (of $\mathcal{B}$-modules)

$$\mathrm{Sym}_\mathcal{B}(\mathcal{B}\otimes_\mathcal{A}\Omega(\mathcal{A}))\simeq\mathcal{B}\otimes_\mathcal{A}\mathrm{Sym}_\mathcal{A}\Omega(\mathcal{A})\simeq\mathcal{B}\otimes_\mathcal{A}\mathrm{gr}(U(\mathcal{A}))\ .$$

Step 6: *Surjectivity of θ.* For this step (only), we use our assumption that I is generated (as an ideal) by a single element, say $I=(\jmath)$, with $\jmath\in\mathcal{A}$. Let $\mathrm{gr}''_k(X')\in I_\mathcal{B}\cap(\mathcal{B}\otimes_\mathcal{A}U_k(\mathcal{A}))/I_\mathcal{B}\cap(\mathcal{B}\otimes U_{k-1}(\mathcal{A}))$. If $X'\in I_\mathcal{B}\cap(\mathcal{B}\otimes U_\ell(\mathcal{A}))$ with $\ell<k$, then $\mathrm{gr}''_k(X')=0=\theta(0)$. Thus we can suppose that $X'\in\mathcal{B}\otimes_\mathcal{A}U_k(\mathcal{A})$ and $X'\notin\mathcal{B}\otimes_\mathcal{A}U_{k-1}(\mathcal{A})$. We show that $\mathrm{gr}''_k(X')=\theta(X)$ for some $X\in(1_\mathcal{B}\otimes\mathrm{d}I)$ of degree k. Since $I_\mathcal{B}$ is the left ideal generated

by $\jmath$ (see Step 4), we can write X' as $X' = Y' \cdot (1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(\jmath))$, where $Y' \in \mathcal{B} \otimes_{\mathcal{A}} U(\mathcal{A})$. Since $\mathcal{B} \otimes_{\mathcal{A}} \mathrm{gr}(U(\mathcal{A}))$ has no non-trivial zero divisors (Step 5), $Y' \in \mathcal{B} \otimes_{\mathcal{A}} U_{k-1}(\mathcal{A})$ and $Y' \notin \mathcal{B} \otimes_{\mathcal{A}} U_{k-2}(\mathcal{A})$. Therefore we can write Y' as $Y' = Y_1' + Y_2'$ where $Y_1' \in \mathcal{B} \otimes_{\mathcal{A}} U_{k-1}(\mathcal{A})$ is of the form $Y_1' = \sum_i b_i \otimes a_i \beta_{\mathcal{A}}(a_{1,i}).\beta_{\mathcal{A}}(a_{2,i}) \dots \beta_{\mathcal{A}}(a_{k-1,i})$ and $Y_2' \in \mathcal{B} \otimes_{\mathcal{A}} U_{k-2}(\mathcal{A})$. Then

$$\begin{aligned} \mathrm{gr}_k''(X') &= \mathrm{gr}_k''(Y' \cdot (1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(\jmath))) = \mathrm{gr}_{k-1}''(Y')\,\mathrm{gr}_1''(1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(\jmath)) \\ &= \mathrm{gr}_{k-1}''(Y_1')\,\mathrm{gr}_1''(1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(\jmath)) \ . \end{aligned}$$

Since $U(\mathcal{A})$ satisfies the PBW theorem, $Y_1' = \mathrm{Id}_{\mathcal{B}} \otimes \mathrm{PBW}_{\mathcal{A}}(Y)$ for some homogeneous element $Y \in \mathcal{B} \otimes_{\mathcal{A}} \mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$ of degree $k-1$. It follows that $\theta_k(Y \cdot (1_{\mathcal{B}} \otimes \mathrm{d}\jmath)) = \mathrm{gr}_k''(Y' \cdot (1_{\mathcal{B}} \otimes \beta_{\mathcal{A}}(\jmath)))$, so we can choose $X = Y \cdot (1_{\mathcal{B}} \otimes \mathrm{d}\jmath)$ to obtain that $\mathrm{gr}_k''(X') = \theta(X)$ with X of degree k. □

In geometrical terms, the theorem is valid for arbitrary Poisson hypersurfaces of any smooth affine Poisson variety. The example which follows is of this form.

