Almost Contact Metric Structures Induced by G_2 Structures

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Abstract

We study almost contact metric structures induced by 2-fold vector cross products on manifolds with G_2 structures. We get some results on possible classes of almost contact metric structures. Finally we give examples.

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1   Introduction

A recent research area in geometry is the relation between manifolds with structure group G_2 and almost contact metric manifolds. A manifold with G_2 structure has a 3-form globally defined on its tangent bundle with some properties. Such manifolds are classified into sixteen classes by Fernández and Gray in [10] according to the properties of the covariant derivative of the 3-form.

On an almost contact metric manifold, there exists a global 2-form and the properties of the covariant derivative of this 2-form yields 2^{12} classes of almost contact metric manifolds, see [3, 9].

Recently Matzeu and Munteanu constructed almost contact metric structure induced by the 2-fold vector cross product on some classes of manifolds with G_2 structures [12]. Arikan et.al. proved the existence of almost contact metric structures on manifolds with G_2 structures [4]. Todd studied almost contact metric structures on manifolds with parallel G_2 structures [14].

Our aim is to study almost contact metric structures on manifolds with arbitrary G_2 structures. We eliminate some classes that almost contact metric structure induced from a G_2 structure may belong to according to properties of characteristic vector field of the almost contact metric structure. In particular, we also investigate the possible classes of almost contact metric structures on manifolds with nearly parallel G_2 structures. In addition, we give examples of almost contact metric structures on manifolds with G_2 structures induced by the 2-fold vector cross product.

2   Preliminaries

Consider $\mathbb{R}^7$ with the standard basis $\{e_1, ..., e_7\}$. The fundamental 3-form on $\mathbb{R}^7$ is defined as

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

where $\{e^1, ..., e^7\}$ is the dual basis of the standard basis and $e^{ijk} = e^i \wedge e^j \wedge e^k$. Then compact, simple and simply connected 14-dimensional Lie group G_2 is

$$G_2 := \{f \in GL(7, \mathbb{R}) \mid f^* \varphi_0 = \varphi_0\}.$$

A manifold with G_2 structure is a 7-dimensional oriented manifold whose structure group reduces to the group G_2. In this case, there exists a global

3-form φ on M such that for all $p \in M$, $(T_pM, \varphi_p) \cong (\mathbb{R}^7, \varphi_0)$. This 3-form is called the fundamental 3-form or the G_2 structure on M and gives a Riemannian metric g, a volume form and a 2-fold vector cross product P on M defined by $\varphi(x, y, z) = g(P(x, y), z)$ for all vector fields x, y on M [8].

Manifolds (M, g) with G_2 structure φ were classified according to properties of the covariant derivative of the fundamental 3-form. The space

$$\mathcal{W} = \{\alpha \in (\mathbb{R}^7)^* \otimes \Lambda^3(\mathbb{R}^7)^* | \alpha(x, y \wedge z \wedge P(y, z)) = 0 \ \ \forall x, y, z \in \mathbb{R}^7\}$$

of of tensors having the same symmetry properties as the covariant derivative of φ was written, and then this space was decomposed into four G_2-irreducible subspaces using the representation of the group G_2 on $\mathcal{W}$. Since

$$(\nabla\varphi)_p \in \mathcal{W}_p = \{\alpha \in T_p^*M \otimes \Lambda^3(T_p^*M) | \alpha(x, y \wedge z \wedge P(y, z)) = 0 \ \ \forall x, y, z \in T_pM\}$$

and there are 16 invariant subspaces of $\mathcal{W}_p$, each subspace corresponds to a different class of manifolds with G_2 structure. For example, the class $\mathcal{P}$, in which the covariant derivative of φ is zero, is the class of manifolds with parallel G_2 structure. A manifold which is in this class is sometimes called a G_2 manifold. $\mathcal{W}_1$ corresponds to the class of nearly parallel manifolds, which are manifolds with G_2 structure φ satisfying $d\varphi = k * \varphi$ for some constant k [10].

Let M^{2n+1} be a differentiable manifold of dimension $2n + 1$. If there is a $(1, 1)$ tensor field ϕ, a vector field ξ and a 1-form η on M satisfying

$$\phi^2 = -I + \eta \otimes \xi, \qquad \eta(\xi) = 1,$$

then M is said to have an almost contact structure (ϕ, ξ, η). A manifold with an almost contact structure is called an almost contact manifold.

If in addition to an almost contact structure (ϕ, ξ, η), M also admits a Riemannian metric g such that

$$g(\phi(x), \phi(y)) = g(x, y) - \eta(x)\eta(y)$$

for all vector fields x, y, then M is an almost contact metric manifold with the almost contact metric structure (ϕ, ξ, η, g). The Riemannian metric g is called a compatible metric. The 2-form Φ defined by

$$\Phi(x, y) = g(x, \phi(y))$$

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for all $x, y \in \Gamma(TM)$ is called the fundamental 2-form of the almost contact metric manifold (M, ϕ, ξ, η, g).

In [9], a classification of almost contact metric manifolds was obtained via the study of the covariant derivative of the fundamental 2-form. Let (ξ, η, g) be an almost contact metric structure on $\mathbb{R}^{2n+1}$. A space

$$\mathcal{C} = \{\alpha \in \otimes_3^0 \mathbb{R}^{2n+1} | \alpha(x, y, z) = -\alpha(x, z, y) = -\alpha(x, \phi y, \phi z)$$

$$+ \eta(y)\alpha(x, \xi, z) + \eta(z)\alpha(x, y, \xi)\}$$

having the same symmetries as the covariant derivative of the fundamental 2-form was given. First this space was written as a direct sum of three subspaces

$$\mathcal{D}_1 = \{\alpha \in \mathcal{C} | \alpha(\xi, x, y) = \alpha(x, \xi, y) = 0\},$$

$$\mathcal{D}_2 = \{\alpha \in \mathcal{C} | \alpha(x, y, z) = \eta(x)\alpha(\xi, y, z) + \eta(y)\alpha(x, \xi, z) + \eta(z)\alpha(x, y, \xi)\}$$

and

$$\mathcal{C}_{12} = \{\alpha \in \mathcal{C} | \alpha(x, y, z) = \eta(x)\eta(y)\alpha(\xi, \xi, z) + \eta(x)\eta(z)\alpha(\xi, y, \xi)\}$$

and then, $\mathcal{D}_1$, $\mathcal{D}_2$ were decomposed into $U(n) \times 1$ irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_4$ and $\mathcal{C}_5, \ldots, \mathcal{C}_{11}$, respectively. Thus there are 2^{12} invariant subspaces, denoted by $\mathcal{C}_1, \ldots, \mathcal{C}_{12}$, each corresponding to a class of almost contact metric manifolds. For example, the trivial class such that $\nabla\Phi = 0$ corresponds to the class of cosymplectic [5] (called co-Kähler by some authors) manifolds, $\mathcal{C}_1$ is the class of nearly-K-cosymplectic manifolds, etc.