Example 3.8. As we have seen in Example 2.3, any pair of complex polynomials (P, Q) defines a Poisson structure on $\mathcal{A} := \mathbb{C}[X_1, X_2, X_3]$ by setting

$$\{X_1, X_2\} := Q\frac{\partial P}{\partial X_3} \ , \quad \{X_2, X_3\} := Q\frac{\partial P}{\partial X_1} \ , \quad \{X_3, X_1\} := Q\frac{\partial P}{\partial X_2} \ . \quad (3.7)$$

Since P is a Casimir function of this Poisson structure, (P) is a Poisson ideal of $\mathcal{A}$ and $\mathcal{B} := \mathbb{C}[X_1, X_2, X_3]/(P)$ is the algebra of functions of a Poisson surface, which may be a singular surface (for example when P is homogeneous of degree at least two). If P is irreducible, so that $\mathcal{B}$ is an integral domain, then according to the above theorem, $\mathcal{B}$ satisfies the PBW theorem. This example covers many classical singular surfaces, such as the well-known Klein surfaces (see [1]).

We also give an example of a Poisson algebra which does not satisfy the conditions of Theorem 3.7, but yet the proof of this theorem can be used, with minor modifications, to show that it satisfies the PBW theorem.

Example 3.9. We pick up again the previous example, but we take now for P the reducible polynomial $P := X_1X_2X_3$. In this case, $\mathcal{B} := \mathbb{C}[X_1, X_2, X_3]/(P)$ is the algebra of functions of a singular Poisson surface, which is the union of the three coordinate planes in $\mathbb{C}^3$. Step 5 in the proof of Theorem 3.7 is not valid anymore, because $\mathcal{B}$ now has non-trivial zero divisors. However, a close inspection of Step 6 in the proof reveils that we only need to prove that $1_{\mathcal{B}} \otimes \mathrm{d}\jmath \in \mathcal{B} \otimes_{\mathcal{A}} \mathrm{Sym}_{\mathcal{A}}\,\Omega(\mathcal{A}) \simeq \mathcal{B}[Y_1, Y_2, Y_3]$ is not a zero divisor, i.e., that

$$\pi(X_2X_3)Y_1 + \pi(X_1X_3)Y_2 + \pi(X_1X_2)Y_3$$

is not a zero divisor of $\mathcal{B}[Y_1, Y_2, Y_3]$, where we recall that $\pi : \mathcal{A} \to \mathcal{B}$ denotes the canonical surjection. Thus, we need to show that if F is a polynomial in Y_1, Y_2, Y_3 with coefficients in $\mathcal{B}$ and

$$F(\pi(X_2X_3)Y_1 + \pi(X_1X_3)Y_2 + \pi(X_1X_2)Y_3) = 0 \ , \tag{3.8}$$

then $F = 0$. To do this, we may assume that F is a homogenous polynomial of degree n in the variables Y_i, say $F = \sum_{i+j+k=n} \pi(F_{i,j,k})Y_1^iY_2^jY_3^k$. If (3.8) holds, then for all $n \geqslant 0$,

$$\sum_{i+j+k=n+1} \pi(F_{i-1,j,k}X_2X_3 + F_{i,j-1,k}X_1X_3 + F_{i,j,k-1}X_1X_2)Y_1^iY_2^jY_3^k = 0 \ ,$$

hence for all $i, j, k \in \mathbb{N}$, the polynomial $X_1X_2X_3$ divides $F_{i-1,j,k}X_2X_3 + F_{i,j-1,k}X_1X_3 + F_{i,j,k-1}X_1X_2$. Since $F_{i,j,k} = 0$ whenever one of the indices i, j, k is negative, this means that X_1 divides every $F_{i,j,k}$. By symmetry, every $F_{i,j,k}$ is also divisible by X_2 and by X_3. It follows that $\pi(F_{i,j,k}) = 0$ for all $i, j, k \geqslant 0$, as was to be shown. We may now conclude, in view of the proof of Theorem 3.7 that $\mathcal{B}$ satisfies the PBW theorem.

3.5. **The symmetrization map.** We now show that if a Poisson algebra $\mathcal{A}$ satisfies the PBW theorem, then the PBW map $\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) \to \mathrm{gr}(U(\mathcal{A}))$, which is a isomorphism of $\mathcal{A}$-algebras can be lifted to an isomorphism of $\mathcal{A}$-modules $\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A})) \to U(\mathcal{A})$. To do this, we need to assume that we can divide elements of our base ring R by arbitrary integers, so we will assume in this paragraph that R contains the field $\mathbb{Q}$ of rational numbers.