In the classification of Chinea and Gonzales, it was shown that the space of quadratic invariants of $\mathcal{C}$ is generated by the following 18 elements:

$$i_1(\alpha) = \sum_{i,j,k} \alpha(e_i, e_j, e_k)^2 \qquad\qquad i_2(\alpha) = \sum_{i,j,k} \alpha(e_i, e_j, e_k)\alpha(e_j, e_i, e_k)$$

$$i_3(\alpha) = \sum_{i,j,k} \alpha(e_i, e_j, e_k)\alpha(\phi e_i, \phi e_j, e_k) \quad i_4(\alpha) = \sum_{i,j,k} \alpha(e_i, e_i, e_k)\alpha(e_j, e_j, e_k)$$

$$i_5(\alpha) = \sum_{j,k} \alpha(\xi, e_j, e_k)^2 \qquad\qquad i_6(\alpha) = \sum_{i,k} \alpha(e_i, \xi, e_k)^2$$

$$i_7(\alpha) = \sum_{j,k} \alpha(\xi, e_j, e_k)\alpha(e_j, \xi, e_k) \qquad i_8(\alpha) = \sum_{i,j} \alpha(e_i, e_j, \xi)\alpha(e_j, e_i, \xi)$$

$$i_9(\alpha) = \sum_{i,j} \alpha(e_i, e_j, \xi)\alpha(\phi e_i, \phi e_j, \xi) \qquad i_{10}(\alpha) = \sum_{i,j} \alpha(e_i, e_i, \xi)\alpha(e_j, e_j, \xi)$$

$$i_{11}(\alpha) = \sum_{i,j} \alpha(e_i, e_j, \xi)\alpha(e_j, \phi e_i, \xi) \qquad i_{12}(\alpha) = \sum_{i,j} \alpha(e_i, e_j, \xi)\alpha(\phi e_j, \phi e_i, \xi)$$

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$$i_{13}(\alpha) = \sum_{j,k} \alpha(\xi, e_j, e_k)\alpha(\phi e_j, \xi, e_k) \quad i_{14}(\alpha) = \sum_{i,j} \alpha(e_i, \phi e_i, \xi)\alpha(e_j, \phi e_j, \xi)$$

$$i_{15}(\alpha) = \sum_{i,j} \alpha(e_i, \phi e_i, \xi)\alpha(e_j, e_j, \xi) \quad i_{16}(\alpha) = \sum_{k} \alpha(\xi, \xi, e_k)^2$$

$$i_{17}(\alpha) = \sum_{i,k} \alpha(e_i, e_i, e_k)\alpha(\xi, \xi, e_k) \quad i_{18}(\alpha) = \sum_{i,k} \alpha(e_i, e_i, \phi e_k)\alpha(\xi, \xi, e_k)$$

where $\{e_1, e_2, ..., e_6, \xi\}$ is a local orthonormal basis. Also following relations among quadratic invariants were expressed for manifolds having dimensions ≥ 7, where $\alpha \in \mathcal{C}$ and $A = \{1, 2, 3, 4, 5, 7, 11, 13, 15, 16, 17, 18\}$:

$\mathcal{C}_1 : i_1(\alpha) = -i_2(\alpha) = -i_3(\alpha) = ||\alpha||^2; \quad i_m(\alpha) = 0 \ (m \geq 4)$

$\mathcal{C}_2 : i_1(\alpha) = 2i_2(\alpha) = -i_3(\alpha) = ||\alpha||^2; \quad i_m(\alpha) = 0 \ (m \geq 4)$

$\mathcal{C}_3 : i_1(\alpha) = i_3(\alpha) = ||\alpha||^2; \quad i_2(\alpha) = i_m(\alpha) = 0 \ (m \geq 4)$

$\mathcal{C}_4 : i_1(\alpha) = i_3(\alpha) = \frac{n}{(n-1)^2}i_4(\alpha) = \frac{n}{(n-1)^2}\sum_{k}^{2n} c_{12}^2(\alpha)(e_k);$

$i_2(\alpha) = i_m(\alpha) = 0 \ (m > 4)$

$\mathcal{C}_5 : i_6(\alpha) = -i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \frac{1}{2n}i_{14}(\alpha);$

$i_{10}(\alpha) = i_m(\alpha) = 0 \ (m \in A)$

$\mathcal{C}_6 : i_6(\alpha) = i_8(\alpha) = i_9(\alpha) = i_{12}(\alpha) = \frac{1}{2n}i_{10}(\alpha);$

$i_{14}(\alpha) = i_m(\alpha) = 0 \ (m \in A)$

$\mathcal{C}_7 : i_6(\alpha) = i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \frac{||\alpha||^2}{2};$

$i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0 \ (m \in A)$

$\mathcal{C}_8 : i_6(\alpha) = -i_8(\alpha) = i_9(\alpha) = -i_{12}(\alpha) = \frac{||\alpha||^2}{2};$

$i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0 \ (m \in A)$

$\mathcal{C}_9 : i_6(\alpha) = i_8(\alpha) = -i_9(\alpha) = -i_{12}(\alpha) = \frac{||\alpha||^2}{2};$

$i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0 \ (m \in A)$

$\mathcal{C}_{10} : i_6(\alpha) = -i_8(\alpha) = -i_9(\alpha) = i_{12}(\alpha) = \frac{||\alpha||^2}{2};$

$i_{10}(\alpha) = i_{14}(\alpha) = i_m(\alpha) = 0 \ (m \in A)$

$\mathcal{C}_{11} : i_5(\alpha) = ||\alpha||^2; i_m(\alpha) = 0 \ (m \neq 5)$

$\mathcal{C}_{12} : i_{16}(\alpha) = ||\alpha||^2; i_m(\alpha) = 0 \ (m \neq 16)$

For details, refer to [9].

We give below most studied classes of almost contact metric structures as direct sum of spaces $\mathcal{C}_i$:

$| \mathcal{C} | =$ the class of cosymplectic manifolds.

$\mathcal{C}_1 =$ the class of nearly-K-cosymplectic manifolds.

$\mathcal{C}_2 \oplus \mathcal{C}_9 =$ the class of almost cosymplectic manifolds.

$$\mathcal{C}_5 = \text{the class of } \alpha\text{-Kenmotsu manifolds.}$$

$$\mathcal{C}_6 = \text{the class of } \alpha\text{-Sasakian manifolds.}$$

$$\mathcal{C}_5 \oplus \mathcal{C}_6 = \text{the class of trans-Sasakian manifolds.}$$

$$\mathcal{C}_6 \oplus \mathcal{C}_7 = \text{the class of quasi-Sasakian manifolds.}$$

$$\mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 = \text{the class of semi-cosymplectic and normal manifolds.}$$

$$\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 = \text{the class of nearly trans-Sasakian manifolds.}$$

$$\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} = \text{the class of quasi-K-cosymplectic manifolds.}$$

$$\mathcal{C}_3 \oplus \mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 = \text{the class of normal manifolds.}$$

$$\mathcal{D}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} = \text{the class of almost-K-contact manifolds.}$$

$$\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11} = \text{the class of semi-cosymplectic manifolds.}$$

Let (M, g) be a 7-dimensional Riemannian manifold with G_2 structure φ and the associated 2-fold vector cross product $\times$ and let ξ be a nowhere zero vector field of unit length on M. Then for

$$\phi(x) := \xi \times x \qquad \eta(x) := g(\xi, x),$$

(ϕ, ξ, η, g) is an almost contact metric structure on M [12, 4]. Throughout this study, (ϕ, ξ, η, g) will denote the almost contact metric structure (a.c.m.s) induced by the G_2 structure φ on M and Φ the fundamental 2-form of the a.c.m.s.

3   Almost contact metric structures obtained from G_2 Structures

Let M be a manifold with G_2 structure φ and ξ a nowhere zero unit vector field on M and (ϕ, ξ, η, g) the a.c.m.s. with the fundamental form Φ induced by the G_2 structure φ.

If $\nabla\varphi = 0$, then it can be seen that $\nabla\Phi = 0$ if and only if $\nabla\xi = 0$ [2, 14].

If ξ is a Killing vector field on a manifold with any G_2 structure, then

$$\begin{aligned} d\eta(x, y) &= \tfrac{1}{2}\{(\nabla_x\eta)(y) - (\nabla_y\eta)(x)\} \\ &= \tfrac{1}{2}\{g(\nabla_x\xi, y) - g(\nabla_y\xi, x)\} \\ &= g(\nabla_x\xi, y), \end{aligned}$$

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which implies

$$d\eta = 0 \Leftrightarrow \nabla\xi = 0.$$

Therefore if the Killing vector field ξ is not parallel, then the a.c.m.s. can not be nearly-K-cosymplectic $(\mathcal{C}_1)$.

To deduce further results, we focus on the covariant derivative of the fundamental 2-form Φ, where the a.c.m.s. (ϕ, ξ, η, g) is obtained from a G_2 structure of any class and ξ is any nonzero vector field. Direct calculation gives

$$(\nabla_x\Phi)(y, z) = g(y, \nabla_x(\xi \times z)) + g(\nabla_x z, \xi \times y). \qquad (3.1)$$

We also compute some of $i_k(\nabla\Phi), (k = 1, ..., 18)$ to understand which class $\nabla\Phi$ may belong to.

Proposition 3.1 *Let φ be a G_2 structure on M of an arbitrary class and (ϕ, ξ, η, g) an a.c.m.s. obtained from φ. Then*

    a. $i_6(\nabla\Phi) = 0$ if and only if $\nabla_{e_i}\xi = 0$ for $i = 1, \cdots, 6$ (Note that $\nabla_\xi\xi$ need not be zero),

    b. $i_{16}(\nabla\Phi) = 0$ if and only if $\nabla_\xi\xi = 0$.

Proof By direct calculation, for any $i, k \in \{1, 2, ..., 6\}$

$$\begin{aligned}
(\nabla_{e_i}\Phi)(\xi, e_k) &= g(\xi, \nabla_{e_i}(\xi \times e_k)) + g(\nabla_{e_i}e_k, \xi \times \xi) \\
&= g(\xi, \nabla_{e_i}(\xi \times e_k)) \\
&= -g(\nabla_{e_i}\xi, \xi \times e_k)
\end{aligned}$$

and thus, we obtain

$$i_6(\nabla\Phi) = \sum_{i,k}((\nabla_{e_i}\Phi)(\xi, e_k))^2 = \sum_{i,k} g(\nabla_{e_i}\xi, \xi \times e_k)^2.$$

Since $\xi \times e_k$ is also a frame element, $i_6(\nabla\Phi) = 0$ if and only if $\nabla_{e_i}\xi$ is zero. Similarly,

$$\begin{aligned}
(\nabla_\xi\Phi)(\xi, e_k) &= g(\xi, \nabla_\xi(\xi \times e_k)) + g(\nabla_\xi e_k, \xi \times \xi) \\
&= -g(\nabla_\xi\xi, \xi \times e_k)
\end{aligned}$$

for any $k \in \{1, 2, ..., 6\}$, and we get

$$i_{16}(\nabla\Phi) = \sum_k (\nabla_\xi\Phi)(\xi, e_k)^2 = \sum_k g(\nabla_\xi\xi, \xi \times e_k)^2.$$

Note that $g(\nabla_\xi \xi, \xi) = 0$ since ξ is of unit length. As a result, $i_{16}(\nabla\Phi) = 0$ if and only if $\nabla_\xi \xi = 0$. ∎

Proposition 3.2 *Let (ϕ, η, ξ, g) be an almost contact metric structure induced by a G_2 structure φ. Then,*

- *$i_{14}(\nabla\Phi) = 0$ if and only if $div(\xi) = 0$.*

- *$i_{15}(\nabla\Phi) = -div(\xi)g(\xi, v)$, where $v = \sum\limits_{j=1}^{6} e_j \times (\nabla_{e_j}\xi)$.*

Proof For any $i, j \in \{1, 2, ..., 6\}$ we have

$$(\nabla_{e_i}\Phi)(\phi e_i, \xi) = g(\xi \times e_i, \nabla_{e_i}(\xi \times \xi)) + g(\nabla_{e_i}\xi, \xi \times (\xi \times e_i))$$
$$= -g(\nabla_{e_i}\xi, e_i)$$
$$= g(\xi, \nabla_{e_i}e_i).$$

On the other hand,

$$\sum_{i=1}^{6} \nabla_{e_i}e_i = -\sum_{i=1}^{6} div(e_i)e_i - div(\xi)\xi - \nabla_\xi \xi$$

and thus

$$g(\xi, \sum_i \nabla_{e_i}e_i) = -g(\xi, \sum_i div(e_i)e_i) - g(\xi, div(\xi)\xi) - g(\xi, \nabla_\xi \xi)$$
$$= -div(\xi).$$

Then

$$i_{14}(\nabla\Phi) = \sum_{i,j}(\nabla_{e_i}\Phi)(\phi e_i, \xi)(\nabla_{e_j}\Phi)(\phi e_j, \xi)$$
$$= \left(g(\xi, \sum_i \nabla_{e_i}e_i)\right)\left(g(\xi, \sum_j \nabla_{e_j}e_j)\right) = (div(\xi))^2.$$

Therefore, $i_{14}(\nabla\Phi)$ is zero if and only if $div(\xi)$ is zero.
Similarly, from equations

$$(\nabla_{e_i}\Phi)(\phi e_i, \xi) = -g(\nabla_{e_i}\xi, e_i) \text{ and } (\nabla_{e_j}\Phi)(e_j, \xi) = g(\nabla_{e_j}\xi, \xi \times e_j)$$