First, the following diagram is a commutative diagram of $\mathcal{A}$-modules:

$$\begin{array}{ccc} T^k_{\mathcal{A}}(\Omega(\mathcal{A})) & \xrightarrow{\ \psi_k\ } & U_k(\mathcal{A}) \\ \big\downarrow{\scriptstyle \tau_k} & & \big\downarrow{\scriptstyle \mathrm{gr}_k} \\ \mathrm{Sym}^k_{\mathcal{A}}(\Omega(\mathcal{A})) & \xrightarrow[\mathrm{PBW}^k_{\mathcal{A}}]{} & \mathrm{gr}^k(U(\mathcal{A})) \end{array}$$

In this diagram, $\mathrm{PBW}^k_{\mathcal{A}}$ is the restriction of $\mathrm{PBW}_{\mathcal{A}}$ to $\mathrm{Sym}^k_{\mathcal{A}}(\Omega(\mathcal{A}))$, the homogenous elements of degree k of $\mathrm{Sym}_{\mathcal{A}}(\Omega(\mathcal{A}))$. The morphism τ_k is the canonical surjection and gr_k is the morphism which was introduced in (3.1). Finally, ψ_k is the extension of the map ψ, defined in (3.2) to the degree k component of the tensor algebra $T_{\mathcal{A}}(\Omega(\mathcal{A}))$, to wit,

$$\psi(a_0\mathrm{d}a_1 \otimes \cdots \otimes \mathrm{d}a_k) = \alpha_{\mathcal{A}}(a_0).\beta_{\mathcal{A}}(a_1)\ldots\beta_{\mathcal{A}}(a_k),$$

for all $a_0, \ldots, a_k \in \mathcal{A}$. Comparing this formula with (3.3), the commutativity of the diagram is obvious.

Let us denote by $T'^k_{\mathcal{A}}(\Omega(\mathcal{A})) \subset T^k_{\mathcal{A}}(\Omega(\mathcal{A}))$ the tensors which are symmetric, that is invariant with respect to the standard action of the symmetric group S_k; the restrictions of ψ_k and τ_k to this subspace are denoted by ψ'_k and τ'_k. Also, the image of ψ'_k is denoted by $U^k(\mathcal{A})$, because it can be viewed in a natural way as the degree k component of a natural grading on $U(\mathcal{A})$. Indeed, the above commutative diagram restricts to a new commutative diagram

$$\begin{array}{ccc} T'^{k}_{\mathcal{A}}(\Omega(\mathcal{A})) & \xrightarrow{\psi'_k} & U^k(\mathcal{A}) \\ \big\downarrow{\scriptstyle \tau'_k} & & \big\downarrow{\scriptstyle \mathrm{gr}_k} \\ \mathrm{Sym}^k_{\mathcal{A}}(\Omega(\mathcal{A})) & \xrightarrow[\mathrm{PBW}^k_{\mathcal{A}}]{} & \mathrm{gr}^k(U(\mathcal{A})) \end{array}$$

where τ'_k is now an isomorphism; since $\mathrm{PBW}^k_{\mathcal{A}}$ is also an isomorphism, it follows that $\mathrm{gr}_k \circ \psi'_k : T'^{k}_{\mathcal{A}}(\Omega(\mathcal{A})) \to \mathrm{gr}^k(U(\mathcal{A}))$ is an isomorphism, and hence that the map ψ'_k in this diagram is an isomorphism between $T'^{k}_{\mathcal{A}}(\Omega(\mathcal{A}))$ and a complement of $U_{k-1}(\mathcal{A})$ in $U_k(\mathcal{A})$. As a corollary,

$$U_k(\mathcal{A}) = U^k(\mathcal{A}) \oplus U_{k-1}(\mathcal{A}) = U^k(\mathcal{A}) \oplus U^{k-1}(\mathcal{A}) \oplus \cdots \oplus U^0(\mathcal{A}) \ ,$$

and so $U(\mathcal{A})$ is graded by $\mathcal{A}$-modules, $U(\mathcal{A}) = \oplus_{i\in\mathbb{N}} U^i(\mathcal{A})$.

Since in the above diagram all maps are isomorphisms (of $\mathcal{A}$-modules), we obtain by composition for every k an isomorphism of $\mathcal{A}$-modules

$$\omega_k : \mathrm{Sym}^k_{\mathcal{A}}(\Omega(\mathcal{A})) \to U^k(\mathcal{A}) \ .$$

Since the inverse of τ'_k is given by

$$\tau'^{-1}_k(a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} a_0 \mathrm{d}a_{\sigma(1)} \otimes \cdots \otimes \mathrm{d}a_{\sigma(k)} \ ,$$

ω_k is given by

$$\omega_k(a_0 \mathrm{d}a_1 \ldots \mathrm{d}a_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \alpha_{\mathcal{A}}(a_0).\beta_{\mathcal{A}}(a_{\sigma(1)}) \cdots \beta_{\mathcal{A}}(a_{\sigma(k)}) \ .$$

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