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we have,

$$i_{15}(\nabla\Phi) = \sum_{i,j}(\nabla_{e_i}\Phi)(\phi e_i, \xi)\ (\nabla_{e_j}\Phi)(e_j, \xi)$$

$$= \sum_{i,j} g(\xi, \nabla_{e_i}e_i)g(\nabla_{e_j}\xi, \xi \times e_j)$$

$$= \Big(g(\xi, \sum_i \nabla_{e_i}e_i)\Big)\Big(\sum_j g(\xi, \nabla_{e_j}(e_j \times \xi))\Big)$$

$$= \Big(g(\xi, -div(\xi)\xi) - g(\xi, \sum_i div(e_i)e_i)\Big)\Big(\sum_j g(\xi, e_j \times \nabla_{e_j}\xi)\Big)$$

$$= -div(\xi).g(\xi, v).$$

∎

Now consider in particular an a.c.m.s. induced by a nearly parallel G_2 structure.

Proposition 3.3 *Let* (ϕ, η, ξ, g) *be an almost contact metric structure induced by a nearly parallel G_2 structure. Then,*

- $i_5(\nabla\Phi) = 0$ *if and only if* $\nabla_\xi\xi = 0$.

- *If* $\nabla_\xi\xi = 0$, *then* $i_{17}(\nabla\Phi) = i_{18}(\nabla\Phi) = 0$.

Proof Since φ is nearly parallel, for any $j, k \in \{1, 2, ..., 6\}$ we have

$$(\nabla_\xi\Phi)(e_j, e_k) = g(e_j, \nabla_\xi(\xi \times e_k)) + g(\nabla_\xi e_k, \xi \times e_j)$$
$$= g(e_j, \nabla_\xi\xi \times e_k) + g(e_j, \xi \times \nabla_\xi e_k) + g(\nabla_\xi e_k, \xi \times e_j)$$
$$= -g(\nabla_\xi\xi, e_j \times e_k).$$

So,

$$i_5(\nabla\Phi) = \sum_{j,k}((\nabla_\xi\Phi)(e_j, e_k))^2 = \sum_{j,k}(g(\nabla_\xi\xi, e_j \times e_k))^2$$

which is zero if and only if $\nabla_\xi\xi$ is zero. Here, $e_j \times e_k$ is also a frame element.

Similarly, For any $i, k \in \{1, 2, ..., 6\}$,

$$(\nabla_{e_i}\Phi)(e_i, \phi e_k) = g(e_i, \nabla_{e_i}(\xi \times (\xi \times e_k))) + g(\nabla_{e_i}(\xi \times e_k), \xi \times e_i)$$
$$= g(e_i, \nabla_{e_i}(-e_k)) + g(\nabla_{e_i}(\xi \times e_k), \xi \times e_i)$$
$$= g(\nabla_{e_i}e_i, e_k) + g(\nabla_{e_i}(\xi \times e_k), \xi \times e_i)$$

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$$(\nabla_\xi \Phi)(\xi, e_k) = g(\xi, \nabla_\xi(\xi \times e_k)) + g(\nabla_\xi e_k, \xi \times \xi)$$
$$= g(\xi, \nabla_\xi \xi \times e_k) + g(\xi, \xi \times \nabla_\xi e_k)$$
$$= -g(e_k, (\nabla_\xi \xi) \times \xi).$$

Then

$$i_{18}(\nabla\Phi) = \sum_{i,k}((\nabla_{e_i}\Phi)(e_i, \phi e_k))((\nabla_\xi \Phi)(\xi, e_k))$$

$$= -\sum_{i,k}\left(g(\nabla_{e_i}e_i, e_k) + g(\nabla_{e_i}(\xi \times e_k), \xi \times e_i)\right)\left(g(e_k, (\nabla_\xi \xi) \times \xi)\right)$$

$$= -\sum_{i,k}\left(g(\nabla_{e_i}e_i, e_k)g(e_k, (\nabla_\xi \xi) \times \xi)\right)$$

$$\quad - \sum_{i,k}\left(g(\nabla_{e_i}(\xi \times e_k), \xi \times e_i)g(e_k, (\nabla_\xi \xi) \times \xi)\right)$$

$$= -\sum_{i,k}\left(g(\nabla_{e_i}e_i, e_k)g(e_k, (\nabla_\xi \xi) \times \xi)\right)$$

$$\quad + \sum_{i,k}\left(g(\nabla_{e_i}e_i, e_k)g(e_k, (\nabla_\xi \xi) \times \xi)\right)$$

$$\quad - \sum_{i,k}\left(g(\xi \times e_k, e_i \times \nabla_{e_i}\xi)g(e_k, (\nabla_\xi \xi) \times \xi)\right)$$

$$= -\sum_i g(\xi \times (\sum_k g((\nabla_\xi \xi) \times \xi, e_k)e_k + g((\nabla_\xi \xi) \times \xi, \xi)\xi), e_i \times \nabla_{e_i}\xi)$$

$$= -\sum_i g(\xi \times ((\nabla_\xi \xi) \times \xi), e_i \times \nabla_{e_i}\xi)$$

$$= -g(\nabla_\xi \xi, \sum_i(e_i \times \nabla_{e_i}\xi)).$$

Thus, if $\nabla_\xi \xi$ is zero, so is $i_{18}(\nabla\Phi)$.

For i_{17} we compute

$$(\nabla_{e_i}\Phi)(e_i, e_k) = g(e_i, \nabla_{e_i}(\xi \times e_k)) + g(\nabla_{e_i}e_k, \xi \times e_i)$$

and

$$(\nabla_\xi \Phi)(\xi, e_k) = g(\xi, \nabla_\xi(\xi \times e_k)) + g(\nabla_\xi e_k, \xi \times \xi)$$
$$= -g(\nabla_\xi \xi, \xi \times e_k)$$
$$= g(e_k, \xi \times (\nabla_\xi \xi))$$

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for any $i, k \in \{1, 2, ..., 6\}$ and obtain

$$
\begin{aligned}
i_{17}(\nabla \Phi) &= \sum_{i,k} ((\nabla_{e_i} \Phi)(e_i, e_k))((\nabla_\xi \Phi)(\xi, e_k)) \\
&= \sum_{i,k} \Big(- g(\nabla_{e_i} e_i, \xi \times e_k) - g(e_k, \nabla_{e_i}(\xi \times e_i)) \Big) \Big(g(e_k, \xi \times (\nabla_\xi \xi)) \Big) \\
&= \sum_{i,k} g(e_k, \xi \times (\nabla_{e_i} e_i)) g(e_k, \xi \times \nabla_\xi \xi) - \sum_{i,k} g(e_k, \xi \times (\nabla_{e_i} e_i)) g(e_k, \xi \times \nabla_\xi \xi) \\
&\quad + \sum_{i,k} g(e_k, e_i \times \nabla_{e_i} \xi) g(e_k, \xi \times \nabla_\xi \xi) \\
&= \sum_{i,k} g(e_k, e_i \times \nabla_{e_i} \xi) g(e_k, \xi \times \nabla_\xi \xi) \\
&= g(\xi \times (\nabla_\xi \xi), \sum_i e_i \times (\nabla_{e_i} \xi))
\end{aligned}
$$

Thus, if $\nabla_\xi \xi = 0$, then $i_{17}(\nabla \Phi) = 0$. ∎

Similarly, if $\nabla \xi$ is zero, then so is $i_{15}(\nabla \Phi)$.

Before giving results on possible classes of a.c.m.s. induced by G_2 structures, note that $\delta \eta = -div(\xi)$. To see this, consider the orthonormal basis $\{e_1, \cdots, e_6, \xi\}$. Then

$$
\begin{aligned}
div(\xi) &= \sum_{i=1}^{6} g(\nabla_{e_i} \xi, e_i) + g(\nabla_\xi \xi, \xi) \\
&= \sum_{i=1}^{6} g(\nabla_{e_i} \xi, e_i).
\end{aligned}
$$

On the other hand, since

$$
\begin{aligned}
(\nabla_{e_i} \eta)(e_i) &= e_i[\eta(e_i)] - \eta(\nabla_{e_i} e_i) \\
&= g(\nabla_{e_i} \xi, e_i) + g(\xi, \nabla_{e_i} e_i) - g(\xi, \nabla_{e_i} e_i) \\
&= g(\nabla_{e_i} \xi, e_i),
\end{aligned}
$$

we have

$$
\delta \eta = - \sum_{i=1}^{6} (\nabla_{e_i} \eta)(e_i) = - \sum_{i=1}^{6} g(\nabla_{e_i} \xi, e_i) = -div(\xi).
$$

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Theorem 1 *Let M be a manifold with a G_2 structure φ and (ϕ, ξ, η, g) be an almost contact metric structure (a.c.m.s.) obtained from φ.*
(a) If $\nabla_\xi \xi \neq 0$, then $\nabla\Phi$ can not be in classes $\mathcal{D}_2, \mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_{11}$.
(b) If $div(\xi) \neq 0$, then the almost contact metric structure can not belong to classes $\mathcal{D}_1$, $\mathcal{C}_i$ for $i = 1, 2, 3, 4, 6, 7, \cdots, 12$ and can not be semi-cosymplectic ($\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11}$).

In following proofs we use the relations below given in [9] together with properties of i_m for each $\mathcal{C}_i$:
If $\alpha \in \mathcal{D}_1$, then $i_m(\alpha) = 0$ for $m \geq 5$.
If $\alpha \in \mathcal{D}_2$, then $i_m(\alpha) = 0$ for $m = 1, 2, 3, 4, 16, 17, 18$.

Proof (a) Let $\nabla_\xi \xi \neq 0$. Then by the proposition [3.1], we have $i_{16}(\nabla\Phi) \neq 0$. This implies $\nabla\Phi \notin \mathcal{D}_2$. In addition, $\nabla\Phi$ can not belong to any of the classes C_i, $i = 1, \ldots, 11$.
(b)If $div(\xi) \neq 0$, then the proposition [3.2] yields that $i_{14}(\nabla\Phi) = (div(\xi))^2 \neq 0$. Hence $\nabla\Phi$ can not satisfy the defining relations of the classes

$$\mathcal{D}_1 = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4,$$

$$\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4, \mathcal{C}_6, \cdots, \mathcal{C}_{12}.$$

Besides, the defining relation of semi cosymplectic manifolds is

$$\delta\Phi = 0 \text{ and } \delta\eta = 0.$$

Since $div(\xi) \neq 0$, $\delta\eta \neq 0$, and thus the a.c.m.s. is not semi cosymplectic. ∎
Note that if $\nabla_\xi \xi \neq 0$, then since $\nabla\Phi \notin \mathcal{D}_2 = \mathcal{C}_5 \oplus \ldots \oplus \mathcal{C}_{11}$, the a.c.m.s. can not be contained in any subclass of $\mathcal{D}_2$. In particular, the a.c.m.s. can not be α-Kenmotsu, α-Sasakian, trans-Sasakian or quasi-Sasakian.
If $div(\xi) \neq 0$, then we have $\nabla\Phi \notin \mathcal{D}_1 = \mathcal{C}_1 \oplus \ldots \oplus \mathcal{C}_4$. In this case, the a.c.m.s. can not be nearly-K-cosymplectic. Also, since the a.c.m.s. can not be semi-cosymplectic, it can not be almost-cosymplectic, α-Kenmotsu, α-Sasakian, trans-Sasakian, normal semi-cosymplectic or quasi-K-cosymplectic.
Note also that the class $\mathcal{C}_{12}$ is not contained in the class of semi-cosymplectic manifolds. We give a proof together with examples in [13]. For completeness, we also remind the proof here.
The defining relation of $\mathcal{C}_{12}$ gives

$$\nabla_\xi \Phi(\xi, x) = -(\nabla_\xi \eta)(\phi(x)),$$

which does not have to be zero. Assume that

$$(\nabla_\xi \eta)(\phi(x)) = 0$$

for all vector fields x. Then replacing x with $\phi(x)$, we get $(\nabla_\xi \eta)(x) = 0$, but then the defining relation of $\mathcal{C}_{12}$ implies $\nabla_x \Phi = 0$ for all x, that is there is no element in $\mathcal{C}_{12}$ which is not in the trivial class. Thus there is a vector field x_0 on M such that

$$(\delta\Phi)(x_0) = -\nabla_\xi \Phi(\xi, x_0) = (\nabla_\xi \eta)(\phi(x_0)) \neq 0.$$

Therefore the class $\mathcal{C}_{12}$ is not semi-cosymplectic.

Consider an a.c.m.s. induced by a nearly parallel G_2 structure. We deduce following results.

Theorem 2 *Let (ϕ, ξ, η, g) be an a.c.m.s. obtained from a nearly parallel G_2 structure φ. If $\nabla_\xi \xi \neq 0$, then $\nabla\Phi$ can not be in classes $\mathcal{D}_1, \mathcal{D}_2, \mathcal{C}_{12}. (\nabla\Phi$ may be contained by the classes $\mathcal{D}_1 \oplus \mathcal{D}_2, \mathcal{D}_1 \oplus \mathcal{C}_{12}, \mathcal{D}_2 \oplus \mathcal{C}_{12}, \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \mathcal{C}_{12}$).*

Proof Let $\nabla_\xi \xi \neq 0$. By proposition [3.3], $i_5(\nabla\Phi) \neq 0$. So, $\nabla\Phi$ can not be in $\mathcal{D}_1$ and $\mathcal{C}_{12}$. Besides, by the proposition [3.1], we have $i_{16}(\nabla\Phi) \neq 0$, then $\nabla\Phi$ can not be in $\mathcal{D}_2$. ■

In particular, the a.c.m.s. can not belong to any subclasses of $\mathcal{D}_1$ and $\mathcal{D}_2$.

Theorem 3 *Let (ϕ, ξ, η, g) be an a.c.m.s. obtained from a nearly parallel G_2 structure φ. Then, $\nabla_\xi \xi = 0$ if and only if M is almost K-contact.*

Proof The defining relation of almost K-contact manifolds is $\nabla_\xi \phi = 0$. Since φ is nearly parallel, for any vector field x,

$$(\nabla_\xi \phi)(x) = \nabla_\xi(\phi x) - \phi(\nabla_\xi x) = \nabla_\xi(\xi \times x) - \xi \times \nabla_\xi x$$
$$= (\nabla_\xi \xi \times x) + (\xi \times \nabla_\xi x) - (\xi \times \nabla_\xi x) = \nabla_\xi \xi \times x,$$

that is zero if and only if $\nabla_\xi \xi$ is zero. ■

Theorem 4 *Let (ϕ, η, ξ, g) be an almost contact metric structure induced by a G_2 structure and $v = \sum_{i=1}^{6} e_i \times \nabla_{e_i} \xi$. If $g(\xi, v) \neq 0$, then $\nabla\Phi$ is not of classes $\mathcal{D}_1, \mathcal{C}_5, \mathcal{C}_7, \mathcal{C}_8, \mathcal{C}_9, \mathcal{C}_{10}, \mathcal{C}_{11}, \mathcal{C}_{12}$.*

Proof First to compute $i_{10}(\nabla \Phi)$, we write

$$
\begin{aligned}
(\nabla_{e_i} \Phi)(e_i, \xi) &= g(e_i, \nabla_{e_i}(\xi \times \xi)) + g(\nabla_{e_i}\xi, \xi \times e_i) \\
&= g(e_i \times \nabla_{e_i}\xi, \xi),
\end{aligned}
$$

and we obtain

$$
i_{10}(\nabla \Phi) = \sum_{i,j=1}^{6} g(e_i \times \nabla_{e_i}\xi, \xi) g(e_j \times \nabla_{e_j}\xi, \xi) = g^2(v, \xi).
$$

Assume that $g(\xi, v) \neq 0$. Then $i_{10}(\nabla \Phi) = g(\xi, v)^2 \neq 0$ and the classes $\mathcal{D}_1, \mathcal{C}_5, \mathcal{C}_7, \mathcal{C}_8, \mathcal{C}_9, \mathcal{C}_{10}, \mathcal{C}_{11}, \mathcal{C}_{12}$ are eliminated similar to previous proofs. ∎

Corollary 5 *If $g(\xi, v) \neq 0$ and $div(\xi) \neq 0$, then $\nabla \Phi$ is not an element of the classes $\mathcal{C}_i$, for $i = 1, \cdots, 12$.*

Next we give examples of a.c.m.s. induced by a parallel G_2 structure and a nearly parallel G_2 structure, respectively. The a.c.m.s. induced by the parallel G_2 structure is in the class $\mathcal{D}_1$, whereas that induced by the nearly parallel G_2 structure is almost-K-contact.

Example 6 *Let (K, g_K) be a 4-dimensional Kähler manifold with an exact Kähler form Ω, i.e. $\Omega = d\lambda$, where λ is a 1-form on K. Consider $\mathbb{R}^3$ with coordinates (x_1, x_2, x_3) and Euclidean metric $h = dx_1^2 + dx_2^2 + dx_3^2$. It is known that $(M = \mathbb{R}^3 \times K, g = h \times g_K)$ admits a parallel G_2 structure*

$$
\varphi = dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge \Omega + dx_2 \wedge Re\theta - dx_3 \wedge Im\theta,
$$

where θ is a volume form on K.
For all $p \in K$, there exist complex coordinates (z_1, z_2) near p such that

$$
g_K = |dz_1|^2 + |dz_2|^2, \qquad \Omega = \frac{i}{2}(dz_1 \wedge d\overline{z}_1 + dz_2 \wedge d\overline{z}_2), \qquad \theta = dz_1 \wedge dz_2
$$

at p. Setting $z_1 = x_4 + ix_5$, $z_2 = x_6 + ix_7$, one has

$$
g_K = dx_4^2 + \ldots + dx_7^2, \qquad \Omega = dx_4 \wedge dx_5 + dx_6 \wedge dx_7,
$$

$$
Re\theta = dx_4 \wedge dx_6 - dx_5 \wedge dx_7, \qquad Im\theta = dx_4 \wedge dx_7 + dx_5 \wedge dx_6
$$

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at p. Thus $g = h \times g_K = dx_1^2 + \ldots + dx_7^2$ and

$$\begin{aligned}
\varphi \;=\; & dx_1 \wedge dx_2 \wedge dx_3 + dx_1 \wedge (dx_4 \wedge dx_5 + dx_6 \wedge dx_7) \\
& + dx_2 \wedge (dx_4 \wedge dx_6 - dx_5 \wedge dx_7) - dx_3 \wedge (dx_4 \wedge dx_7 + dx_5 \wedge dx_6).
\end{aligned}$$

(see [11]).

Consider the a.c.m.s. (ϕ, ξ, η, g) on M induced by the parallel G_2 structure φ, for $\xi = x_2 \partial x_1$ and $\phi(x) = \xi \times x$. For the covariant derivative of the metric on a product manifold, see [6].

$$\begin{aligned}
(\nabla_{\partial x_2} \Phi)(\partial x_2, \partial x_3) \;=\; & g(\partial x_2, \nabla_{\partial x_2}(x_2 \partial x_1 \times \partial x_3)) + g(\nabla_{\partial x_2} \partial x_3, x_2 \partial x_1 \times \partial x_2) \\
=\; & -g(\partial x_2, \nabla_{\partial x_2}(x_2 \partial x_2)) \\
=\; & -g(\partial x_2, \partial x_2 [x_2] \partial x_2) \\
=\; & -1,
\end{aligned}$$

and thus the structure is not cosymplectic. We show that this structure is in the class $\mathcal{D}_1$. Since $\nabla_\xi \xi = x_2 \partial x_1 [x_2] \partial x_1 = 0$, we have

$$\begin{aligned}
(\nabla_\xi \Phi)(y, z) \;=\; & g(y, \nabla_\xi(\xi \times z)) + g(\nabla_\xi z, \xi \times y) \\
=\; & g(y, (\nabla_\xi \xi) \times z) + g(y, \xi \times \nabla_\xi z) + g(\nabla_\xi z, \xi \times y) \\
=\; & g(y, (\nabla_\xi \xi) \times z) \\
=\; & 0.
\end{aligned}$$

In addition, $\nabla_x \xi = \nabla_x x_2 \partial x_1 = x[x_2]\partial x_1 + x_2 \nabla_x \partial x_1$ for any vector field x. It can be seen that $\nabla_x \partial x_1 = 0$ for any $x \in M = \mathbb{R}^3 \times K$ and thus $\nabla_x \xi = x[x_2]\partial x_1$. Then

$$\begin{aligned}
(\nabla_x \Phi)(\xi, y) \;=\; & g(\xi, \nabla_x(\xi \times y)) + g(\nabla_x y, \xi \times \xi) \\
=\; & g(\xi, (\nabla_x \xi) \times y) + g(\xi, \xi \times \nabla_x y) \\
=\; & -g((\nabla_x \xi) \times \xi, y) \\
=\; & -x[x_2]x_2 g(\partial x_1 \times \partial x_1, y) \\
=\; & 0.
\end{aligned}$$

As a result, $\nabla \Phi \in \mathcal{D}_1$ by the definition of the space $\mathcal{D}_1$.

Example 7 *A Sasakian manifold is a normal contact metric manifold, or equivalently, an almost contact metric structure (ϕ, ξ, η, g) such that*

$$(\nabla_x \phi)(y) = g(x, y)\xi - \eta(y)x,$$

see [5]. A 7-dimensional 3-Sasakian manifold is a Riemannian manifold (M, g) equipped with three Sasakian structures $(\phi_i, \xi_i, \eta_i, g)$, $i = 1, 2, 3$ satisfying

$$[\xi_1, \xi_2] = 2\xi_3, \quad [\xi_2, \xi_3] = 2\xi_1, \quad [\xi_3, \xi_1] = 2\xi_2$$

and

$$\phi_3 \circ \phi_2 = -\phi_1 + \eta_2 \otimes \eta_3, \quad \phi_2 \circ \phi_3 = \phi_1 + \eta_3 \otimes \eta_2,$$

$$\phi_1 \circ \phi_3 = -\phi_2 + \eta_3 \otimes \eta_1, \quad \phi_3 \circ \phi_1 = \phi_2 + \eta_1 \otimes \eta_3,$$

$$\phi_2 \circ \phi_1 = -\phi_3 + \eta_1 \otimes \eta_2, \quad \phi_1 \circ \phi_2 = \phi_3 + \eta_2 \otimes \eta_1.$$

There exists a local orthonormal frame $\{e_1, \cdots, e_7\}$ such that $e_1 = \xi_1$, $e_2 = \xi_2$ and $e_3 = \xi_3$. The corresponding coframe via the Riemannian metric is denoted by $\{\eta_1, \cdots, \eta_7\}$. The differentials $d\eta_i$, $i = 1, 2, 3$ are

$$d\eta_1 = -2(\eta_{23} + \eta_{45} + \eta_{67}), \quad d\eta_2 = 2(\eta_{13} - \eta_{46} + \eta_{57}), \quad d\eta_3 = -2(\eta_{12} + \eta_{47} + \eta_{56}).$$

The 3-form

$$\varphi = \frac{1}{2}\eta_1 \wedge d\eta_1 - \frac{1}{2}\eta_2 \wedge d\eta_2 - \frac{1}{2}\eta_3 \wedge d\eta_3$$

is a nearly parallel G_2 structure (i.e. $d\varphi_i = -4 * \varphi_i$) on M, constructed in [1]. For properties and examples of Sasakian and 3-Sasakian manifolds [7] is a good reference. The brackets of $e_1, \cdots, e_7$ are computed by using the differentials $d\eta_i$ and by the equation $d\eta(e_i, e_j) = -\eta([e_i, e_j])$ for frame elements e_i, e_j.

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = 2e_2,$$

$$[e_4, e_5] = 2e_1, \quad [e_6, e_7] = 2e_1, \quad [e_4, e_6] = 2e_2,$$

$$[e_5, e_7] = -2e_2, \quad [e_4, e_7] = 2e_3, \quad [e_5, e_6] = 2e_3.$$

By the Kozsul formula we obtain $\nabla_{e_i} e_i = 0$ and

$$\nabla_{e_1} e_2 = e_3, \nabla_{e_1} e_3 = -e_2, \nabla_{e_1} e_4 = -e_5, \nabla_{e_1} e_5 = e_4, \nabla_{e_1} e_6 = -e_7, \nabla_{e_1} e_7 = e_6,$$

$$\nabla_{e_2} e_1 = -e_3, \nabla_{e_2} e_3 = e_1, \nabla_{e_2} e_4 = -e_6, \nabla_{e_2} e_5 = e_7, \nabla_{e_2} e_6 = e_4, \nabla_{e_2} e_7 = -e_5,$$

$$\nabla_{e_3} e_1 = e_2, \nabla_{e_3} e_2 = -e_1, \nabla_{e_3} e_4 = -e_7, \nabla_{e_3} e_5 = -e_6, \nabla_{e_3} e_6 = e_5, \nabla_{e_3} e_7 = e_4,$$

$$\nabla_{e_4} e_1 = -e_5, \nabla_{e_4} e_2 = -e_6, \nabla_{e_4} e_3 = -e_7, \nabla_{e_4} e_5 = e_1, \nabla_{e_4} e_6 = e_2, \nabla_{e_4} e_7 = e_3,$$

$$\nabla_{e_5} e_1 = e_4, \nabla_{e_5} e_2 = e_7, \nabla_{e_5} e_3 = -e_6, \nabla_{e_5} e_4 = -e_1, \nabla_{e_5} e_6 = e_3, \nabla_{e_5} e_7 = -e_2,$$

$$\nabla_{e_6}e_1 = -e_7, \nabla_{e_6}e_2 = e_4, \nabla_{e_6}e_3 = e_5, \nabla_{e_6}e_4 = -e_2, \nabla_{e_6}e_5 = -e_3, \nabla_{e_6}e_7 = e_1,$$

$$\nabla_{e_7}e_1 = e_6, \nabla_{e_7}e_2 = -e_5, \nabla_{e_7}e_3 = e_4, \nabla_{e_7}e_4 = -e_3, \nabla_{e_7}e_5 = e_2, \nabla_{e_7}e_6 = -e_1.$$

By the local expression of

$$\begin{aligned}
\varphi &= \tfrac{1}{2}\eta_1 \wedge d\eta_1 - \tfrac{1}{2}\eta_2 \wedge d\eta_2 - \tfrac{1}{2}\eta_3 \wedge d\eta_3 \\
&= \eta_{123} - \eta_{145} - \eta_{167} + \eta_{246} - \eta_{257} + \eta_{347} + \eta_{356},
\end{aligned}$$

the 2-fold vector cross products of frame elements are

$$e_1 \times e_2 = e_3, e_1 \times e_3 = -e_2, e_1 \times e_4 = -e_5, e_1 \times e_5 = e_4, e_1 \times e_6 = -e_7, e_1 \times e_7 = e_6,$$

$$e_2 \times e_3 = e_1, e_2 \times e_4 = e_6, e_2 \times e_5 = -e_7, e_2 \times e_6 = -e_4, e_2 \times e_7 = e_5,$$

$$e_3 \times e_4 = e_7, e_3 \times e_5 = e_6, e_3 \times e_6 = -e_5, e_3 \times e_7 = -e_4,$$

$$e_4 \times e_5 = -e_1, e_4 \times e_6 = e_2, e_4 \times e_7 = e_3, e_5 \times e_6 = e_3, e_5 \times e_7 = -e_2, e_6 \times e_7 = -e_1.$$

Consider the a.c.m.s. (ϕ, ξ, η, g) on M induced by the 2-fold vector cross product of the nearly parallel G_2 structure φ, where $\xi = e_1 = \xi_1$, $\eta = d\eta_1$ and $\phi(x) = \xi \times x$. First, since

$$(\nabla_x \Phi)(y, z) = g(y, \nabla_x(e_1 \times z)) + g(\nabla_x z, e_1 \times y),$$

we have

$$(\nabla_{e_2}\Phi)(e_1, e_2) = 1 \neq 0$$

and thus the a.c.m.s. is not cosymplectic. In addition,

$$\nabla_{e_2}\Phi(\xi, e_2) = \nabla_{e_2}\Phi(e_1, e_2) = 1,$$

implying that $\nabla\Phi \notin \mathcal{D}_1 = \mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_4$. On the other hand

$$\begin{aligned}
(\nabla_\xi \Phi)(x, y) &= g(x, \nabla_{e_1}(e_1 \times y)) + g(\nabla_{e_1}y, e_1 \times x) \\
&= g(x, (\nabla_{e_1}e_1) \times y) + g(x, e_1 \times \nabla_{e_1}y) + g(\nabla_{e_1}y, e_1 \times x) \\
&= \varphi(e_1, \nabla_{e_1}y, x) + \varphi(e_1, x, \nabla_{e_1}y) \\
&= 0,
\end{aligned}$$

which gives that the a.c.m.s. is almost-K-contact, that is, an element of the class

$$\mathcal{D}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_6 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10}.$$

Moreover,

$$\delta\Phi(e_1) = -\sum_{i=2}^{7}(\nabla_{e_i}\Phi)(e_i, e_1) - (\nabla_{e_1}\Phi)(e_1, e_1) = -2$$

yields the a.c.m.s. is not semi-cosymplectic ($\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_7 \oplus \mathcal{C}_8 \oplus \mathcal{C}_9 \oplus \mathcal{C}_{10} \oplus \mathcal{C}_{11}$). The a.c.m.s. is not trans-Sasakian ($\mathcal{C}_5 \oplus \mathcal{C}_6$):

$$(\nabla_{e_2}\Phi)(e_1, e_2) = 1,$$

whereas

$$\frac{1}{3}\{g(e_2, e_1)\eta(e_2) - g(e_2, e_2)\eta(e_1)\} = -\frac{1}{3},$$

i.e., the defining condition of being trans-Sasakian is not satisfied. In particular, the a.c.m.s. is not α-Sasakian or α-Kenmotsu. Note that we started with a Sasakian structure on a manifold, then we used the 2-fold vector cross product of the nearly parallel G_2 structure φ, however the induced a.c.m.s. is not Sasakian.

In fact, consider the a.c.m.s. induced by φ, where $\xi = ae_1 + be_2 + ce_3$ for constants a, b, c. Assume that this structure is Sasakian. Then we have

$$(\nabla_x\phi)(y) = g(x, y)\xi - \eta(y)x$$

for all vector fields x, y. For $x = e_1$, $y = e_2$, we get $b = \pm\frac{1}{3}$. If $x = e_1$, $y = e_4$, we obtain $b = 0$. Thus the a.c.m.s. is not Sasakian. Note that since $(\nabla_{e_1}\Phi)(e_1, e_2) = b$, $(\nabla_{e_1}\Phi)(e_1, e_3) = c$ and $(\nabla_{e_2}\Phi)(e_2, e_1) = a$, $\nabla\Phi \neq 0$ unless ξ is zero. Thus the a.c.m.s. is not cosymplectic.

By direct calculation, it can be seen that for $\xi = ae_1 + be_2 + ce_3$, one has $\nabla_\xi\xi = 0$. By Theorem [3], the a.c.m.s. is almost-K-contact.

References

[1] Agricola, I. and Friedrich, T., 3-Sasakian Manifolds in Dimension Seven, Their Spinors and G_2 Structures, J. Geom. Phys. 60 (2010) 326-332.

[2] Aktay, Ş., On Deformations of Parallel G_2 Structures and Almost Contact Metric Structures, in print.

[3] Alexiev, V. and Ganchev, G., On the Classification of the Almost Contact Metric Manifolds, Math. and Educ. in Math., Proc. of the XV Spring Conf. of UBM, Sunny Beach, 155 (1986).

[4] Arikan, M. F., Cho, H. and Salur, S., Existence of Compatible Contact Structures on G_2 Manifolds, Asian J. Math., (2) 17 (2013), 321-334.

[5] Blair, D.E., Riemannian Geometry of Contact and Symplectic Manifolds, Birkhäuser, Switzerland, 2002.

[6] Blair, D.E. and Oubina, J. A., Conformal and related changes of metric on the product of two almost contact metric manifolds, Publ. Mat., 34, (1), (1990) 199-207.

[7] Boyer, C. P and Galicki, K., Sasakian Geometry, Oxford Mathematical Monogrphs, Oxford University Press, 2008.

[8] Bryant, R., Metrics with Exceptional Holonomy, Ann. of Math. 126 (1987), 525-576.

[9] Chinea, D. and Gonzales, C., A Classification of Almost Contact Metric Manifolds, Ann. Mat. Pura Appl., (4) 156 (1990), 15-36.

[10] Fernández, M. and Gray, A., Riemannian Manifolds with Structure Group G_2, Ann. Mat. Pura Appl., (4) 132 (1982), 19-25.

[11] Joyce, D.D., Compact Manifolds with Special Holonomy, Oxford University Press, New York, 2000.

[12] Matzeu, P. and Munteanu, M-I., Vector Cross Products and Almost Contact Structures, Rendiconti di Matematica, Serie VII 22 (2002), 359-376.

[13] Özdemir, N., Aktay, Ş. and Solgun, M., Almost Hermitian Structures on the Products of Two Almost Contact Metric Manifolds, Preprint

[14] Todd, A. J., An Almost Contact Structure on G_2 Manifolds, arXiv:1501.06966.